Gravitational Effects from Amplitudes for String-Brane Interactions

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Abstract

In this thesis we examine both high and low energy gravitational phenomena using amplitudes derived from a quantum description of interactions between strings and branes. We demonstrate that the coupling of branes to the massless states of the closed string exactly match the couplings of these branes at low energy to the supergravity fields associated with these states. We examine the scattering of massive closed strings from a brane at high energy and large impact parameters and it is concluded that this process can be well approximated by an eikonal description.

Declaration

The work in this thesis is based on research carried out at the Centre for Research in String Theory, Queen Mary University of London. Except where specifically acknowledged in the text the work in this thesis is the original work of the author and based upon the joint-publications [1], written with Rodolfo Russo and David Turton, as well as [2] written with Cristina Monni.

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Contents

Chapter 1

Introduction

At this time, it is understood that there exist four kinds of interactions between fundamental particles which can be identified experimentally. Of these four, there are three for which the microscopic behaviour is well understood in terms of a quantum field theory; electromagnetism, the weak force and the strong force are all accurately described up to the limits of experimental precision by renormalisable gauge field theories with broken symmetry. Yet very little is known about gravitational interactions at small distances, either experimentally or theoretically. Even the classical description of gravity afforded by Newton's law has only been experimentally verified down to a length scale of 10^{-5}m [3], leading to proposals such as that of [4] which suggests that large extra spatial dimensions could exist that are only accessible to gravity leading to its apparent weakness and our inability to detect these dimensions. If this were true then it raises the exciting possibility that exotic physics may soon be seen in the coming generations of experiments. Attempts to quantise gravity as a gauge theory, as was done for the other forces, fail due to the irrelevance of the gravitational coupling which results in extremely weak interactions at low energies, but these grow in strength with energy thus rendering the naive perturbative description nonrenormalisable. It has been suggested in a proposal known as *Asymptotic Safety* that this only occurs due to a poor choice of variables for the perturbative expansion and that a better choice would lead one to a renormalisable quantum description of gravity [5]. However, it has also been claimed that the stark difference between the density of states in gravity, as computed from the Bekenstein-Hawking entropy, and the density of states in any renormalisable quantum field theory rules this out as a possibility [6]. While the renormalisability of gravity may remain an open question, it is this resistance to conventional description that led to the proposal that string theory may provide a solution to this problem in a way which united all four kinds of interaction in a single framework.

String theory was originally formulated in terms of scattering amplitudes as a means to describe the strong force [7] on account of its ability to incorporate a large variety of particles of different masses and spins, but was eventually superseded after the development of quantum chromodynamics. However, shortly after this it was realised that the S-matrix of this model could be interpreted as a description of interactions between string-like objects by using a Lagrangian [8–10]. Instead of treating the fundamental interacting entities of the theory as point-like, this action describes the dynamics of an object that is extended in one spatial direction with a world-sheet, a surface representing the motion of this object through spacetime. The coordinates describing the location of the world-sheet as an embedding in spacetime then formed the fields of a two-dimensional quantum field theory on the world-sheet. Boundary conditions for this theory dictated that there exist two distinct types of string — those with end-points are known as open strings whereas when these end-points are identified we obtain strings with a periodicity in the spatial dimensions, these being referred to as closed strings. The Lagrangian formulation indicated that there were many possibilities for building different string theories, yet common to all of them was the existence of a massless spin-2 state in the spectrum of the string which couples to other strings in a manner consistent with the graviton [11]. Furthermore, it was discovered that the loop amplitudes of string theory were UV finite and there are good grounds for believing that this is generally true. Subsequently the development of string theory as a theory of quantum gravity began in earnest, yielding the enormous framework we have now which connects seemingly disparate ideas from both physics and mathematics.

Today string theory is not a single model, rather it is a complex beast composed of myriad components taken from all across physics. Because of this it is quite malleable and boasts tremendous computational power, yet it may also be some indication that a more appropriate framework exists in which this disjunction does not occur. Indeed, it is a major motivation in the analysis of gravity using string theory that we may find some clue as to the correct degrees of freedom to use in its exposition. One of the most significant discoveries in this direction was the realisation of D-branes as dynamical objects of the non-perturbative spectrum of string theory [12, 13]. These first arose when it was realised that the end-points of open string could satisfy two possible boundary conditions for each spacetime direction: Neumann (N) boundary conditions implied that the end-points were free to move but this must be constrained such that they follow a null trajectory in spacetime. The alternative was to impose Dirichlet boundary conditions, effectively fixing the position of the end-points to a single locus in that direction; a Dirichlet-brane, or D-brane, is then the hypersurface in spacetime on which these end-points are fixed. These were initially considered to be static and unphysical due to their breaking of Lorentz invariance in flat spacetime, however, as previously stated they have proved to be physical entities in their own right which can be described for small values of the string coupling in terms of the open strings that live on them. At that time it was already understood that for energies much smaller than the string scale only massless degrees of freedom are relevant and string theory should be accurately portrayed by field theories describing point particles; in this limit it was realised that the D-branes are represented by the classical solutions to the equations of motion for these field theories which carry some conserved charges [14]. Recently, this multifaceted nature of string theory has been key to advancements in our understanding of black hole physics, where the dual-properties of D-branes as classical solutions of supergravity and as fundamental objects of string theory are beneficially employed [15]. Furthermore, the expansion of results for string theory amplitudes in powers of the string length parameter are commonly compared with the analogous point particle amplitudes, both as a convincing check and as a guide to obtaining new results.

Thus, presented with the tools of string theory, how might one use this to learn about gravitational interactions? The primary means for investigating any theory of particle physics has traditionally been scattering phenomena and this was the method used for some of the original work done on this topic. In order to see new physics the traditional approach is to consider the energy of the probes to be high such that we may resolve processes going on at short distance scales; particularly in the case of quantum gravity one would like to be able to resolve the classical singularities seen in black hole solutions, as well as the quantum infinities produced by the naive quantisation of general relativity. With string theory we have the additional incentive that it allows us to overlook some of the global aspects of spacetime. This is because one of the most notable features of string theory is that it places consistency requirements on the dimensionality of spacetime and perversely this number is greater than that which we observe. Any phenomenologically useful model must eventually compactify these dimensions to leave four of them extended (for a discussion of these issues see reviews such as [16] or the recent book [17]), but the precise details of this compactification are expected to have little effect on high energy physics. The reason for this is that high energy processes probe short distances and, provided that the energy is high enough, this distance will be smaller than the scale of compactification; therefore these processes are insensitive to the specific nature of the compactification.

Initial studies of string theory as a theory of quantum gravity which can deal with high energy processes focused on two principle forms of scattering: the first was hard scattering [18–20] in which the projectiles exchange large amounts of momenta; equivalently we could say that the projectiles approach one another closely during scattering. In these early examinations it was discovered that at each order in the perturbative expansion, the high energy behaviour of all string scattering amplitudes is dominated by a saddle point in the moduli space of the corresponding Riemann surfaces. These results demonstrated a remarkably soft exponential fall-off of the amplitude with the energy completely unlike anything seen in quantum field theory, a demonstration of how the finite size of the string naturally cuts off processes which result in infinities in the case of point particles. What is more, this behaviour was found to be universal, applying equally to all string states, and these processes have a simple interpretation in terms of a classical string trajectory in spacetime. However, in the limit for which these results are valid it is found that all orders of the perturbative expansion make significant contributions to the S-matrix and so convergence becomes an issue which remains unresolved [21].

The second branch of investigation considered the soft exchange of gravitons, in which the momenta transferred are small [22–24]. For this it was found that interesting features arise at different values of the transferred momentum t. For small t, corresponding to large distance processes, it was found that elastic scattering dominates and this is mediated by the long range exchange of gravitons. As t is increased one encounters a semiclassical eikonal description in which the strings appear to move as if in a gravitational field produced by the target string. This description is required by the break down of partial-wave unitarity for the scattering amplitudes which contain increasingly large divergences in the energy at each order of the perturbative expansion. In the field theory limit in which the size of the string is neglected it was noted that all of these terms may be resummed to give an impact parameter b dependent phase to the S-matrix, $S = e^{i\delta(b)}$, which ensures unitarity. The analogous resummation of the full string scattering amplitudes treats the nth order term in the series as an amplitude describing the exchange of $(n+1)$ soft gravitons but with all possible states being allowed to interpolate between these exchanges. These intermediate states modify the eikonal phase δ of the S-matrix, shifting the impact parameter by the strings spacetime coordinates $\hat{X}$ which, being operators themselves, forced the promotion of the eikonal phase to the eikonal operator $\hat{\delta}(b + \hat{X})$. If the value of t is increased further quantum processes begin to contribute significantly due to the production of inelastic and diffractive states. These states contain excitations of the string not present in the initial states and since these internal oscillations dictate the mass of these string states then the final states generally contain strings strings of nonzero mass. Beyond this, the analysis could be extended into the regime similar to that of hard scattering and in this regime it was noted that processes involving the capture of the probe string begin to dominate.

As has already been stated, one of the most important recent discoveries for the analysis of quantum gravity has been that of the existence of D-branes as nonperturbative objects in the spectrum of string theory. From the supergravity point of view, D-branes are solutions which can demonstrate singularities and, in the case of D-brane configurations with more charges, by an appropriate compactification of the extra spacetime dimensions they can be made to resemble a classical black hole with a horizon [25, 26]. The world-sheet description then provides us with a microscopic description of this black hole, opening up many exciting possibilities. Currently there is much interest in the use of these ideas to resolve the information paradox, attempting to to account for the entropy of black holes as being due to the microstates of these string-brane systems.

The aim of this thesis is twofold, this being the investigation of gravitational phenomena using string theory both for high energy and low energy configurations. At low energies we will consider a two-charge black hole-like configurations of a D1-brane and D5-brane wrapped around a compact spacetime dimension, both of which carry a travelling wave. The supergravity solutions describing these configurations are well known [27–31] but not all of these can really be a low energy description of some string configuration since these configurations may require states that are physical unacceptable. We demonstrate that it is possible to compute the asymptotic values of those solutions which do accept a microscopic description in terms of open strings from string amplitude calculations based on work presented in [1]. We describe them as black hole-like since the microstates with two particular values of charge have a degeneracy that scales in the same way as a black hole with two charges, however their description in supergravity fail to have a horizon of nonzero area and so don't really describe macroscopic black holes. The two D-brane configurations considered here are in fact related to each other via T-duality, which describes the relationship between string theories defined on spaces with compact dimensions of different radii, and there are many other systems that one may build which are also dual to these and one another. Spacetime supersymmetry may be used to guarantee that the degeneracy of the microstates in these two-charge configurations is independent of the duality frame, however, the nature of the supergravity solutions representing them can vary greatly; in [32] it was observed that while some duality frames admit this black hole-like description, for others the supergravity solutions are smooth and horizonless and no duality frame may be described by both types. A similar analysis as presented here was performed in the D1-D5 duality frame consisting of a D1-brane and D5-brane bound together [33] and for these systems it may be argued, from the scaling of the degeneracy, that they should only admit smooth solutions. These both provide examples of the unusual fact that calculations done with strings on a flat background somehow generate background fields. The techniques employed in [1, 33] were applied to two-charge systems since they are simple and tractable, however they have since been successfully extended to the case of three-charge systems [34] which unlike the two-charge system has a nonzero horizon. This allows the possibility of applying a comparison of black hole entropy with microstate counting for the case of a macroscopic black hole.

At high energies we return to the age-old method of string scattering and consider the high energy scattering interaction between a massive string and a D-brane at tree level. Yet this is a fundamentally new type of scattering in comparison with that discussed above, here we are taking a soliton of the non-perturbative string spectrum as our scattering target and probing it using the technology of perturbative string theory. In many ways this allows an analysis which is much cleaner than usual, since at weak coupling when the tools of perturbation theory are applicable these solitons are extremely massive and therefore largely unaffected by the probe string, thus the properties of these scattering amplitudes which may be attributed to the brane are easily isolated since they must be independent of the energy of the probe. The work presented here was originally laid out in [2] and in it is calculated the tree-level scattering amplitudes describing the interaction of an energetic closed string with a stack of N parallel Dp-branes for small scattering angles; the closed string may be any of a particular class of bosonic states referred to as the leading Regge trajectory, strings of maximum possible spin for their mass and we allow the possibility that the interaction causes the initial string to be scattered into a different state of this class. These amplitudes allow us to carry out a check on the validity of a recent proposal regarding the resummed amplitudes for high energy string-brane scattering with small scattering angles which states that these have an expression in terms of an eikonal operator [35]. It is found that the factorisation of higher-order diagrams necessary for the eikonal operator appears to be correct and that the eikonal description is accurate.

