Quantum states to brane geometries via fuzzy moduli space

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with Jurgis Pasukonis + background ...

Intro : AdS/CFT

String theory on $AdS_5 \times S^5$ with five-form flux N, radius R and string coupling g_s.

is equivalent to $N = 4$ SYM with gauge group $U(N)$, coupling g_{YM}^2

$$\frac{R}{l_s} = (g_{YM}^2 N)^{1/4}$$
$$g_s = g_{YM}^2$$

Intro : super-algebra

The $SU(2,2|4)$ supersymmetry algebra of the theory contains 32 supercharges, including 16 Q's and 16 S's. The Q's : as in any SUSY gauge theory. The S's only in super-conformal. The algebra helps classify gauge invariant operators.

The gauge theory has 6 scalars (under the Lorentz group) which are hermitian $N \times N$ matrices : $X_1 \cdots X_6$. Useful to combine them into 3 complex combinations

$$
\begin{aligned}
X &= X_1 + iX_2 \\
Y &= X_3 + iX_4 \\
Z &= X_5 + iX_6
\end{aligned}
$$

Intro : Operators $\rightarrow$ states

In **radial quantization**, we pick a point in Euclidean space-time, and the radial direction plays the role of time.

Gauge-invariant operators , e.g single traces trZ^n and their products, correspond to quantum states.

The radial scaling operator plays the role of the Hamiltonian, which is the translation operator for global time in AdS.

BPS operators/states

Generally, states belong to lowest weight repesentations of $SU(2,2|4)$. The lowest weight states are annihilated by the S's.

Q's are conjugate to the S's in radial quantization and generate the states in the representation.

For short representations, a fraction of the Q's also annihilate the LWS.

shortest representations : half-BPS operators

Contain lowest weight states annihilated by half the Q's

The lowest weight states are holomorphic gauge-invariant functions of just one complex matrix $Z = X_5 + iX_6$

$$\begin{aligned} L_0 = 1 \quad &: \quad trZ \\ L_0 = 2 \quad &: \quad trZ^2, (trZ)^2 \\ L_0 = 3 \quad &: \quad trZ^3, trZ^2 trZ, (trZ)^2 \end{aligned}$$

BPS condition implies that $L_0 = L_{56}$

BPS states : small traces and gravitons

In the regime of small L_0, these correspond to gravitons in the $AdS_5 \times S^5$. The S^5 :

$$x_1^2 + x_2^2 + \cdots x_5^2 + x_6^2 = R^2$$

has $SO(6)$ isometries, L_{ij}.

The states above are dual to graviton states – quantized small perturbations of the $SO(4,2) \times SO(6)$ background metric – which carry angular momentum for rotations in the 56 plane.

Single traces and multi-traces are orthogonal in planar limit $n \ll N$

Finite N effects

For large n, the states trZ^n are no longer dual to gravitons, rather BMN background at $n \sim \sqrt{N}$.

For $n \sim N$, the trace basis is no longer the simplest way to describe the BPS operators because single traces and multi-traces can mix

$$trZ^{N+1} = trZ^N trZ + \cdots$$

Cayley-Hamilton relations

A qualitative change in the description of the states : both in gauge theory and gravity.

Intro : Young diagram basis for half-BPS

A better basis is the Young diagram basis $\chi_R(Z)$ where R is Young diagram, described by lengths of rows $R = [r_1, r_2, \cdots]$. For $L_0 = n$, we have Young diagrams with n boxes.

A linear combination of multi-traces determined by Young diagram, e.g

$$\chi_{[2]}(Z) = \frac{1}{2}((trZ)^2 + trZ^2)$$
$$\chi_{[1^2]}(Z) = \frac{1}{2}((trZ)^2 - trZ^2)$$

Intro : Half BPS $\rightarrow$ Giant gravitons

Half-BPS states in $ADS_5 \times S^5$ also arise from giant gravitons. The S^5 can be thought as S^3 fibered over a disc in the $(5, 6)$ plane.

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 - x_5^2 - x_6^2$$

3-branes can wrap the S^3 and rotate around a circle in the disc. They will carry non-zero angular momentum L_{56} (like gravitons) but are described by a Dirac-Born-Infeld brane action.

$$\int d^4\sigma\sqrt{g} + \int C^{(4)}$$

Mc Greevy, Susskind, Toumbas, 2000

Intro : Half BPS $\rightarrow$ giant gravitons

There are also branes rotating around the (56) circle, which form an S^3 embedded in ADS_5 rather than S^5.