The main body of this work is arranged as follows — in Chapter 2 we review the basic tools necessary to perform world-sheet calculations in string theory. String theory is introduced as a free superconformal field theory and the reader is reminded of some of its properties, these are then applied to the case of an isolated closed string in order to for us to note some features of the spectrum to be used later on. The S-matrix for string theory is presented with a short exposition of its form before moving on to discuss the inclusion of D-branes, non-perturbative objects in string theory. Chapter 3 goes on to discuss another vital component of string theory, its representation at low energy by supergravity. We discuss the massless spectrum of the preceding chapter and the field theory action describing their interactions at low energy where the massive states are no longer dynamical. After this, solutions of the supergravity equations are considered and we review those describing black hole-like objects, the p-brane solutions. These are identified with the D-branes of the world-sheet theory. In Chapter 4 we examine a more complex brane system which may be considered to represent a D-brane configuration with 2 charges. The world-sheet description of this system is used to compute the one-point functions for the supergravity fields which determine their asymptotic behaviour and this method is confirmed to produce the correct results. Chapter 5 instead considers the interactions of strings and branes in scattering processes at high energies. First of all we review the recent proposal of an eikonal operator which reproduces the elements of the S-matrix for string-brane scattering [35]. Then we make use of effective operator methods to calculate the high energy limit of tree-level scattering amplitudes describing the scattering of a massive string state from a Dp-brane, this is then compared with the analogous quantity computed using the eikonal operator and the two are found to be in perfect agreement. This is a good indication that the eikonal operator captures all of the relevant terms which are most divergent high energy and does not miss any other contributions from the moduli space. Finally, in Chapter 6 we review the results of this thesis and propose future avenues of investigation which may be of interest.

Chapter 2

Scattering Amplitudes in String Theory

In this chapter we shall be considering a description of strings propagating on a flat background which has the benefit of being tractable and will allow us to gather all of the core ingredients we will need for the calculation of the string amplitudes analysed in Chapters 4 and 5. It will be discussed in later chapters how, even if this is the case, that gravitational phenomena are unavoidable in string theory as soon as soon as closed string appear as external states or propagate in some internal channel. In Section 2.1 we introduce the classical description of the string as an object which maps out a two-dimensional surface in spacetime via an action which describes the coordinates of this surface, this is then extended to the superstring which generalises this action by introducing superpartners to the world-sheet coordinates. In Section 2.2 we discuss the quantisation of this action via the path integral, which involves the inclusion of Faddeev-Popov ghosts for the purposes of fixing the world-sheet symmetries, and the isolation of the physical states of the Hilbert space of the superstring by BRST invariance. This is followed by a discussion of the spectrum of closed superstrings in Section 2.3, these being the states which will most interest us in later chapters and then a discussion of the local world-sheet operators which can be used to represent these states, vertex operators, in Section 2.4. In Section 2.5 we define the S-matrix as it appears in string theory and give a general discussion of its subtleties and finally we finish by discussing the importance of world-sheets with boundaries and their relationship to D-branes.

2.1 The string action

When initially considering the classical dynamics of a string in D-dimensions, characterised by spacetime coordinates X^μ, we are led to an action that is proportional to the area swept out by a string in spacetime. The result of this is the bosonic string, which is most conveniently described by the so-called Polyakov action,

$$S_{X,g} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{2.1}$$

where the indices take the values $a, b \in \{0, 1\}$. The string coordinates are parameterised by a two dimensional Lorentzian manifold, the *world-sheet*, with coordinates $(\tau, \sigma) \equiv (\sigma^0, \sigma^1)$ and metric g_{ab}. Both the world-sheet metric and the spacetime metric both take a 'mostly plus' signature. The length parameter α' specifies the scale of the strings which determines when their extended nature should become relevant to any phenomena under consideration. The action (2.1) is favoured for the study of most aspects of string theory by virtue of its linearity in the spacetime coordinates X. However, since there is no kinetic term for the metric in this action we may eliminate it by substituting in the solution to the equations of motion for g, one then recovers the original string action of Nambu and Goto [36],

$$S_{NG} = \frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2} = \frac{1}{2\pi\alpha'}\int dA, \tag{2.2}$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau} \qquad X'^\mu = \frac{\partial X^\mu}{\partial\sigma}. \tag{2.3}$$

This action is classically equivalent to (2.1) and makes clear the relationship to the area of the world-sheet, but its nonlinearity often makes it impractical.

Equation (2.1) enjoys both local coordinate invariance on the world-sheet and is also invariant under the Weyl transformation $g_{ab} \to \exp[2\omega(\sigma^i)]g_{ab}$. In the absence of boundaries on the world-sheet we can generalise this action without disturbing these symmetries by including an Einstein-Hilbert term for the world-sheet metric,

$$S_\lambda = -\frac{\lambda}{4\pi}\int d^2\sigma\sqrt{-g}R. \tag{2.4}$$

The tensorial structure of this term makes it clear that it is invariant under local diffeomorphisms, and under a Weyl rescaling the integrand transforms as $\sqrt{-g'}R' = \sqrt{-g}\left(R - 2\nabla^2\omega\right)$; since the two differ by just a total derivative this would imply that S_λ respects Weyl symmetry. Of course, if one considers world-sheets with boundaries this will not be the case but Weyl invariance can be recovered for S_λ by the inclusion of a boundary term of the form

$$\frac{\lambda}{2\pi}\int_{\partial M} ds\, k \tag{2.5}$$

where the boundary is parameterised by the proper time (or length) s, and k is the *geodesic curvature* of the boundary

$$k = \pm t^a n_b \nabla_a t^b, \tag{2.6}$$

while n^a is an outward pointing normal unit vector, t^a is a tangent unit vector and the upper and lower signs correspond to timelike and spacelike boundaries respectively.

The action $S = S_X + S_\lambda$ is not only that of a two dimensional object in D dimensional spacetime, it also describes two dimensional gravity coupled to D scalar fields, X^μ. It is the most general such action compatible with world-sheet diffeomorphism and Weyl invariance along with spacetime Poincaré invariance. Even with the inclusion of the term S_λ in the action the world-sheet metric does not become dynamical since the symmetries allow us to choose coordinates such that the world-sheet is always locally flat; instead, S_λ only ever contributes a topological factor and so we will give it no further attention until Section (2.5). It is often convenient to choose world-sheet coordinates such that the metric is fixed in conformal gauge, $g_{ab} = \exp[\omega(\sigma^i)]\eta_{ab}$, then equation (2.1) describes a free conformal field theory in two dimensions, permitting many aspects of the theory to be easily explored.

The quantised theory of a string propagating in a flat background which can be obtained from equation (2.1) is rich, but despite its many interesting properties it also has many shortcomings, most notably a total lack of fermionic states and the presence of tachyonic states, as such we would like to generalise (2.1). There are many possibilities for this and throughout this work we will make use of the first known consistent approach, the RNS model [37–39], in which additional world-sheet degrees of freedom are inserted into the Polyakov action such that there is a world-sheet supersymmetry, hence we introduce the two dimensional Majorana spinors Ψ^μ as the superpartner of the world-sheet coordinate field.

In order to consistently couple spinors in a locally coordinate invariant way we require the two dimensional curved space Dirac matrices $\gamma^a(\sigma^i)$ appropriate to the world-sheet. These are properly defined by introducing an orthonormal vielbein e_a^α such that $g_{ab} = \eta_{\alpha\beta}e_a^\alpha e_b^\beta$ thus allowing us to write $\gamma^a(\sigma^i) = e_\alpha^a(\sigma^i)\gamma^\alpha$, where now the γ^α are the traditional Dirac matrices of a locally Lorentzian frame and satisfy $\{\gamma^\alpha, \gamma^\beta\} = -2\eta^{\alpha\beta}$. Here we will choose the basis

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.7}$$

for these matrices. Since these matrices are purely imaginary the Dirac operator $i\rho^a\partial_a$ will be real and the Majorana condition allows us to take the spinor components to be real.

Of course, in introducing a supersymmetric partner for the world-sheet coordinate field we should strictly speaking do the same for the world-sheet metric. Indeed, the requirement of local supersymmetry forces us to include a world-sheet *gravitino* χ_a, a Majorana spinor with a vector index. Much like the metric field this is nondynamical in two dimensions since a kinetic term for the gravitino would be proportional to $\chi_a \gamma^{[a}\gamma^b\gamma^{c]}\nabla_b \chi_c$ which necessarily vanishes as a result of antisymmetry in the world-sheet indices.

The simplest action we can write down including X^μ, Ψ^μ, e_a, χ_a which retains the coordinate and Weyl invariance of the Polyakov action while also demonstrating our desired supersymmetry is the following*[40]

$$S = -\frac{1}{2\pi}\int d^2\sigma\, e\left[\frac{1}{2\alpha'}g^{ab}\partial_a X^\mu \partial_b X_\mu - i\bar{\Psi}^\mu \gamma^a \partial_a \Psi_\mu + \bar{\chi}_a \gamma^a \gamma^b \Psi^\mu \left(\sqrt{\frac{2}{\alpha'}}\partial_b X_\mu - \bar{\chi}_b \Psi_\mu\right)\right]. \tag{2.8}$$

To be specific, the local supersymmetry transformations that leave (2.8) invariant are

$$\begin{aligned} \delta X^\mu &= \sqrt{2\alpha'}\bar{\epsilon}\Psi^\mu, & \delta\Psi^\mu &= -i\gamma^a \epsilon\left(\frac{1}{\sqrt{2\alpha'}}\partial_a X^\mu - \bar{\Psi}^\mu \chi_a\right) \\ \delta\chi_a &= \nabla_a \epsilon, & \delta e^a_\alpha &= -2i\bar{\epsilon}\gamma^a \chi_\alpha. \end{aligned} \tag{2.9}$$

Furthermore, the Weyl symmetry is modified since we must specify how the new superpartners are transformed, the suitable behaviour being

$$\begin{aligned} \delta e^\alpha_a &= \omega e^\alpha_a, & \delta\chi_a &= \frac{1}{2}\omega\chi_a \\ \delta\Psi^\mu &= -\frac{1}{2}\omega\Psi^\mu, & \delta X^\mu &= 0. \end{aligned} \tag{2.10}$$

In addition to the symmetries that the action (2.8) was constructed to have, it also possesses one more local fermionic symmetry determined by Majorana spinor parameter ξ acts only on the gravitino characterised by

$$\begin{aligned} \delta\chi_a &= i\gamma_a \xi \\ \delta e^\alpha_a = \delta\Psi^\mu &= \delta X^\mu = 0. \end{aligned} \tag{2.11}$$

We have now an action with three local bosonic symmetries; these are the invariance under reparameterisation for the two world-sheet coordinates and one for Weyl transformations. These allow us to fix the form of the world-sheet metric (at least locally). Likewise, the world-sheet supersymmetry and superconformal symmetries allow us to choose the components of the gravitino. Hence we are free to pick the conformal gauge,

$$g_{ab} = e^\omega \eta_{ab}, \quad \chi_a = i\gamma_a \xi, \tag{2.12}$$

where ω is some scalar field and ξ is some Majorana spinor. Classically we may use super-Weyl transformations to gauge these away to leave $g_{ab} = \eta_{ab}$ and $\chi_a = 0$.

*Note that for two dimensional Majorana spinors the covariant derivative reduces to the partial derivative since Fermi statistics forces the spin connection term to vanish.

With these choices the action simplifies to

$$S = -\frac{1}{2\pi}\int d^2\sigma \left(\frac{1}{2\alpha'}\eta^{ab}\partial_a X^\mu \partial_a X_\mu - i\bar{\Psi}^\mu \gamma^a \partial_a \Psi_\mu\right), \tag{2.13}$$

which exhibits a residual global supersymmetry along with conformal invariance. The equations of motion for the dynamic fields are particularly simple in this gauge, these are

$$\partial_a \partial^a X_\mu = 0, \quad \gamma^a \partial_a \Psi^\mu = 0. \tag{2.14}$$

In addition to these, the equations of motion for the world-sheet metric and gravitino yield the constraint equations which are used to eliminate negative norm states from the spectrum of the superstring. In analogy with the bosonic string, the variation of the vielbein defines the world-sheet energy momentum tensor which we assign the normalisation $T_{ab} \equiv -\frac{2\pi}{e}\frac{\delta S}{\delta e^b_\alpha} e_{a\alpha}$,

$$T_{ab} = \frac{1}{2\alpha'}\partial_a X^\mu \partial_b X_\mu - \frac{\eta_{ab}}{4\alpha'}\partial_a X^\mu \partial^a X_\mu - \frac{i}{2}\bar{\Psi}^\mu \gamma_{(a}\partial_{b)}\Psi_\mu = 0 \tag{2.15}$$

while varying the gravitino results in the supercurrent,

$$J_a \equiv -\frac{\pi}{e}\frac{\delta S}{\delta \chi^a} = \frac{1}{\sqrt{2\alpha'}}\gamma^b \gamma_a \Psi^\mu \partial_b X_\mu = 0 \tag{2.16}$$

2.2 The quantum superstring

When quantising the superstring one finds that the Hilbert space of states is actually much larger than the space of physical states and we will require some method to identify such states. The modern approach to quantising the superstring makes use of BRST quantisation and it manages this in a natural manner, by identifying unphysical states as arising from gauge symmetries of the RNS action and removing them by the inclusion of *ghost* fields. Furthermore the RNS superstring has the peculiar property that due to the double-valued nature of the spinor fields the spectrum of states requires truncation in order to yield a local theory, eliminating the tachyonic modes and resulting in a spacetime supersymmetry.