We will call these AdS_5 giants as opposed to S^5 giants.

The size of these branes grows as the angular momentum increases

Intro : Matters of size.

Important consequences :

- The brane has to be sufficiently large for the analysis of quantum fluctuations around the brane solution to be valid. Need angular momenta ($L_0 = L_{56}$) of order N.
- Single AdS_5 giant graviton branes can have arbitrarily large L_{56}.
- But in S^5 they have a cutoff N on the angular momentum. This was dubbed the stringy exclusion principle.

  Maldacena, Strominger, 1998

Intro : Half BPS giants and Young diagrams

The Young diagram description of operators gives a neat description of finite N cutoffs :

$$c_1(R) \leq N$$

And naturally offers dual operators for these different giants.

- A single long column with $L \sim N$ boxes is an S^5-giant with $L_{56} = L_0 = L$. Cut-off at N, explained by $\chi_{1^{N+1}}(Z) = 0$.
- A single row of length $L \sim N$ is an AdS_5 giant with $L_{56} = L_0 = L$.
- Multiple columns : Multiple S giants.
- Multiple rows : Multiple AdS giants.

Corley, Jevicki, Ramgoolam , 2001

Balasubramanian, Berkooz, Naqvi, Strassler, 2001

Intro : Young diagram basis - properties

- Diagonalize the inner product (from Yang-Mills theory)

$$\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle \sim \delta_{RS}$$

  which is not renormalized, and at zero coupling comes from $\langle Z^i_j (Z^\dagger)^k_l \rangle = \delta^i_l \delta^j_k$
- connected to S_n representation theory and to $U(N)$ representations-
- These are related by Schur-Weyl duality : which can be used to construct infinitely many charges in the gauge theory (commuting with L_0) which characterize the Young diagram operators.
- 3-point function of these operators : Littlewood-Richardson coefficients.

Intro : Young diagram basis - properties

- Can be modified using $S_n \to S_{n_1} \times S_{n_2} \cdots$ projectors, to describe strings attached to these branes.
- Counting of these modified operators agrees with expectations from Gauss Law for branes on compact spaces.

  Berenstein, Balasubramanian, Feng, Huang , 2004
- The Young diagram picture connects with free fermions and LLM geometries for operators of L_0 order N^2.
- Integrability found in studies of one-loop dilatation operator acting on strings attached to giants.

  Berenstein + collaborators ;

  de Mello Koch + collaborators ... 2007 -2011

Eighth-BPS case – The giants.

Focus on giant gravitons, which are large in S^5. Now require only that the brane solutions preserve $1/8$ of the Q's rather than one half.

There is an elegant construction of these by Mikhailov in terms of polynomials in 3 complex variables.

$$P(x, y, z) = 0$$

which define a 2-complex dimensional surface in $\mathbb{C}^3$.

The giant worldvolume is the intersection of this surface with $S^5 \subset \mathbb{C}^3$.

Time evolution simply

$$P(e^{it}x, e^{it}y, e^{it}z) = 0$$

Eighth-BPS : the operators

The gauge theory operators which are eighth-BPS at zero coupling $g_{YM}^2 = 0$ are holomorphic gauge invariant operators made from 3 complex matrices.

$$tr(X^2 Y^2 Z), tr(X^2 YZYZ), \cdots$$

At weak coupling, where g_{YM}^2 is non-zero but small, the spectrum of eighth-BPS operators is given by the Kernel of the 1-loop dilatation operator

$$\mathcal{H} = tr[X, Y][\check{X}, \check{Y}] + [X, Z][\check{X}, \check{Z}] + [Y, Z][\check{Y}, \check{Z}]$$

The X, Y part of this hamiltonian in the planar limit is related to the Heisenberg spin chain. (Minahan, Zarembo)

Explicit forms of these BPS operators at finite N are not known explicitly in general, but have been characterized in terms of S_n group theory and explicit results for low n derived. (general N)

Pasukonis, Ramgoolam : 2010

Tom Brown 2010

They are symmetrized traces in the leading large N limit, but have $1/N$ corrections which can be constructed using S_n group theory.

Relation to chiral ring

The spectrum of Kernel states is isomorphic, as a vector space, to the space of chiral operators.