Before proceeding to describe the quantum dynamics of the superstring we will find it convenient to analytically continue the action (2.8) to a Euclidean world-sheet. To achieve this we redefine the coordinate describing time-like directions on the world-sheet by $\tau_E = i\tau$ and scale the resulting action such that the statistical weight it contributes to the Euclidean path integral is $e^{-S_E} = e^{iS_L}$, where S_L is the original Lorentzian action. As is usual in quantum field theory, the Euclidean path integral is better defined than the Lorentzian path integral; in this case it is because world-sheets of non-trivial topology may have a singular Lorentzian metric but will be finite with a Euclidean signature. That the two theories are equivalent can be demonstrated and amounts to an analytic continuation of the path integral.

Returning to the matter at hand, the BRST method is focussed upon the partition function, which for the Euclideanised variant of the action (2.8) is

$$Z = \int DgD\chi DXD\Psi\, e^{-S[g,\chi,X,\Psi]-S_\lambda}; \tag{2.17}$$

with this we can define the correlation function of any operator $\mathcal{O}$ to be

$$\langle\mathcal{O}\rangle = \int DgD\chi DXD\Psi\; \mathcal{O}[g,\chi,X,\Psi]\; e^{-S[g,\chi,X,\Psi]}. \tag{2.18}$$

However, the partition function defined above is, in actuality, ill defined due to the large amount of symmetry possessed by the action. Since neither the metric nor the gravitino field are dynamical we have seen previously that they may be gauge fixed and it is the integration over gauge equivalent configurations in (2.17), resulting in an infinite overcounting, which renders the partition function inappropriate as it stands. To overcome this we may employ the Faddeev-Popov method to factor out the volume of the space of gauge transformations and replace the integral over g_{ab} with a pair of anticommuting ghost fields, b_{ab} and c^a, and replace the integral over χ_a with a pair of commuting ghosts, β_a and γ [41]. Denoting the gauge-fixed matter action containing the X and Ψ coordinates by $S_m = S_X + S_\Psi$ and denoting the action for the ghost fields by $S_g = S_{bc} + S_{\beta\gamma}$ we can write a well defined partition function,

$$Z = \int DXD\psi DbDcD\gamma D\beta\; e^{-S_m[X,\Psi]-S_g[b,c,\beta,\gamma]}. \tag{2.19}$$

We will note the specific form of these actions in Sections (2.2.1) and (2.2.2) and review some of their properties as superconformal field theories.

In equation (2.19) we have been somewhat cavalier in neglecting the global properties of the world-sheet which are determined by its topology. For the time being we will focus on local properties until Section (2.5) where we will review how to include these global aspects in order to define the string S-matrix. We will see that they result in a path integral which has been decomposed into a sum over all possible topologies with integrals over metrics which are not equivalent under diffeomorphisms and Weyl transformations.

Finally it should be emphasised that in gauge-fixing the path integral (2.17) one should take care of the possibility that these symmetries are anomalous in the quantum theory, that is to say, do all of these symmetries survive regularisation? For example, we can regularise (2.19) in a manifestly diffeomorphism and Poincaré invariant way by the introduction of extremely massive coordinate fields. These terms will not, however, be Weyl invariant and so any Weyl anomaly must be checked explicitly. It is perhaps one of string theories most notable features that the retention of Weyl symmetry sets a requirement upon the dimensionality of spacetime; for the superstring this condition is satisfied for ten dimensional spacetime.

2.2.1 Fields of the bosonic string

First we shall consider the world-sheets fields which are present in both bosonic and supersymmetric string theories, these are the spacetime coordinate fields X^μ and the ghosts associated with world-sheet reparameterisation invariance and Weyl symmetry b_{ab} and c_a. Having picked a choice of gauge in equation (2.12) we are now considering a conformally flat world-sheet with a metric $ds^2 = e^\omega \left(d\tau_E^2 + d\sigma^2\right)$. However, we will find that a vast array of powerful techniques become available to us if we adopt complex coordinates defined by $z = e^{\tau_E + i\sigma}$ and $\bar{z} = e^{\tau_E - i\sigma}$, then by choosing ω such that $e^\omega = |z|^{-2}$ one obtains a flat world-sheet metric $ds^2 = dzd\bar{z}$. Hence, we may express the gauge-fixed action for the coordinate fields X in these complex coordinates as

$$S_X = \frac{1}{2\pi\alpha'} \int d^2z \, \partial X^\mu \bar{\partial} X_\mu. \tag{2.20}$$

Henceforth, for notational clarity, we shall denote derivatives with respect to z by ∂ and those with respect to $\bar{z}$ by $\bar{\partial}$. Together with S_Ψ this will define a free superconformal field theory (SCFT) of D bosons and D fermions which, couched in terms of complex coordinates, allows us to bring to bear all of the tools of complex analysis†. Though we will examine it more in the next section, the action (2.20) is invariant under conformal transformations and together with the terms given by (2.30), (2.36), (2.41) forms an action invariant under both conformal transformations and supersymmetric transformations, which combine to give the larger superconformal group. In two dimensions conformal transformations are coordinate transformations $z \to f(z)$ such that

$$ds^2 \to \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) ds^2. \tag{2.21}$$

Primary fields are operators which transform under this coordinate change in the following manner,

$$A'(z', \bar{z}') = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} A(z, \bar{z}), \tag{2.22}$$

where h and $\bar{h}$ are real numbers which specify the *conformal weight* $(h, \bar{h})$ of the primary field A. Notice that these transformation properties strongly resemble the standard tensor transformation properties for coordinate transformations.

† A thorough introduction to the use of complex analysis in two dimensional conformal field theory may be found in [42] and its applications to string theory are elucidated in [43].

In this new framework our first order of business is to solve the equation of motion (2.14), which in complex coordinates are written in the rather suggestive form,

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0, \tag{2.23}$$

It is clear from this that ∂X^μ should be a *holomorphic* function of z whilst $\bar{\partial}X^\mu$ should be an *antiholomorphic* function of $\bar{z}$. This allows us to uniquely identify the conformal weights of $\partial X^\mu(z)$ and $\bar{\partial}X^\mu(\bar{z})$, which are $(1,0)$ and $(0,1)$ respectively since (2.20) must be invariant under conformal transformations and the transformation of d^2z is determined by (2.21). As such the derivatives of the X^μ coordinate admit local Laurent expansions

$$\partial X^\mu(z) = -i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n=-\infty}^{\infty}\frac{\alpha_n^\mu}{z^{n+1}}, \quad \bar{\partial}X^\mu(\bar{z}) = -i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n=-\infty}^{\infty}\frac{\tilde{\alpha}_n^\mu}{\bar{z}^{n+1}}, \tag{2.24}$$

where by convention we have normalised these expressions with a factor of $-i(\alpha'/2)^{1/2}$. By including the conformal weight in the exponent of z and $\bar{z}$ in these expansions we ensure that the scaling dimension of the modes α_n^μ, $\tilde{\alpha}_n^\mu$ will be n; the result of this is that the inner product we shall define on the string Hilbert space later will lead to the convenient Hermitian conjugates $\alpha_n^{\mu\dagger} = \alpha_{-n}^\mu$ and $\tilde{\alpha}_n^{\mu\dagger} = \tilde{\alpha}_{-n}^\mu$.

Upon integration if we insist that $X^\mu(z,\bar{z})$ is to be single valued we require that $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. Though this need not be true for all string backgrounds it certainly holds for the simplest configurations that we will consider here; globally flat worldsheets without boundaries describing closed strings propagating in an uncompactified Minkowski space are our primary interest for this chapter until Section 2.6. By writing the expansion of the coordinate fields in terms of the world-sheet coordinates τ and σ then it can be seen that the identification of the zero modes arises from the imposition of a periodicity condition on $X^\mu(\tau,\sigma)$ under $\sigma \to \sigma + 2\pi$. In a similar fashion, open strings which are the subject of Section 2.6 are described by globally flat world-sheets with a single boundary when propagating in isolation and must satisfy the same constraint on their zero modes. In their case, varying the action to obtain the equations of motion produces accompanying boundary conditions to be imposed on the coordinate fields. Through the expansion of the coordinate fields in terms of modes one can show that the boundary conditions are equivalent to a denumerable set of relations identifying α_n^μ modes with $\tilde{\alpha}_n^\mu$ modes, one of which is $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. This example is a Neumann boundary condition and together with Dirichlet boundary conditions, which have the form $\alpha_n^\mu = -\tilde{\alpha}_n^\mu$ these are the simplest boundary conditions commonly encountered.

In both the cases of closed strings and open strings with Neumann boundary conditions described above these zero modes can be related to the spacetime momentum of the string. Since

(2.20) is invariant under the spacetime translation $X^\mu \to X^\mu + \epsilon a^\mu$ there exists an associated Noether current $a_\mu j^\mu_a$ where $j^\mu_a = i\partial_a X^\mu/\alpha'$, the corresponding conserved charge will be the spacetime momentum and this is easily computed due to the holomorphicity of $j^\mu = i\partial X^\mu/\alpha'$ and antiholomorphicity of $\tilde{j}^\mu = i\bar{\partial}X^\mu/\alpha'$,

$$p^\mu = \frac{1}{2\pi i}\oint_C (dz j^\mu - d\bar{z}\tilde{j}^\mu) = \Big(\frac{2}{\alpha'}\Big)^{1/2}\alpha^\mu_0 = \Big(\frac{2}{\alpha'}\Big)^{1/2}\tilde{\alpha}^\mu_0. \tag{2.25}$$

Thus we find the mode expansion of the coordinate fields,

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n\neq 0}\frac{1}{n}\Big(\frac{\alpha^\mu_n}{z^n} + \frac{\tilde{\alpha}^\mu_n}{\bar{z}^n}\Big). \tag{2.26}$$

Our assumption concerning the single-valuedness of X^μ is appropriate for most of the cases we will consider with the notable of exception of strings with winding, a case which will return to in Chapter 4.

The mode operators α^μ_n, $\tilde{\alpha}^\mu_n$ provide us with a description of the string complimentary to that offered by the SCFT of coordinate fields and it is often enlightening to consider have we may translate from one to the other. Furthermore, it is often the case that one of these will lend itself more conveniently to a solution and so we should be familiar with both.If we return now to equation (2.23), since it should be properly considered an operator equation it will be modified by delta function contact terms in the presence of other operators. Taking advantage of this allows us to compute the *operator product expansion* (OPE) of the operators $X^\mu(z,\bar{z})$ with each other,

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2; \tag{2.27}$$

here we include in the OPE only terms which remain significant as the operators approach the same point. In actuality, on their own the coordinate fields X^μ are not primary fields and so do not appear in any well behaved expressions, however they do appear within exponents to give well defined conformal fields and their derivatives also give holomorphic and antiholomorphic fields of definite conformal weight. As such it is often helpful to consider equation (2.27) as a useful mnemonic from which the OPEs for all these other possible conformal operators can be obtained. After the comments made above it is to be expected that these OPEs can be translated into statements about mode operators; indeed, by taking the contour integral over z of the residue for $X^\mu(z,\bar{z})^\mu\partial X^\nu(w)$ at w one may deduce the commutation relations for them,

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n}, \tag{2.28}$$

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{2.29}$$

with all other commutators vanishing. These are identical to those that would be obtained via the canonical quantisation of the action (2.20).

In addition to the spacetime coordinates X, the other coordinates found in the bosonic string path integral (2.19) are the anticommuting bc ghosts. They have the gauge fixed action

$$S_{bc} = \frac{1}{2\pi} \int d^2 z \left(b\bar{\partial} c + \widetilde{b} \partial \widetilde{c} \right). \tag{2.30}$$

The equations of motion for these fields are simple, we can write them as

$$\bar{\partial} c = \bar{\partial} b = 0, \tag{2.31a}$$

$$\partial \widetilde{c} = \partial \widetilde{b} = 0. \tag{2.31b}$$

Once again we see that analyticity is an important property of these conformal fields, these equations requiring that the operators b, c be holomorphic functions, whilst the operators $\widetilde{b}$, $\widetilde{c}$ must be antiholomorphic functions. For brevity, from this point forward we shall only state results explicitly for the holomorphic fields since they are easily extrapolated to analogous antiholomorphic expressions. In this case, the conformal weights of the bc fields are not uniquely fixed by the form of the action (2.30), instead they may have independent conformal weights $(h_b, 0)$ and $(h_c, 0)$ such that $h_b = \lambda$ and $h_c = 1 - \lambda$ for any real constant λ. This constant is fixed by the origins of the bc ghosts from the gauge-fixing of the bosonic world-sheet symmetries via the Faddeev-Popov procedure. During this process, the parameter ϵ^a from the coordinate transformation $\sigma^a \to \sigma^a + \epsilon^a$ is replaced by the vector ghost field c^a which inherits its tensorial properties but is a grassmann number; these properties allow us to deduce that the c field should have conformal weight $(-1, 0)$ leading to the conclusion that b has conformal weight $(2, 0)$, a conclusion which can be confirmed by a similar analysis of the Faddeev-Popov procedure where b originates from a traceless symmetric world-sheet tensor which is replaced by b_{ab}.