This is formed by gauge-invariant operators modulo the relations

$$[X, Y] = [Y, Z] = [X, Z] = 0$$

This is in turn the space of functions of diagonal matrices X, Y, Z, invariant under permutations of the N entries.

Holomorphic functions on $S^N(\mathbb{C}^3)$.

Also isomorphic to Hilbert space of states constructed from N bosons in a harmonic oscillator potential in 3 dimensions.

HO states from giant gravitons

It was argued that the moduli space of giant gravitons (large in S^5) derived, starting from the Mikhailov polynomials, is a limit of projective spaces.

Geometric quantization of these projective spaces gives the Hilbert space of harmonic oscillators.

Biswas, Gaiotto, Lahiri , Minwalla; 2006 (BGLM)

Outline

BPS operators $\leftrightarrow$ oscillator states $\leftrightarrow$ giant graviton geometries.

- Review of Mikhailov and BGLM.
- Fuzzy geometry and oscillator states to branes
- Fluctuations of some specific brane worldvolumes.
- Group theoretic labels and Gauge theory operators.
- Open problems.

Mikhailov : Polynomials and giants

Take a polynomial in 3 variables, say degree is up to d

$$P(x,y,z) = \sum_{\substack{i,j,k \\ i+j+k \leq d}} c_{i,j,k} x^i y^j z^k = 0$$

Take the intersection of this with $|x|^2 + |y|^2 + |z|^2 = R^2$.
Mikhailov shows that this an eighth-BPS brane worldvolume.

The space of solutions is parametrized by the $c_{i,j,k}$. There are $n(d) = \frac{(d+1)(d+2)(d+3)}{6}$ of them.

First guess is $\mathbb{CP}^{n(d)-1}$. Projective space because scaling $c_{i,j,k} \sim \lambda c_{i,j,k}$ leaves equation invariant.

Simple case : subtlety

Simplest case,

$$P = (cz + 1) = 0$$

The intersection with the sphere S^5 is an S^3 :

$$|x|^2 + |y|^2 = 1 - 1/|c|^2$$

When $c \to \infty$ the 3-sphere has maximal size. When $c < 1$, there is no intersection with the S^5.

BGLM : A $\mathbb{CP}^1$ moduli space yet

SO the physical moduli space is the space $|c| > 1$.

BGLM show that the symplectic form, coming from the 3-brane action, vanishes on $|c| = 1$. They define a map to $\mathbb{CP}^1$ parametrized by another coordinate w – closed off the holes. And they find that the sympectic form on $\mathbb{CP}^1$ is $N\omega_{FS}$ where ω_{FS} is the standard Fubini-Study one.

BGLM : $\mathbb{CP}^{n(d)-1}$ moduli space

Consider the set of polynomials of degree up to d

$$P(x,y,z) = \sum_{\substack{i,j,k \\ i+j+k \leq d}} c_{i,j,k} x^i y^j z^k$$

The dimension of this space is $n(d) = \frac{(d+1)(d+2)(d+3)}{6}$

BGLM explain that the moduli space is still $\mathbb{CP}^{n(d)-1}$. There is a non-trivial transformation between the c_{ijk} to w_{ijk} so that the symplectic form is in the same cohomology class as $N\omega_{FS}$.

Geometric quantization : Hilbert space $\mathcal{H}^{(d)}$

The Hilbert space from geometric quantization of this moduli space is $\mathcal{H}^{(d)}$: the space of holomorphic sections of the line bundle of degree N over $\mathbb{CP}^{n(d)-1}$. Concretely :

$$\prod_{i,j,k} w_{i,j,k}^{n_{i,j,k}}$$

$$\sum_{i,j,k} n_{i,j,k} = N$$

This is isomorphic to the Hilbert space of N harmonic oscillators in 3 dimensions, which written in second-quantized form

$$\prod_{i,j,k} (a^{\dagger}_{i,j,k})^{n_{i,j,k}} |0\rangle$$

Harmonic oscillators

$$\prod_{i,j,k} (a^\dagger_{i,j,k})^{n_{i,j,k}} |0\rangle$$

The 3D harmonic oscillator hamiltonian separates into a sum of 3 HO Hamiltonians for 3 directions x, y, z. This state has $n_{i,j,k}$ particles in the state with i units of energy in the x ; j units in the y and k units in the z-direction.