As before, the product bc must satisfy the equations of motion up to a contact term and from this we may obtain the OPEs,

$$b(z)c(w) \sim \frac{1}{z - w}. \tag{2.32}$$

As noted above, the fields b, c are both holomorphic, thus allowing us to express them as Laurent series. Single-valuedness requires that the exponents of z in the expansion be integer valued. The result is

$$b(z) = \sum_n \frac{b_n}{z^{n+2}}, \qquad c(z) = \sum_n \frac{c_n}{z^{n-1}}, \tag{2.33}$$

where the modes have anticommutation relations

$$\{b_m, c_n\} = \delta_{m+n}. \tag{2.34}$$

The additional conformal weight factors present in the exponents of z for each series are conventionally included such that the scaling dimension of the modes b_n, c_n will be n, as for the α_n^μ and $\widetilde{\alpha}_n^\mu$.

2.2.2 Fields of the supersymmetric extension

Having dealt with the fields of the bosonic string, let us turn to the fields which are their superpartners in the RNS superstring. From here on in it shall be useful to work with the spinor components directly, hence we shall write

$$\Psi^\mu = \begin{pmatrix} \widetilde{\psi}^\mu \\ \psi^\mu \end{pmatrix} \tag{2.35}$$

which, along with the definitions (2.7), leads us to the gauge-fixed action

$$S_\Psi = \frac{1}{2\pi} \int d^2z \left(\psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right); \tag{2.36}$$

The equations of motion resulting from this are

$$\bar{\partial} \psi^\mu = \partial \widetilde{\psi}^\mu = 0; \tag{2.37}$$

from which we conclude that the ψ are holomorphic functions while the $\widetilde{\psi}$ are antiholomorphic functions. As a result we will again refrain from writing the results for the antiholomorphic fields with the understanding that they may be simply determined from the appropriate holomorphic result. The OPE for the Fermionic matter fields are

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}, \tag{2.38}$$

with the OPE between $\psi^\mu(z)$ and $\widetilde{\psi}^\nu(w)$ being regular at $z = w$.

Given that ψ and $\widetilde{\psi}$ are components of a two-dimensional spinor, all physically relevant quantities will be expressed in terms of their bispinor products and this allows for them to be double valued; as such the modes in the Laurent expansion will carry an index valued in either the integers or half-integers,

$$\psi^\mu(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{\psi_r^\mu}{z^{r+1/2}}, \tag{2.39}$$

where ν takes either the value 0 or $1/2$. The anticommutators can then be determined from equation (2.38)

$$\{\psi_r^\mu, \psi_s^\nu\} = \{\widetilde{\psi}_r^\mu, \widetilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}. \tag{2.40}$$

Finally we shall summarise some basic properties of the bosonic $\beta\gamma$ ghosts which arise from the gauge fixing of the world-sheet gravitino. These have the action

$$S_{\beta\gamma} = \frac{1}{2\pi}\int d^2z \left(\beta\bar{\partial}\gamma + \widetilde{\beta}\partial\widetilde{\gamma}\right). \tag{2.41}$$

This action bears a heavy resemblance to (2.30) and so it is not surprising that the equations of motion take a similar form,

$$\bar{\partial}\gamma = \bar{\partial}\beta = 0, \tag{2.42a}$$

$$\partial\widetilde{\gamma} = \partial\widetilde{\beta} = 0. \tag{2.42b}$$

Likewise, the OPE for the $\beta\gamma$ fields looks familiar, but differs from (2.32) by a sign due to their different statistics,

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w}, \quad \gamma(z)\beta(w) \sim \frac{1}{z-w}. \tag{2.43}$$

Note that this differs by a sign from the OPE of fermionic fields with the same action as a result of the differing statistics.

Despite the similarity of the actions, the differences between the $\beta\gamma$ and bc fields becomes more manifest when we consider the mode expansions. Whereas the bc fields were components of a world-sheet tensor, the $\beta\gamma$ fields are components of a world-sheet spinor and so their behaviour under world-sheet a rescaling is markedly different. This is because the $\beta\gamma$ fields arise as the ghosts of a fermionic symmetry thus making them components of a two-dimensional spinor, allowing them to be double-valued. As with the bc ghosts, the conformal weights of the $\beta\gamma$ ghosts are only restricted by the action to take the form $(h_\beta, 0)$ and $(h_\gamma, 0)$ where $h_\beta = \lambda'$ and $h_\gamma = 1 - \lambda'$, but in this case γ is the ghost field that replaces the Majorana spinor parameterising the local supersymmetry transformations in the Faddeev-Popov procedure and so we can conclude that $h_\gamma = -1/2$ and therefore $h_\beta = 3/2$. Hence for the mode expansions of these fields we take the sum over an index r taken from either the integers or half-integers,

$$\beta(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+3/2}}, \qquad \gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-1/2}}. \tag{2.44}$$

Finally, using these expansions, the OPE tells us that the commutation relations for these mode operators are

$$[\gamma_r, \beta_s] = \delta_{r+s}. \tag{2.45}$$

Before moving on we will note a fact which will be required when we discuss the vertex operators of physical string states, this is the process of replacing the $\beta\gamma$ ghost system with an equivalent theory of conformal fields known as *bosonisation*. The possibility of this is a highly nontrivial property which we will state here without proof, to do it one introduces three new fields, φ, η, ξ along with the identifications

$$\beta(z) = e^{-\varphi(z)}\partial\xi(z), \quad \gamma(z) = e^{\varphi(z)}\eta(z). \tag{2.46}$$

The chiral scalar field φ has the OPE

$$\varphi(z)\varphi(w) \sim -\ln(z-w) \tag{2.47}$$

while the anticommuting fields η, ξ are independent of the scalar and form a system similar to the bc ghosts with the only singular term in their OPEs being

$$\eta(z)\xi(w) \sim \frac{1}{z-w}. \tag{2.48}$$

This equivalence may be extended to all local operators. For a more complete discussion of bosonisation in the context of superstring theory the interested reader may consult [43].

2.2.3 Superconformal symmetry

Having summarised the basic properties of the world-sheet fields of the superstring in gauge-fixed complex coordinates we shall briefly return to discuss the world-sheet energy momentum tensor and supercurrent. It has been mentioned in passing that the gauge-fixed action $S = S_m + S_g$ is a superconformal field theory, that is to say it is invariant under superconformal transformations. This is the residual symmetry left over after gauge-fixing the diffeomorphism and Weyl symmetry which consists of those transformations that leave the metric in its gauge-fixed form. Having removed the metric as a dynamical object we can reinterpret the energy-momentum tensor (2.15) and the supercurrent (2.16) as conserved Noether currents; the energy momentum tensor arises from conformal invariance while the supercurrent is associated with supersymmetry.

From (2.15) we can see that the energy momentum tensor should be traceless, and after translating the world-sheet theory to z-coordinates we find that this condition becomes $T_{z\bar{z}} = T_{\bar{z}z} = 0$. The result of this is that the conservation of the energy momentum tensor, $\partial^a T_{ab} = 0$, can be reduced to the conditions $\bar{\partial}T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$. This implies that the components of the energy momentum tensor are expressible as a holomorphic function $T(z) \equiv T_{zz}$ and an antiholomorphic function $\widetilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$; indeed, by taking the liberty of rescaling the energy momentum tensor we can write the contribution from the matter fields as

$$T^m(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu, \tag{2.49a}$$

$$\widetilde{T}^m(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X_\mu - \frac{1}{2}\widetilde{\psi}^\mu\bar{\partial}\widetilde{\psi}_\mu. \tag{2.49b}$$

Similarly, the supercurrent also breaks into a holomorphic component and an antiholomorphic component which we will write as

$$J^m(z) = i\frac{\sqrt{2}}{\alpha'}\psi^\mu \partial X_\mu, \tag{2.50a}$$

$$\tilde{J}^m(\bar{z}) = i\frac{\sqrt{2}}{\alpha'}\tilde{\psi}^\mu \bar{\partial} X_\mu. \tag{2.50b}$$

Since we are now considering the quantum theory we must also include the contribution to these currents from the ghost fields and these are given by

$$T^g(z) = (\partial b)c - 2\partial(bc) + (\partial\beta)\gamma - \frac{3}{2}\partial(\beta\gamma), \tag{2.51a}$$

$$\widetilde{T}^g(\bar{z}) = (\bar{\partial}\widetilde{b})\widetilde{c} - 2\bar{\partial}(\widetilde{b}\widetilde{c}) + (\bar{\partial}\widetilde{\beta})\widetilde{\gamma} - \frac{3}{2}\bar{\partial}(\widetilde{\beta}\widetilde{\gamma}), \tag{2.51b}$$

$$J^g(z) = -\frac{1}{2}(\partial\beta)c + \frac{3}{2}\partial(\beta c) - 2b\gamma, \tag{2.51c}$$

$$\widetilde{J}^g(\bar{z}) = -\frac{1}{2}(\bar{\partial}\widetilde{\beta})\widetilde{c} + \frac{3}{2}\bar{\partial}(\widetilde{\beta}\widetilde{c}) - 2\widetilde{b}\widetilde{\gamma}. \tag{2.51d}$$

We shall use the superscript to distinguish between these contributions and refer to the full currents without, that is $T = T^m + T^g$ and $J = J^m + J^g$.

The variation of any operator under a superconformal transformation may be determined from its OPE with these currents and it is to be expected that we can associate with them charges which generate these transformations. As with the fields contained in the string action, the energy momentum tensor and supercurrent both allow a mode expansion and these modes fulfill the role of superconformal charges. As for the other fields, we write these expansions as

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}, \quad \widetilde{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\widetilde{L}_n}{\bar{z}^{n+2}}, \\ J(z) &= \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}, \quad \widetilde{J}(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \frac{\widetilde{G}_r}{\bar{z}^{r+3/2}}. \end{aligned} \tag{2.52}$$

A simple residue calculation will confirm that the generators from the holomorphic and antiholomorphic fields form two copies of the following superalgebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}, \tag{2.53a}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s}, \tag{2.53b}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r}, \tag{2.53c}$$

which we refer to as the *Ramond algebra* for integer valued r, s and the *Neveu-Schwarz algebra* for half-integer valued r, s. The c-number seen above is the central charge of the algebra c, dependent upon the field content of the particular SCFT under consideration it is intimately connected with the Weyl anomaly; our earlier requirement that this anomaly vanish requires that c too must be zero, and this is true provided the dimensionality of spacetime is ten. It is worth noting that within the algebra above the operators L_{-1}, L_0, L_1 and their antiholomorphic counterparts form closed subalgebras. For the closed string these generate the global conformal transformations specified by the group $SL(2,\mathbb{C})$ (but in the presence of a boundary on the world-sheet we must identify the antiholomorphic operators with the holomorphic ones and this is reduced to the group $SL(2,\mathbb{R})$).

By inserting the mode expansions of X^μ, ψ^μ and $\widetilde{\psi}^\mu$ into ((2.49)) and ((2.50)) we can determine expressions for the generators of superconformal transformations in terms of these modes, for both algebras these are

$$L_n^m = \frac{1}{2}\sum_{k\in\mathbb{Z}}\alpha_{n-k}\cdot\alpha_k + \frac{1}{4}\sum_{r\in\mathbb{Z}+\nu}(2r-n)\psi_{n-r}\cdot\psi_r + a^m\delta_n, \tag{2.54a}$$

$$L_n^g = \sum_{k\in\mathbb{Z}}(n+k)b_{n-k}c_k + \sum_{r\in\mathbb{Z}+\nu}\frac{n+2r}{2}\beta_{n-r}\gamma_r + a^g\delta_n, \tag{2.54b}$$

$$G_r^m = \sum_{n\in\mathbb{Z}}\alpha_n\cdot\psi_{r-n}, \tag{2.54c}$$

$$G_r^g = \sum_{n\in\mathbb{Z}}\left(\frac{2r+n}{2}\beta_{r-n}c_n + 2b_n\gamma^{r-n}\right), \tag{2.54d}$$

with analogous expressions holding for the modes of antiholomorphic fields. The c-numbers a^m, a^g are normal ordering constants which arise from the ambiguity in defining the product of mode operators which neither commute nor anticommute. These numbers depend upon the algebra under consideration, for the Ramond (R) sector these are $a^m = 5/8$ and $a^g = -5/8$ while for the Neveu-Schwarz (NS) sector they are $a^m = 0$ and $a^g = -1/2$.

The operators we have summarised above provide the necessary tools required for us to identify the physical Hilbert space of the superstring. It is a well known requirement of string theory that one must carry out such a process since the space of states generated by all of the mode operators in Sections (2.2.1) and (2.2.2) is not unitary; physical states in this space are subject to the additional constraint that they satisfy the operator equations which are the quantum analog of equations (2.15) and (2.16). As such, in the next section we shall review such a process known as BRST quantisation in which we identify the spectrum of physical states with the cohomology of the BRST charge, the generator of the BRST symmetry which is constructed from the superconformal generators and the ghost modes. With this in place we will finally be able to move on to discuss the nature of the string states which are central to this work.

2.2.4 BRST quantisation

It would appear from (2.19) that our original gauge symmetry has simply disappeared, but it is merely in a new guise; the action $S = S_m + S_g$ is invariant under the *Becchi-Rouet-Stora-Tyutin (BRST) transformation*; this symmetry provides us with an elegant method by which we can identify the space of physical states [44]. The generator of BRST transformations is given by

$$Q_B = \frac{1}{2\pi i}\oint_C (dz j_B - d\bar{z}\tilde{j}_B), \tag{2.55}$$

where the BRST current is given in terms of the energy momentum tensor and supercurrent of the matter and ghost systems, which are expressed in terms of primary fields in equations (2.49-2.51),

$$\begin{aligned} j_B &= cT^m + \gamma J^m + \frac{1}{2}(cT^g + \gamma J^g) \\ &= cT^m + \gamma J^m + bc\partial c + \frac{3}{4}(\partial c)\beta\gamma + \frac{1}{4}c(\partial\beta)\gamma - \frac{3}{4}c\beta\partial\gamma - b\gamma^2. \end{aligned} \tag{2.56}$$

A similar expression may be written for the antiholomorphic BRST current. Using another contour argument one can show that these imply that the BRST charge may be expanded as a sum of the superconformal generators and ghost modes,

$$\begin{aligned} Q_B = \sum_n c_{-n}L_n^m + \sum_r \gamma_{-r}G_r^m - \sum_{m,n}\frac{1}{2}(n-m)b_{-m-n}c_m c_n \\ + \sum_{m,r}\left(\frac{1}{2}(2r-m)\beta_{-m-r}c_m\gamma_r - b_{-m}\gamma_{m-r}\gamma_r\right) + a^g c_0. \end{aligned} \tag{2.57}$$

Here we see that much like the generators of conformal transformations, the BRST charge requires normal ordering.

The BRST charge has many interesting properties, not least is that it is nilpotent $Q_B^2 = 0$ provided that the central charge of the SCFT vanishes, but the most important to us here is that all physical states (those that are invariant under a change in the gauge-fixing condition) must satisfy the operator equation $Q_B |\varphi\rangle = 0$, such states are closed. Due to nilpotence this is trivially satisfied by exact states which are closed states of the form $|\varphi\rangle = Q_B |\phi\rangle$, but these have zero-norm so we are interested in those states which cannot be written in this way. Since any two physical states which differ by a zero-norm state will have the same inner products they are physically equivalent and as a result we can reduce the problem of determining the space of physical states to the task of finding the equivalence classes of solutions to $Q_B |\varphi\rangle = 0$.

Examining equation (2.57) it is obvious that the BRST symmetry treats each member of the two pairs of ghosts quite differently, on account of their differing conformal weights. The result of this is that we must impose one last condition to well and truly isolate physically relevant states;

we must identify the appropriate ghost vacuum of the Hilbert space. For this purpose there exist ghost number operators which measure the ghost charge according to the assignments $c, \gamma \to +1$ and $b, \beta \to -1$. In the case of the bc ghosts this conserved charge results from the invariance of the ghost CFT under the variations $\delta b = -i\epsilon b$ and $\delta c = i\epsilon c$ which are associated with the current

$$j_{bc} = - : bc :, \tag{2.58}$$

where the modes operators of the ghost fields should be normal ordered. It is simple to confirm that an analogous symmetry exists for the $\beta\gamma$ ghosts with a current of the same form. The ghost number is then given by the residue of j_{bc} at $z = 0$,

$$Q_{bc} = \frac{1}{2\pi i} \oint_C dz j_{bc}(z). \tag{2.59}$$

We can take a normal ordering scheme such that the ghost number then takes the form

$$Q_{bc} = \frac{1}{2}\left(c_0 b_0 - b_0 c_0\right) + \sum_{n=1}^{\infty} \left(c_{-n} b_n - b_{-n} c_n\right), \tag{2.60}$$

with a similar expression being valid for the $\beta\gamma$ fields when the modes take integer indices, that is, in the Ramond sector (in the NS sector we find there are no zero modes). These charges are conserved by both the Hamiltonian and BRST transformations. Notice that there is an ambiguity in the normal ordering of the zero modes since they induce a degenerate ground state for the Hilbert space. If we focus our attention on the bc ghosts, we can label the two possible ground as $|\uparrow\rangle$ and $|\downarrow\rangle$ such that they are annihilated by c_0 and b_0 respectively, while $c_0 |\downarrow\rangle = |\uparrow\rangle$ and $b_0 |\uparrow\rangle = |\downarrow\rangle$. If these ground states are to describe physically propagating states then they must both be annihilated by all of the positive frequency operator modes: $b_n |\downarrow\rangle = b_n |\uparrow\rangle = c_n |\downarrow\rangle = c_n |\uparrow\rangle = 0$ for $n > 0$. We can assign the state $|\downarrow\rangle$ the ghost charge $-1/2$, thus implying that $|\uparrow\rangle$ has ghost charge $+1/2$. It is reasonable to expect that physical states should be free of ghosts and so should be proportional to one of these two states and the correct choice one should make is that they be proportional to $|\downarrow\rangle$. With this choice the BRST condition $Q |\varphi\rangle = 0$ is equivalent to the following conditions on the superconformal generators,

$$(L_0) |\varphi\rangle = 0, \tag{2.61}$$

which leads to a string mass-shell condition, and

$$\begin{aligned} L_m |\varphi\rangle &= 0, \\ G_r |\varphi\rangle &= 0, \end{aligned} \qquad m, r > 0 \tag{2.62}$$

which go on to yield conditions for the allowed polarisations of the state. These conditions are sufficient to ensure that (2.15) and (2.16) be satisfied as operator equations, whereas the other choice does not yield this complete set.

It is interesting that the choice of the ground state specified above is not invariant under global conformal transformations. Indeed, it was defined such that $L_{-1}\,|\downarrow\rangle \neq 0$. Instead this is the state of lowest energy. One can also define the $SL(2,\mathbb{C})$ invariant vacuum state $|\,\mathbb{1}\rangle$ which we will encounter again in Section 2.4 in the same way, with the difference being a shift in the Fermi sea-level which changes those ghost modes which annihilate it

$$b_n\,|\,\mathbb{1}\rangle = 0, \quad n \geq -1, \qquad c_n\,|\,\mathbb{1}\rangle = 0, \quad n \geq 2, \tag{2.63}$$

$$\beta_r\,|\,\mathbb{1}\rangle = 0, \quad r \geq -\frac{1}{2}, \qquad \gamma_r\,|\,\mathbb{1}\rangle = 0, \quad r \geq \frac{3}{2}. \tag{2.64}$$

The relation between the two vacua $|\downarrow\rangle$ and $|\,\mathbb{1}\rangle$ can be written formally as $|\downarrow\rangle = \delta(c_1)\delta(\gamma_{1/2})\,|\,\mathbb{1}\rangle$, where the delta functions $\delta(c_1)$, $\delta(\gamma_{1/2})$ impose the appropriate behaviour on $|\downarrow\rangle$ by virtue of the operator analog of the familiar delta function property $x\delta(x) = 0$. The fermionic nature of c_1 means that its action within the delta function can be practically realised by setting $\delta(c_1) = c_1$, giving the desired result for $|\downarrow\rangle$ under the action of the annihilation modes b_n, c_n, $n > 0$. The bosonic nature of $\gamma_{1/2}$, however, precludes the possibility of a simple algebraic expression for $\delta(\gamma)$; instead it must expressed in terms of the bosonised field $\varphi(z)$, as will be noted later in equation (2.81) and the discussion following that. For now we will simply write

$$|\downarrow\rangle = c_1\delta(\gamma_{1/2})\,|\,\mathbb{1}\rangle\,. \tag{2.65}$$

For a discussion of the use of the delta function operator as introduced here, one may consult the text [45] or the recent review [46].

2.3 The closed superstring spectrum

In this section we shall examine how the conditions of BRST invariance discussed in the previous section are employed to determine the spectrum of physical states accessible to the superstring and use them in particular to determine vertex operators representing both massless and massive closed strings. Specifically we shall go on to consider the bosonic spectrum of massless closed strings which are to be encountered again in Chapter 4, as well as the class of states known as the leading Regge trajectory composed of strings with the maximum possible spin for their mass; the latter are a convenient subset of the spectrum for use in studying the interactions of massive strings as we shall do in Chapter 5.

Thus far we have been coy about the global aspects of the world-sheet analysis by maintaining a discussion of local properties. Hence forth we can afford to be more concrete and for this section we consider the states of a single string in isolation, these are described by compact and connected world-sheets of genus zero which may have at most one boundary. We shall restrict our attention here to oriented world-sheets since the spectrum of the unoriented string can be obtained from that of the oriented string by a projection onto states that are invariant under a world-sheet parity transformation. Thus fundamentally there are two distinct forms of string — the closed string which lacks end points and is described by world-sheets which are conformally equivalent to the Riemann sphere and the open string described by world-sheets with a single boundary representing the end points which are conformally equivalent to the unit disk. In practice we will find it convenient to describe the closed and open strings using the complex plane $\mathbb{C}$ and the upper-half of the complex plane $\mathbb{H}_+$ respectively.

There is very little modification required for our discussion of solutions to the equations of motion for the coordinate fields X and bc ghosts of the closed string since the only requirement imposed is that of periodicity in the argument of z, $\bar{z}$ which is already implicit in their analyticity. Similarly, the Ψ fields and $\beta\gamma$ ghosts may be periodic or antiperiodic due to the spinorial nature; periodicity of these fields is associated with the Neveu-Schwarz sector while antiperiodicity is associated with the Ramond sector. Of course, we are free to choose these conditions independently for the holomorphic and antiholomorphic fields and so the spectrum of the closed strings can be divided into four distinct sectors which we label NS-NS, R-R, NS-R and R-NS. Of these, the first two sectors contain the bosonic spectrum of the closed string whilst the latter two contain the fermions. Since in the work that follows we will have a requirement for closed string states alone we shall restrict the content here to these, however, the effects of boundaries on the world-sheet will be of great interest. Even in processes which have only closed strings in their initial and final configurations, the presence of a boundary indicates intermediate states involving open strings and these break the independence of the holomorphic and antiholomorphic fields. We shall return to these matters in Section 2.6.

Let us begin by describing the Hilbert space of the closed superstring. Having expanded all world-sheet fields in terms of modes with oscillator algebras we can divide these up into creation and annihilation operators which act upon some ground state to generate the Hilbert space. Due to their independence this space will be the direct product of spaces built using holomorphic and antiholomorphic modes and so we will restrict ourselves to the specifics of the holomorphic modes where suitable to avoid repetition. In addition to the oscillator modes there is also a noncommuting pair of operators for the centre of mass position and momentum so conventionally we further specify the ground state with the momentum of the string. For the NS sector there are no zero modes beyond the bc ghosts, which were discussed previously, and so the ground state is that which is annihilated by the modes α_m^μ, b_m, c_m for $m > 0$ and by ψ_r^μ, β_r, γ_r for $r > 0$; we shall label this $|0;p\rangle$ for a string with momentum p. For the R sector the ground state is specified by the same conditions but will have further structure as the ψ^μ, β and γ fields will have zero modes. In analogy with the bc system, the BRST quantisation procedure dictates that we group β_0 with the annihilation operators and γ_0 with the creation operators, however, the ψ_0^μ are more complicated. These have the algebra $\{\psi_0^m, \psi_0^\nu\} = \eta^{\mu\nu}$ and so furnish a representation of the Clifford algebra, therefore the ground states of the R sector are spacetime spinors which we will label $|\alpha;p\rangle$ where α represents the spinor components. Since the ground state of the NS sector is a Lorentz singlet and all mode operators carry only Lorentz indices we can conclude that the NS-NS sector consists only of bosonic states while the states of the NS-R and R-NS sectors carry spinor indices and so represent fermions, states of the R-R sector are direct products of spinor representations and so overall carry integer spin thus making them bosons in agreement with our earlier claim.