This is a giant graviton state with angular momenta

$$L_{12} = \sum_i i n_{i,j,k}$$
$$L_{34} = \sum_j j n_{i,j,k}$$
$$L_{56} = \sum_k k n_{i,j,k}$$

This is also the Einstein solid model !!

BGLM : CP

To get the full spectrum of eighth-BPS operators, we take the limit of $d \to \infty$

$$\cdots \mathcal{H}^{(d)} \to \mathcal{H}^{(d+1)} \to \cdots$$

Now each particle in the harmonic oscillator system can have i, j, k with the only constraint $i + j + k \geq 0$.

If we restrict to Polynomials in one variable we recover a Hilbert space spanned by

$$(a_i^\dagger)^{n_i}|0\rangle$$

This is isomorphic to the space of Young diagrams.

" $a_i^\dagger$ creates a row of length i". The n_i is the number of rows of length i. Total number of rows (including length 0) is fixed at N.

states to specific brane geometries ?

Specific geometries of branes are specific points on w-space. ($\mathbb{CP}$)

Important tool is fuzzy geometry of CP. Fuzzy geometry starts with the endomorphism algebras of a sequence of Hilbert spaces and approaches the algebra of functions (not holomorphic) of some geometry – in this case CP

A lot of literature on this with the view that the CP (CP^2) is part of space-time or part of a brane worldvoulme. Here the interpretation is different, but maths is the same.

Toric geometry will also show up.

Fuzzy geometry of $\mathbb{CP}^{M-1}$

Take the Hilbert space $\mathcal{H}$ of degree N holomorphic polynomials in M variables W_I, with $I = 0, 1 \cdots, M-1$

Arising from quantization of $\mathbb{CP}^{M-1}$ with symplectic form $N\omega_{FS}$.

States (wavefunctions) are

$$\prod_I W_I^{n_I}$$

Will write W_I here for $w_{i,j,k}$ from previous.

Fuzzy geometry : Endomorphism algebras

The endomorphism algebras $End(\mathcal{H}_N)$ are spanned by operators of the form

$$W_{I_1} W_{I_2} \cdots W_{I_n} \partial_{W_{J_1}} \partial_{W_{J_2}} \cdots \partial_{W_{J_n}}$$

And transform as $V_{n,\bar{n}}$ under the $SU(M)$ isometry group of the $\mathbb{CP}^{M-1}$ – a "composite" made of n fundamentals and n anti-fundamentals.

There is a cutoff at $n = N$, because the Hilbert space has polynomials of degree exactly N in these homogeneous coordinates of the $\mathbb{CP}$.

Fuzzy geometry : $SU(M)$ harmonics

The space $\mathbb{CP}^{M-1}$ is a coset $SU(M)/U(M-1)$ and has a function space which can be decomposed under $SU(M)$, with representations

$$Fun(\mathbb{CP}^{M-1}) = \bigoplus_{n=0}^{\infty} V_{n,\bar{n}}$$

These are spanned by functions of the form

$$\frac{W_{I_n} \cdots W_{I_1} \bar{W}_{J_n} \cdots \bar{W}_{J_1}}{|W|^{2n}}$$

Truncated spectrum of harmonics and star product

The Endomorphism algebra $\mathcal{H}_N$ gives a finite approximation of the classical algebra of functions.
The map between operators and functions takes the form

$$\begin{gathered} W_{I_1} W_{I_2} \cdots W_{I_n} \partial_{W_{J_1}} \partial_{W_{J_2}} \cdots \partial_{W_{J_n}} \\ \downarrow \\ \frac{W_{I_1} W_{I_2} \cdots W_{I_n} \bar{W}_{J_1} \bar{W}_{J_2} \cdots \bar{W}_{J_n}}{|W|^{2n}} \end{gathered}$$

There is a star product on the functions which makes this map an algebra homomorphism. At large N, the star product just becomes the ordinary product.

Fuzzy to toric geometry

The algebra of functions is generated by $W_I \partial_{W_J}$ which are the $SU(M)$ Lie algebra which acts on $\mathbb{CP}^{M-1}$.

Specific wavefunctions can be characterized by the eigenvalues of the Cartan of this $SU(M)$.

$$W_I \partial_{W_I} \leftrightarrow W_I \bar{W}_I$$

In the toric description of the $\mathbb{CP}$, these magnitudes of the homogeneous coordinates parameterize a simplex in $\mathbb{R}^{M-1}$. The angles parametrize the T^{M-1} fibres of the toric fibration.