Of course, the Hilbert space generated by acting with creation operators on the various ground states will not be unitary, a fact easily realised by considering states built using the timelike modes. Instead, the physical, unitary Hilbert space is identified with the *cohomology* of the BRST charge operator Q_B. This means that each physically distinct state is represented by an equivalence class of the BRST invariant states generated with the creation operators, two states $|\psi\rangle$ and $|\psi'\rangle$ being equivalent if they differ by a BRST exact state, $|\psi'\rangle = |\psi\rangle + Q_B |\phi\rangle$. It is possible to show that this procedure does indeed yield a positive definite space and the interested reader can find examples of this in the literature [47–49].

It is customary to classify distinct particles by their representation under the Poincaré group which amounts to a specification of the mass M and spin J of the state; the mass of the string state is entirely determined by the mass-shell relation which is best expressed by a linear combination of the L_0 and $\tilde{L}_0$ constraints and expresses the mass in terms of the number of excited oscillator modes,

$$\frac{\alpha'}{2}M^2 = \sum_{m=1}^{\infty} (\alpha_{-m} \cdot \alpha_m + \widetilde{\alpha}_{-m} \cdot \widetilde{\alpha}_m) + \sum_{r>0} r \left(\psi_{-r} \cdot \psi_r + \widetilde{\psi}_{-r} \cdot \widetilde{\psi}_r\right) + a + \widetilde{a}. \tag{2.66}$$

Another, linearly independent, combination of the L_0 and $\widetilde{L}_0$ yields a level matching condition, a strict requirement relating the number of holomorphic and antiholomorphic modes allowed in a physical states,

$$\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r>0} r\psi_{-r} \cdot \psi_r + a = \sum_{m=1}^{\infty} \widetilde{\alpha}_{-m} \cdot \widetilde{\alpha}_m + \sum_{r>0} r\widetilde{\psi}_{-r} \cdot \widetilde{\psi}_r + \widetilde{a}. \tag{2.67}$$

It is possible to take physical states as those without excited ghost modes and so we have ignored their contribution in (2.66) and (2.67). The only contribution to the ghosts is in ensuring that we have the correct normal ordering constant.

With these two conditions, in addition to the correct choice of the ground state as discussed following equation (2.60), one may determine all possible combinations of oscillators allowed to produce states at any particular mass level; the BRST conditions will then yield a finite number of conditions which remove unphysical polarisations of these states. In general, the result of this procedure describes states in highly reducible representations of the Poincaré group and so one must extract from this the irreducible representations in order to determine the various spins of the particles characterised by such states. Of course, care must be taken to identify the subset of BRST exact states from within those found since they do not contribute to amplitudes.

As an example, let us use this to proceed sector by sector for the closed string to illustrate the lowest lying states. In the NS-NS sector the lowest lying state is $|0;k\rangle \otimes |0;k\rangle$[‡] which has the mass $M^2 = -2/\alpha'$ and has spin zero; this is a tachyonic state. Next, by acting upon this state with the lowest frequency fermionic modes we obtain the massless state $\varepsilon_{\mu\nu}\psi^{\mu}_{-\frac{1}{2}}|0;k\rangle \otimes \widetilde{\psi}^{\nu}_{-\frac{1}{2}}|0;k\rangle$. This may be divided up into irreducible representations describing a spin two, spin one and spin zero states — these are the graviton, Kalb-Ramond two-form and dilaton.

The NS-R and R-NS sectors both generate the same states and so we consider them together. Due to the inclusion of the R sector all of these states form a spinorial representation, the lightest state being a fermionic tachyon $|0;k\rangle \otimes |\alpha;k\rangle$ with $M^2 = -1/\alpha'$, where $|\alpha;k\rangle$ is a Majorana spinor of $SO(1,9)$. Once again there is a massless state of the form $\psi^{\mu}_{-\frac{1}{2}}|0;k\rangle \otimes |\alpha;k\rangle$; this state has spin $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$, implying that these states contain both the gravitino and the spin-$\frac{1}{2}$ superpartner of the Kalb-Ramond two-form, this becoming clear after the GSO projection which will be discussed shortly. Further action with creation operators will go on to yield massive fermions.

The R-R sector contains no tachyonic states and the lowest lying state is the massless bispinor $|\alpha;k\rangle \otimes |\beta;k\rangle$. This and all other states in this sector can be decomposed into various bosonic states.

It might be noted at this point that the string spectrum retains some undesirable features; it contains tachyons which we would like to eliminate whilst keeping the important massless states such as the gauge boson and graviton. What's more, the presence of both the NS and R sectors will in general lead to a non-local conformal field theory. The GSO projection mentioned earlier

[‡]Here we employ a bookkeeping technique that will be employed later on in which we ascribe a momentum $k = p/2$ each to the holomorphic and antiholomorphic fields

presents a method for taming the RNS model of the superstring and was first proposed by Gliozzi, Scherk and Olive [50, 51]. This consists of projecting out undesirable states from the superstring spectrum in such a way that we are left with a consistent and local theory. This also introduces an unexpected but desirable property of string theory — spacetime supersymmetry.

The basis of the GSO projection is that the both the NS and R sectors may be further decomposed into two subsectors consisting of the eigenstates of the fermion number operator F (strictly speaking this is really the spinor number operator once we include contributions from the ghost fields, but we use the conventional name). The action of the GSO projection in the NS sector is simple — we initially consider a bosonic state $|\varphi\rangle$ that is a component of a state we would like to retain in our spectrum, such as the graviton. We can imagine acting on this state with n anticommuting operators,

$$\psi^{\mu_1}\psi^{\mu_2}\dots\psi^{\mu_n}\,|\varphi\rangle\,, \tag{2.68}$$

which unsurprisingly results in another bosonic state since the ψ^{μ} operators, despite being components of a world-sheet spinor, are spacetime vectors. For even n this is not unusual since the ψ^{μ_i} may be paired up into commuting operators, but if n is odd then the product of these operators will be an anticommuting operator. To eliminate such states we can define the chirality operator $(-1)^F$ and discard all states for which $(-1)^F = -1$. This operator anticommutes with all world-sheet spinors and is built from the world-sheet fermion number operator which we decompose into a matter and a ghost contribution, $F = F_m + F_g$. It is simple to show that by defining

$$F_m = \sum_{r=1/2}^{\infty} \psi_{-r}\cdot\psi_r \tag{2.69}$$

then F_m will have the desired properties. The analogous quantity for the spinor ghosts is actually already familiar to us, we can just use the ghost charge for the $\beta\gamma$ ghosts, $Q_{\beta\gamma}$, the bc version of which was introduced in equation (2.60). In this case we can write

$$F_g = Q_{\beta\gamma} = -\sum_{r=1/2}^{\infty}\left(\gamma_{-r}\beta_r + \beta_{-r}\gamma_r\right). \tag{2.70}$$

Later on in Section (2.4), where we will discuss the local world-sheet operator associated with the NS ground state of the superstring, it will be shown that the ghost charge associated with the $\beta\gamma$ system for this ground state is $Q_{\beta\gamma}\,|0;0\rangle = -\,|0;0\rangle$. As a result, the ghost contributions to the ground state in the NS sector ensures that $(-1)^F\,|0;k\rangle = -\,|0;k\rangle$ and thus we can eliminate both tachyons encountered above.

A similar trick in the R sector requires a little more care due to the spinorial nature of the ground state, but briefly we state that as a spinorial representation of $SO(1,9)$ the space of R sector states is reducible to two Weyl representations which are distinguished by their eigenvalue under the chirality operator Γ constructed from the Dirac algebra. As we have already noted, the ψ_0 furnish a representation of the Dirac algebra and thus by writing $\Gamma^\mu = i\sqrt{2}\psi_0^\mu$ we can express the chirality operator in its usual form

$$\Gamma \equiv \Gamma^0\Gamma^1 \cdots \Gamma^9. \tag{2.71}$$

It is simple to deduce that this operator will count the number of ψ_0^μ operators used to create a state, modulo 2, and so in analogy with the case of the NS sector we also introduce an R sector representation of the operator $F = F_m + F_g$ which will count the number of all the other world-sheet spinor modes. This is given by

$$F_m = \sum_{r=1}^{\infty} \psi_{-r} \cdot \psi_r \tag{2.72}$$

and

$$F_g = -\frac{1}{2}\left(\gamma_0\beta_0 + \beta_0\gamma_0\right) - \sum_{r=1/2}^{\infty} \left(\gamma_{-r}\beta_r + \beta_{-r}\gamma_r\right). \tag{2.73}$$

Defining $\bar{\Gamma} = i\Gamma(-1)^F$, where the factor i cancels the contribution of the ghost ground state to ensure Hermiticity, we find that this modified chirality operator anticommutes with the fermionic modes, $\{\bar{\Gamma}, \psi^\mu\} = 0$, and so fulfills the task of differentiating between 'odd' and 'even' states. In this case we have no preference for fixing $\bar{\Gamma}\,|\alpha\rangle = \pm\,|\alpha\rangle$ and both choices will lead to a consistent theory. In fact, for a single R sector these two choices are physically equivalent, but when we take its product with the space of antiholomorphic states then the two independent choices cause distinguishable differences.

Having discerned four sectors which we can label as NS$\pm$, R$\pm$ then there are potentially sixteen ways in which one could pair them to give sectors of the closed string and we are free to consider any combination of these sectors to build a string theory. The great utility of the GSO projection is that it narrows these choices down to those combinations which are self-consistent, and there are remarkably few such theories. We have already remarked that the elimination of the tachyon forces the removal of NS$-$ from our considerations, other considerations include further restrictions which reduces the pool of potential string models to just two physically distinct spectra for the closed string; these are the type II strings that differ only by the relative chirality of the holomorphic and antiholomorphic R sectors,

Considering what remains of the massless states following the GSO projection, since in the NS-NS sector these were all states of positive chirality they are unaffected and we retain the graviton, dilaton and two-form. Likewise the NS-R and R-NS states retain fermions of spin $3/2$

IIA:	(NS+,NS+)	(R+,NS)	(NS+,R-)	(R+,R-),
IIB:	(NS+,NS+)	(R+,NS)	(NS+,R+)	(R+,R+).

and $1/2$ but now the spinor ground state of the R sector is explicitly Majorana-Weyl and hence these have the appropriate number of degrees of freedom to be superpartners for the bosonic states. For the type IIA string both pairs of these states have opposite chirality whereas for the type IIB string they will be the same. The most distinct difference between the two closed string models is found in the R-R sector for which the ground state now transform in the product of two Weyl representations instead of two Dirac representations. For ten dimensional spacetime both Weyl representations have sixteen dimensions, while for type IIB alone they have the same chirality, and so the massless R-R states decompose as follows

$$\text{IIA:} \quad \mathbf{16} \times \mathbf{16'} = [0] + [2] + [4]$$

$$\text{IIB:} \quad \mathbf{16} \times \mathbf{16} = [1] + [3] + [5]_+$$

where $[n]$ is representative of a rank n antisymmetric tensor and, in particular, the rank 5 tensor is self-dual. These are the n-form field strengths of the R-R sector.

Let us now proceed to examine the massive states of the NS-NS sector with the aim of identifying the string states of the leading Regge trajectory. The mass-shell condition (2.66) for the NS-NS sector after the GSO projection becomes

$$\alpha' M^2 = 4n, \tag{2.74}$$

with $n = 0, 1, 2....$ Obviously we have already elaborated upon the case $n = 0$, the various massive states are then obtained for each level by acting on the ground state with an appropriate number of oscillators to satisfy equation (2.74) whilst also satisfying the level-matching condition (2.67). For the first massive level ($n = 1$) the mass-shell condition (2.74) can be satisfied by taking products of the following states

$$|\phi_\varepsilon\rangle = \varepsilon_{\mu\nu}\alpha^\mu_{-1}\psi^\nu_{-\frac{1}{2}}\,|0;k\rangle\,, \tag{2.75a}$$

$$|\phi_A\rangle = \mathcal{A}_{\mu\nu\rho}\psi^\mu_{-\frac{1}{2}}\psi^\nu_{-\frac{1}{2}}\psi^\rho_{-\frac{1}{2}}\,|0;k\rangle\,, \tag{2.75b}$$

$$|\phi_B\rangle = B_\mu\psi^\mu_{-\frac{3}{2}}\,|0;k\rangle\,. \tag{2.75c}$$

In general this state does not satisfy the BRST condition and so is not physical; Furthermore, any physical states derived from these may contain elements which are spurious, these are the BRST