The toric base can be viewed as the weight spac of this $SU(M)$ and the wavefunctions form a discrete set of points on the simplex.

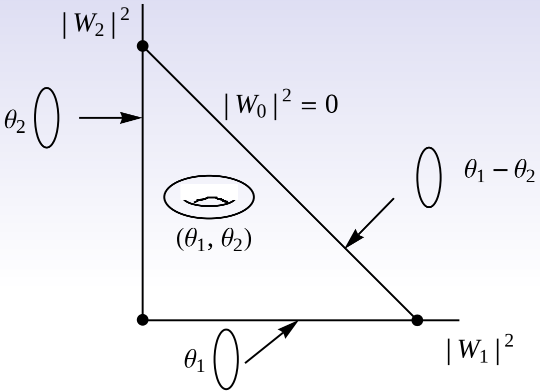

Figure: $\mathbb{CP}^2$ as a toric fibration (e.g half-BPS states deg up to 2).The base is the triangle (2-simplex) parametrized by $|W_1|^2, |W_2|^2$ and the fiber is the torus (θ_1, θ_2). The fiber degenerates to a circle on the edges of the triangle and to a point in the corners.

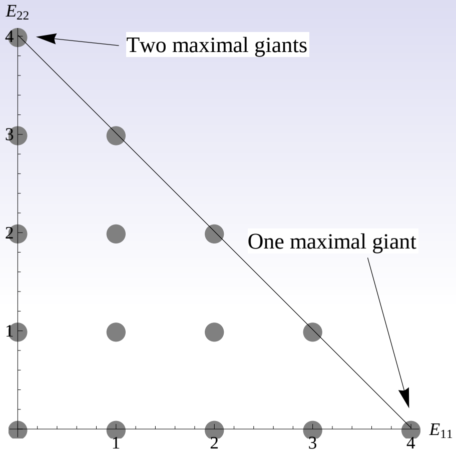

Figure: States on fuzzy $\mathbb{CP}^2_N$ with $N = 4$. The three corners of the base correspond to the points where T^2 shrinks to a point, and the states there are localized in all $\mathbb{CP}^2$ directions.

Density matrix for a pure state

The density matrix associated with a wavefunction ψ is the projector $P_\psi = \psi >< psi|$ so that

$$tr_{\mathcal{H}}(P_\psi \mathcal{O}) = \langle \psi | \mathcal{O} | \psi \rangle$$

P_ψ is in $End(\mathcal{H}_N)$. For ψ equal to

$$\prod_I W_I^{n_I}$$

it becomes the function

$$\frac{\prod_I (W_I \bar{W}_I)^{n_I}}{|W|^{2 \sum_I n_I}}$$

Special states at the vertices

Generic states are localized at points in the base simplex and are uniformly spread along the tori.
At the vertices of the base simplex, are special points corresponding to states

$$(W_I)^N$$

where I runs over the triples (i, j, k) describing the monomials.
These correspond to the giant graviton geometry described by

$$P(x, y, z) = x^i y^j z^k = 0$$

These are the simplest states which can be mapped cleanly to a polynomial. For more general states, we have to deal with the subtleties of the map between c's and w's – this could be bypassed here due to the fact that these points on the moduli space are special points invariant under $U(1)^3$

The perturbations around these states, i.e nearby HO states have a nice factorization property which can be understood physically.

In some cases, this factorization can be exhibited by taking limits of the partition function of the BPS states.

Perturbations around the vacuum

The vacuum state is

$$w_{0,0,0}^{N}$$

where $c_{0,0,0}$ is the coefficient of the identity in the polynomial.

The perturbations are generated by

$$A^{\dagger}_{i,j,k} = w_{i,j,k}\partial_{w_{0,0,0}}$$

with $i + j + k$ small and positive. These are bulk states (gravitons).

When we analyze perturbations around the state

$$w^N_{1,1,1}$$

corresponding to $xyz = 0$, which can be interpreted as a composite of $x = 0, y = 0.z = 0$ branes.

There are excitations

$$\begin{aligned}
A^{\dagger}_{i,j,k} &= w_{1+i,1+j,1+k}\partial_{w_{1,1,1}}; i + j + k \geq 0 \\
A^{\dagger}_{-1,j,k} &= w_{0,1+j,1+k}\partial_{w_{1,1,1}} \\
A^{\dagger}_{i,-1,k} &= w_{1+i,0,1+k}\partial_{w_{1,1,1}} \\
A^{\dagger}_{-1,j,k} &= w_{1+i,1+j,0}\partial_{w_{1,1,1}}
\end{aligned}$$

The first line is a set that has the same quantum numbers as excitations around vacuum. They are bulk gravitons.

The second line are worldvolume excitations on the x-brane. The x-brane is extended along y, z planes so can have worldvolume (open string) excitations. Similarly for 3rd and 4-th lines.

There are also $A_{-1,-1,k}$ excitations which open string excitations at the string which lies at intersection of two 3-branes.

These open string excitations should be visible by doing worldvolume analysis of the branes ; or by considering open strings by worldsheet methods.

The simplest predictions around 1/2-BPS verified by "restricted schur" operator constructions.

Also by local analysis of the symplectic form near the specified monomials.

geometric Young diagram labels for eighth BPS sector

Another outcome of analysing the CP-structure of the HIlbert space is a Young diagram labelling for the eighth-BPS sector states.
The half-BPS Young diagrams (with up to d columns) and maximum of N rows, come from quantizing the giant gravitons from polynomials

$$P(x, y, z) = c_{0,0,0} + c_{0,0,1} z + c_{0,0,2} z^2 + \cdots$$

$(w_{0,0,i})^{n_i}$ creates n_i rows of length i. So degree decomposition of the holomorphic wavefunction gives the Young diagram.

degree decomposition for $1/8$ BPS

Same thing can be done in the full 1/8 BPS case. for the states coming from degree polynomials of degree up to d in 3 variables, we can organize the states according to how many came from each degree.

Now at each degree there are complete representations of $U(3)$.

This leads to (Λ, Y) labels : pairs of $U(3) \times U(N)$ YOung diagrams.

For explicit small Λ (e.g $[2, 2], [3, 2]$ at any N we can explicit construct such operators)

More generally, not known how to organize the Kernel of 1-loop dilatation operator in this geometrical way.

$$\mathcal{H}_N^{l;d} = \bigoplus_{Y(N_1,N_2,\cdots,N_d;N)} \bigoplus_{\Lambda\in Reps(U(3))} V_\Lambda \otimes \left(\bigoplus_{\Lambda_1\cdots\Lambda_d} V^{\Lambda}_{\Lambda_1,\cdots,\Lambda_d} \otimes_{k=1}^{d} V^{h;k;N_k}_{\Lambda_k} \right)$$

This means that there is a labelling of BPS states by

$$|\Lambda, M_\Lambda, Y, \Lambda_1, \cdots, \Lambda_d, a, b_1, \cdots, b_d\rangle$$

where $\Lambda, \Lambda_1, \cdots, \Lambda_d$ are irreps of $U(3)$, a is a Littlewood-Richardson multiplicity for Λ appearing in $\Lambda_1 \otimes \Lambda_2 \cdots \otimes \Lambda_d$. The labels b_k (for $k = 1 \cdots d$) run over the multiplicities of Λ_k appearing in $\mathrm{Sym}^{N_k}(\mathrm{Sym}^k(V_3))$. The numbers N_k obey $N_1 + \cdots + N_d \leq N$ and determine the Young diagram Y of $U(N)$.

$w_{p,q,r}$ of fixed degree k form $Sym^k(V_3)$. Polynomials in $w_{p,q,r}$ form another symmetric product.

Open problems

- Match giant graviton geometries and oscillator states to kernel of one-loop dilatation op.
- Find the (Λ, Y) basis from gauge theory. There is a (Λ, R, τ) basis at zero coupling. But one-loop dilatation operator mixes the R, τ labels.
- What are the $1/8$ BPS generalizations of the Littlewood-Richardson coeffs. ?
- Integrability for strings attached to the $1/8$ BPs states ?
- Is there a geometric quantization picture for the zero coupling BPS states ?

Open problems

- Apply some of the fuzzy geometry constructions to giant gravitons for more general AdS/CFT examples – where S^5 is replaced by Sasaki-Einstein. Chiral ring states (analog of harmonic oscillator partition functions) from gauge theory known.
- Sufficiently large non-BPS perturbations of eighth BPS states are finite horizon black holes. Would like to understand aspects black hole dynamics from the combinatorics and correlators of gauge theory operators.