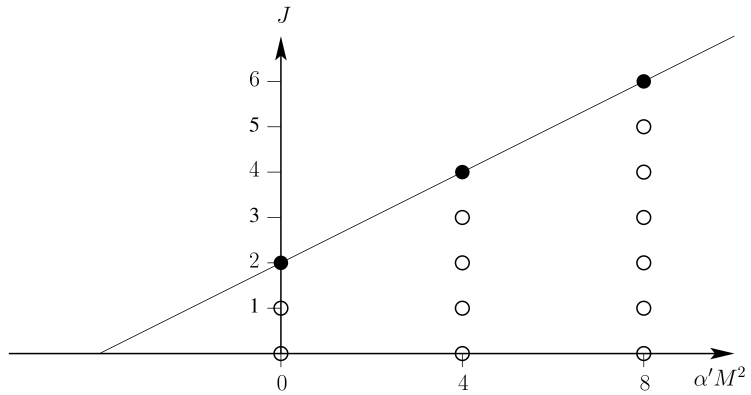

Figure 2.1: Diagrammatic depiction of the type II string spectrum indicating the mass and spin of each physical state, represented by circles. Filled circles show states on the leading Regge trajectory.

exact states. To remedy these problems we take the linear combination,

$$|\phi\rangle = \left(\varepsilon_{\mu\nu}\alpha^{\mu}_{-1}\psi^{\nu}_{-\frac{1}{2}} + \mathcal{A}_{\mu\nu\rho}\psi^{\mu}_{-\frac{1}{2}}\psi^{\nu}_{-\frac{1}{2}}\psi^{\rho}_{-\frac{1}{2}} + B_{\nu}\psi^{\nu}_{-\frac{3}{2}}\right)|0;k\rangle\,, \tag{2.76}$$

and impose the condition $Q\,|\phi\rangle = 0$ with Q as given by (2.57). It was shown in [52] that we can gauge away the scalar, the two-form and the vector states. The requirement of BRST invariance for the remaining states implies that longitudinal polarisations are forbidden

$$k^{\mu}\epsilon_{(\mu\nu)} = 0, \quad k^{\mu}\mathcal{A}_{[\mu\nu\rho]} = 0. \tag{2.77}$$

The result is that we are left with two physical states which may be combined to describe closed strings: the state $\varepsilon_{(\mu\nu)}\alpha^{\mu}_{-1}\psi^{\nu}_{-\frac{1}{2}}\,|0;0\rangle$ has a polarisation $\varepsilon_{(\mu\nu)}$ which is a completely symmetric, traceless tensor invariant under the litte group $SO(9)$, so it carries 44 degrees of freedom. The state $\mathcal{A}_{[\mu\nu\rho]}\psi^{\mu}_{-\frac{1}{2}}\psi^{\nu}_{-\frac{1}{2}}\psi^{\rho}_{-\frac{1}{2}}$ has a polarization $\mathcal{A}_{[\mu\nu\rho]}$ which is a three-form of $SO(9)$, corresponding to 84 degrees of freedom. Together, these two states have 128 degrees of freedom, which is the full bosonic content of the holomorphic sector of the first massive level, as explained in [53].

Continuing beyond the first massive level to higher mass levels, it can be difficult to identify the physical states together with the BRST exact states on account of the proliferation of states at each level. Nevertheless it is technically feasible, a practical approach would be to consider the states in light-cone gauge to avoid consideration of unphysical oscillations; then one need only rearrange the states of any particular mass level into irreducible representations of the little group $SO(9)$ and the task is complete. This method has been demonstrated in the work [52] for the first two levels.

In this thesis we shall focus on states belonging to the leading Regge trajectory. These states take a particularly simple form, making them useful for carrying out calculations in which one wishes to understand the generic behaviour exhibited by massive string states. These states are those with maximum possible spin J for their mass level $\alpha' M^2/4 = n$ and are constructed as follows — first build the state for an open string state which is totally symmetric in its polarisation. In general the tensor product of two such states will give a closed string state in a reducible representation containing physical states of the same mass but different spin J, this J being the number of totally symmetric indices carried by the polarisation of a state in an irreducible representation. By symmetrising over all indices of the polarisation one is left with a state of spin $J = (\alpha' M^2 + 4)/2$. This is the maximum possible spin in the type II string spectrum for a fixed mass level and it is the set of all such states that comprises the leading Regge trajectory. This can be seen pictorially in figure 2.1. Written in terms of oscillators, such an open string state will have the form $|\phi_n\rangle = \varepsilon_{\mu_1\ldots\mu_n\alpha} \prod_{i=1}^{n} \alpha_{-1}^{\mu_i} \psi^{\alpha}_{-\frac{1}{2}} |0;k\rangle$, and this will give the following closed string state,

$$|\phi_n\rangle^{closed} = \epsilon_{\mu_1\ldots\mu_n\,\nu_1\ldots\nu_n(\alpha\beta)} \left[\prod_{i=1}^{n} \alpha_{-1}^{\mu_i} \widetilde{\alpha}_{-1}^{\nu_i}\right] \psi^{\alpha}_{-\frac{1}{2}} \widetilde{\psi}^{\beta}_{-\frac{1}{2}} |0;p\rangle \,, \tag{2.78}$$

where $\epsilon_{\mu_1\ldots\mu_n\alpha\nu_1\ldots n_n\beta} = \varepsilon_{\mu_1\ldots\mu_n\alpha} \otimes \widetilde{\varepsilon}_{\nu_1\ldots\nu_n\beta}$. From here on in it shall be implicitly understood that the polarisation ϵ is symmetric in all indices unless stated otherwise.

2.4 Vertex Operators

It has already been remarked upon in Section 2.2 that for any description we are able to offer using the mode operators of the superstring, it should be possible to pose an alternative description in terms of world-sheet fields. This holds true also for the string spectrum, allowing us to freely describe the states of the string in terms of local operators which are in one-to-one correspondence with these states. These are the *vertex operators* and they are a critical tool in the analysis of string scattering amplitudes. In general, it is difficult to define an amplitude between initial and final states separated by a finite world-sheet time in a way which is consistent with local world-sheet symmetries, but it is possible to form well-defined amplitudes using asymptotic states, these being the scattering amplitudes. On the z-plane the limit $\tau \to -\infty$ corresponds to $z = 0$ and in an SCFT we can act on the ground state with primary fields $\mathcal{A}_i$ to create asymptotic 'in' states $|i\rangle = \mathcal{A}_i(0)|0\rangle$ with energy $L_0|i\rangle = h_i|i\rangle$ and such that $L_n|i\rangle = 0$. The states $|i\rangle$ are thus annihilated by all of the lowering operators of the Virasoro algebra and are highest weight states. However, we have already encountered such states for the superstring in Section 2.3, these are precisely the physical states of the spectrum; therefore there must exist local operators which generate physical states, $|A\rangle = V_A(k,0)\,|0;0\rangle$, the *vertex operators*.

Since not only are we considering a two dimensional SCFT, but also a free field theory, it is not particularly difficult to compute the vertex operator associated with a particular state or vice versa. Consider a conserved charge Q acting upon the state $|\mathcal{O}\rangle$; we can use the operator product expansion to evaluate the contour integral of Q about an operator insertion of $\mathcal{O}(0)$ at the origin. To begin with let us identify the state $|\,\mathbb{1}\rangle$ corresponding to the unit operator; all fields will remain holomorphic at the origin and will be single-valued, indicating that the unit operator lies within the NS sector (Once again we restrict our attention to the holomorphic sector for brevity unless stated otherwise). Specifically, since ψ^μ, ∂X^μ, b, c, β and γ are all holomorphic at the origin then the contour integrals which invert the mode expansions of Section 2.2 will have no poles for modes with frequency greater than some minimum. This implies that the state $|\,\mathbb{1}\rangle$ is annihilated by the matter modes α^μ_m, ψ^μ_r for $m, r > 0$, ghost modes c_m, γ_r for $m, r \geq 3/2$ and antighost modes b_m, β_r for $m, r \geq -1$. This is the NS vacuum $|0;0\rangle$ for the matter space, while the shift in exponents for the Laurent expansion of the ghosts maps the unit operator to a ghost state that differs from the ground state; for the bc ghosts it is found that the identity operator maps to the state $b_{-1}\,|{\downarrow}\rangle$, the $\beta\gamma$ are more complicated and require bosonisation after which the identity then maps to $e^{\varphi(0)}\,|{\downarrow}\rangle$. We can bestow momentum upon the vacuum state in the traditional way by acting upon it with a translation operation,

$$|0;k\rangle = e^{ik\cdot X(0,0)}\,|0;0\rangle \tag{2.79}$$

Having identified the state corresponding to the identity operator we can now work in reverse to map the states from Section 2.3 to their local operators. To do so we again consider the mode operators as before, but now we note that the negative frequency modes will have poles, and therefore a nonzero residue, in the contour integrals about the identity operator. This allows us to make the following identifications,

$$\alpha^\mu_{-m} \to \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!}\partial^m x^\mu(0), \qquad m \geq 1, \tag{2.80a}$$

$$\psi^\mu_{-r} \to \frac{1}{(r-1/2)!}\partial^{r-1/2}\psi^\mu(0), \qquad r > 0, \tag{2.80b}$$

$$b_{-m} \to \frac{1}{(m-2)!}\partial^{m-2} b(0), \qquad m \geq 2, \tag{2.80c}$$

$$c_{-m} \to \frac{1}{(m+1)!}\partial^{m+1} c(0), \qquad m \geq -1, \tag{2.80d}$$

$$\beta_{-r} \to \frac{1}{(r-3/2)!}\partial^{r-3/2}\beta(0), \qquad r > 1, \tag{2.80e}$$

$$\gamma_{-r} \to \frac{1}{(r+1/2)!}\partial^{r+1/2}\beta(0), \qquad r > -1. \tag{2.80f}$$

These continue to hold even when the operators act upon a general state $|\mathcal{O}\rangle$ since if $:\mathcal{O}:$ is a normal-ordered operator it may well have singularities in the operator product expansion but the contour integral of the contractions will not have a pole, hence we can write $\mathcal{A}_{-m} : \mathcal{O} :=: \mathcal{A}_{-m}\mathcal{O} :$ for modes $\mathcal{A}_{-m}$.

With these we can now relate the states $|0;0\rangle$ and $|\,\mathbb{1}\rangle$ with a local operator,

$$|0;0\rangle = c(0)e^{-\varphi}\,|\,\mathbb{1}\rangle\,. \tag{2.81}$$

The curious form of this relation results from our freedom to choose the ghost vacuum. Whereas we have chosen the vacuum to be the state of lowest energy, the other natural choice would be a ground state invariant under global conformal transformations, this is the state $|\,\mathbb{1}\rangle$ picked out by the state-operator isomorphism. For the bc ghosts one may obtain the whole array of vacuum states simply by applying the oscillator modes to the vacuum $|0;0\rangle$ or $|\,\mathbb{1}\rangle$. For the $\beta\gamma$ ghosts this process is complicated by their bosonic nature, the different vacua generate inequivalent representations of the ghost algebra. These vacua can be labelled by their superghost charge q which is measured using the current $j_{\beta\gamma} = -\beta\gamma$,

$$Q_{\beta\gamma} = \oint \frac{dz}{2\pi i} j, \tag{2.82a}$$

$$Q_{\beta\gamma}\,|q\rangle = q\,|q\rangle\,. \tag{2.82b}$$

To shift between different $\beta\gamma$ vacua we must use coherent state operators composed of the bosonised scalar ϕ; the operator $e^{q\varphi}$ carries a charge q and so shifts the charge of the vacuum by the same amount, hence we write

$$|q\rangle = e^{q\varphi}\,|0\rangle\,. \tag{2.83}$$

From this it becomes obvious that the vacuum $|0;0\rangle$ contains the superghost vacuum of charge $q = -1$ whereas $|\,\mathbb{1}\rangle$ carries no charge. The same state can be represented using different vacua and these charges are then passed onto the vertex operators via the factors $e^{q\varphi}$ and we will see later on that we cannot generally write string amplitudes using just one of these vacua since the world-sheet itself carries a background superghost charge which must be cancelled using the vertex operators. Of course, an analogous construction can be built for the bc ghosts which also plays an important role in the global considerations of world-sheet amplitudes.

From the preceding discussion it is simple to compute the vertex operator corresponding to the BRST invariant state of equation (2.76) with superghost charge -1, this is

$$V_{-1} = ce^{-\varphi}\left(\frac{i}{\sqrt{2\alpha'}}\varepsilon_{\mu\nu}\partial X^{\mu}\psi^{\nu} + \mathcal{A}_{\mu\nu\rho}\psi^{\mu}\psi^{\nu}\psi^{\rho} + B_{\nu}\partial\psi^{\nu}\right)e^{ik\cdot X}. \tag{2.84}$$

As mentioned under (2.76) we may gauge away the scalar, vector and two-form states from this state, while BRST invariance requires that we impose the conditions (2.77) to the polarisations ε and A. The term containing ε, when taken in a product with its antiholomorphic partner, gives the vertex for the first massive state on the leading Regge trajectory. For the state in the mass level n with superghost charges $(q, \widetilde{q})$ we indicate this vertex operator with the notation $W^{(n)}_{(q,\widetilde{q})}$ and then we have $W^{(1)}_{(-1,-1)} = V_{-1}\widetilde{V}_{-1}$. It is often said that such vertices are in the $(-1,-1)$ *picture*. If we continue this process to higher mass levels we obtain the generic vertex operator describing the states (2.78) superghost charge $(-1,-1)$,

$$W^{(n)}_{(-1,-1)}(k, z, \bar{z}) = \epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta}V^{\mu_1\ldots\mu_n\alpha}_{-1}(k, z)\widetilde{V}^{\nu_1\ldots\nu_n\beta}_{-1}(k, \bar{z}), \tag{2.85}$$

with

$$V^{\mu_1\ldots\mu_n\alpha}_{-1}(k, z) = \frac{1}{\sqrt{n!}}\left(\frac{i}{\sqrt{2\alpha'}}\right)^{n} e^{-\varphi(z)}\left(\prod_{i=1}^{n}\partial X^{\mu_i}\right)\psi^{\alpha}e^{ik\cdot X(z)}. \tag{2.86}$$

In an abuse of notation the string coordinate field $X(z, \bar{z})$ has been decomposed into $X(z, \bar{z}) = (X(z) + \widetilde{X}(\bar{z}))/2$ such that the decomposed fields match with the conventional open string coordinate field, and as previously stated each of these carry k which is half of the physical momentum of the string, $p = 2k$. This division is a technical convenience which is particularly useful for calculations on world-sheets with boundaries as conducted in this thesis.

The vertex (2.85) carries a superghost charge of $(-1,-1)$ but we will also need vertices for the same state with charge $(0,0)$. If we were to write the $(-1,-1)$ picture vertex as $W_{(-1,-1)} = e^{-\varphi-\widetilde{\varphi}}\mathcal{O}$ then the appropriate $(0,0)$ vertex is given by $W_{(0,0)} = G_{-\frac{1}{2}}\widetilde{G}_{-\frac{1}{2}}\mathcal{O}$. If this is used to change pictures for the states of the leading Regge trajectory we obtain

$$W^{(n)}_{(0,0)}(k, z, \bar{z}) = -\epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta}V^{\mu_1\ldots\mu_n\alpha}_{0}(k, z)\widetilde{V}^{\nu_1\ldots\nu_n\beta}_{0}(k, \bar{z}) \tag{2.87}$$

where

$$V_0^{\mu_1\dots\mu_n\alpha}(k,z) = \frac{1}{\sqrt{n!}}\left(\frac{i}{\sqrt{2\alpha'}}\right)^{n+1}\left(\partial X^{\mu_n}\partial X^{\alpha} - 2i\alpha' k\cdot\psi\partial X^{\mu_n}\psi^{\alpha} - 2\alpha' n\partial\psi^{\mu_n}\psi^{\alpha}\right)\left(\prod_{i=1}^{n-1}\partial X^{\mu_i}\right)e^{ik\cdot X(z)}. \tag{2.88}$$

Before moving on we should make one point; strictly speaking, the vertices given in equations (2.88) and (2.86) can only be applied to the cases $n > 0$, however we shall extend their use to the case $n = 0$ with the understanding that henceforth when one encounters a product of the form $\prod_{i=1}^{0} a_i$ it is to be replaced by unity.

2.5 The S-matrix in string theory

Thus far we have only considered the local properties of the string world-sheet, the spacetime interpretation for this is that we have simply been studying the free isolated string. It is one of the original attractions of string theory that it describes interactions entirely in terms of the global topology of the world-sheet; it is this which greatly simplifies the perturbative expansion by reducing the number of possible diagrams in comparison to the Feynman diagrams of point particle theories, but more importantly, it also implies the absence of short-distance divergences which plague these other models of fundamental physics. The partition function (2.17) can be used to determine the amplitudes describing world-sheets connecting some given initial and final curves which represent the observed strings and it is in this way that interactions enter string theory; the path integral sums over all of the distinct world-sheets that can connect these strings and the interactions between two strings are represented by the splitting or joining of two parts of the world-sheet. This type of interaction is decidedly nonlocal, at no point of the world sheet can one distinguish it from the case of a free string and it is only from global properties that one can discern that an interaction takes place. This is clearly distinct from the case of point particles since a world-line which branches into two will be of a different nature to that without a branch point.

The sum over metrics in the path integral should really be a sum over distinct world-sheet metrics not related to one another by local symmetries, hence the need for using the Faddeev-Popov method in Section 2.2, but it is difficult to define such a sum for arbitrary initial and final states which is consistent with these local symmetries. One case which is well understood is that in which these states are asymptotic and are allowed to propagate for an extremely large time before and after the interaction; these amplitudes precisely correspond to elements of the string

S-matrix. For closed strings these asymptotic states look like infinitely long cylinders attached to the world-sheet, while for open strings they are long strips attached to a boundary of the world-sheet, and by a conformal transformation these world-sheets can be made into compact Riemann surfaces with the cylinders and strips mapped to points within the surface and on the boundary respectively. These points on the world-sheet are the string sources and so at each point we insert a vertex operator to represent the on-shell external string.

Given these considerations one would then define the connected S-matrix in superstring theory describing scattering between n physical states described by vertex operators $V_{\varphi_i}(k_i, \sigma_i)$ by

$$S(k_1, \ldots, k_n) = \sum_{\substack{\text{compact} \\ \text{topologies}}} \int \frac{DgD\chi DXD\Psi}{V_{\text{sym}}} e^{-S[g,\chi,X,\Psi]-\lambda\chi} \prod_{i=1}^{n} \int d\sigma_i^2 \sqrt{-g}\, V_{\varphi_i}(k_i, \sigma_i). \tag{2.89}$$

Note that to make the vertex operator insertions coordinate invariant we must integrate them over the world-sheet, while we have also factored out the volume of the local symmetry gauge group in the quantity V_{sym} as an indication that we are yet to perform the gauge fixing process. For each summand in (2.89) the action S_λ with possible boundary terms given by (2.4-2.5) is evaluated to give the Euler number χ of that Riemann surface, this being determined by the genus, g, the number of boundaries, b, and the number of crosscaps c,

$$\chi = 2 - 2g - b - c. \tag{2.90}$$

While it is easy to visualise the genus of a surface as being equal to the number of handles attached to it, and the number of boundaries tells us how many edges this surface has, it is a little more difficult to do so for crosscaps. Crosscaps are features associated with unoriented surfaces, to describe them one must cut a hole in a surface before identifying opposite points on the boundary thereby introduced; this gluing of the of the edge of the hole causes it to become nothing more than the boundary of a coordinate patch and so no local features are left behind in this process, however, it now prevents us from picking a consistent choice of orientation for each coordinate patch giving an unoriented surface. For further discussion of crosscaps and their use in string theory the reader may consult [54].

The sum in (2.89) can then be taken over all integers $g, b, c \geq 0$ such that only one of g and c is nonzero, this condition ensuring we don't overcount topologically equivalent surfaces. It is possible to restrict this sum depending on which string theory one is interested in. In this work we are primarily interested in the type II closed strings which are oriented and so we consider world-sheets for $c = 0$. As will be discussed in Section 2.6 we can still add boundaries to world-sheets describing closed strings, these then represent interactions between the strings and D-branes.

Furthermore, here we shall describe the process which is strictly speaking only appropriate for the quantisation of the bosonic string, but it is applicable to the tree and one-loop amplitudes of the superstring provided one inserts appropriate superghost charges via the vertex operators in order to cancel the background charge of these world-sheets. The technology necessary for extending the S-matrix for the superstring beyond one-loop, written in terms of path integrals over super-Riemann surfaces, was introduced in a series of papers by D'Hoker and Phong [55–58] but this goes beyond what is necessary for this work[§].

The path integrals in (2.89) contain an integral over all metrics of a given topology in each term and these require gauge-fixing. The result should resemble (2.19) but this was written with the implicit assumption that all metrics were gauge equivalent, something which is not true in most cases. Generally the space of metrics of topology r may only be reduced to a space of equivalence classes consisting of metrics related by diffeomorphisms and Weyl transformations, this is the *moduli space* F which can be parameterised by a finite number of moduli. This space itself has a fair amount of structure, it may retain a residual discrete symmetry described by a group of order n_M on account of 'large coordinate transformations'; this is the modular group, obtained from the quotient of the diffeomorphism group with its connected component. There may also be a subgroup of the diffeomorphisms and Weyl transformations which do not change the metric, these form the conformal Killing group (CKG). This symmetry is not fixed by the choice of a gauge (this was encountered earlier for the case of the complex plane where the CKG manifest itself as the conformal symmetry), the simplest way to deal with it is to further specify the gauge by fixing some of the coordinates for the vertex operators in the path integral. In the case where one does not have a sufficient number of vertex operators then the volume of the CKG must be divided out explicitly. The transformations of the CKG will be generated by κ conformal Killing vectors (CKVs) and if there are μ moduli then the Riemann-Roch theorem can relate these numbers to the Euler number of the world-sheet by

$$\mu - \kappa = -3\chi. \tag{2.91}$$

Given these considerations, if the Faddeev-Popov procedure is then applied to the heuristic

[§]During the writing of this work a set of very accessible reviews concerning supermanifolds, super Riemann surfaces and their application to superstring perturbation theory were published [46, 59, 60]. The interested reader is encouraged to look at these.

form of the S-matrix given by (2.89) one finds the result below,

$$S(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} e^{-\lambda\chi}\int_F \frac{d^\mu t_k}{n_M}\int DX\,D\psi\,Db\,Dc\,e^{-S_m-S_g}$$
$$\times \prod_{(a,i)\notin f}\int d\sigma_i^a \prod_{k=1}^{\mu}\frac{1}{4\pi}(b,\partial_k g)\prod_{(a,i)\in f} c^a(\hat{\sigma}_i)\prod_{i=1}^{n}\sqrt{g}\,V_{\varphi_i}(k_i,\sigma_i), \quad (2.92)$$

where for each topologically distinct world-sheet we integrate over the μ distinct moduli, we eliminate any residual discrete group of symmetries by dividing by its order n_R and we have fixed a set of coordinates $\hat{\sigma}_i^a$ which we label by f to remove the CKG. The action for the matter fields is given by

$$S_m = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}(\partial_a X^\mu\partial^a X_\mu - i\bar{\psi}^\mu\rho^a\partial_a\psi_\mu) \quad (2.93)$$

and the action for the ghost fields is given by

$$S_g = \frac{1}{2\pi}\int d^2\sigma\sqrt{g}g^{ab}b_{ac}\nabla_b c^c \quad (2.94)$$

The inner product $(b,\partial_k g)$ is defined for all traceless symmetric tensors of equal rank, in this case it takes the following form

$$(b,\partial_k g) = \int d^2\sigma\sqrt{g}b_{ab}\partial_k g^{ab}. \quad (2.95)$$

If the vertex operators in equation (2.92) are fixed then, together with the c ghost factors, they will be exactly the BRST invariant vertex operators of Section 2.4 when evaluated using z, $\bar{z}$ coordinates in conformally flat coordinate patches. The integrated vertices lack the ghost factors and so are not BRST invariant, however their variation is just a total derivative which vanishes upon integration.

With (2.92), evaluating S-matrix elements is then simple in principle since the gauge-fixed action describes a set of free fields. The path integrals can be expressed as the expectation value of the vertex operators which is then to be computed using the Green's functions appropriate to that world-sheet. Of course in practice this is not the case when dealing with complicated world-sheets and the process is not well understood beyond world-sheets of genus one.

It is worth noting that, despite not being derived as such, equation (2.92) has the form of a perturbative expansion. Each term in the sum over topologies is weighted by a factor $e^{-\lambda\chi}$ and adding a handle to the world-sheet adds a factor of $e^{2\lambda}$ to this weighting; since this corresponds to the emission and reabsorption of a closed string then the amplitude for the emission of a closed string is proportional to e^{λ} and this must be proportional the string coupling constant g_s. Likewise, adding a strip to the world-sheet adds a factor of e^{λ} and represents the emission and reabsorption of an open string, hence the amplitude for the emission of an open string is proportional to $e^{\lambda/2}$ and so the coupling of open strings is proportional to $g_s^{1/2}$.

2.6 Open strings and D-branes — boundary states

Let us return again to the question of boundary conditions for the world-sheet fields first raised in Section 2.3. The simplest world-sheet with a boundary is an infinitely long strip which describes a single propagating open string and this is conformally equivalent to the upper half of the complex plane, $\mathbb{H}$. In this case the boundary is the real axis and the world-sheet fields are functions of z and $\bar{z}$ where we restrict z to take values such that $\operatorname{Im} z \geq 0$ while $\bar{z}$ is required to take $\operatorname{Im} z \leq 0$. The variation of the matter fields within S_m then yields boundary conditions which accompany the equations of motion, these may be summarised by

$$\left[\bar{\partial}\widetilde{X}^{\mu} \pm \partial X^{\mu}\right]_{z=\bar{z}} = 0, \tag{2.96a}$$

$$\left[\widetilde{\psi}^{\mu} \pm \psi^{\nu}\right]_{z=\bar{z}} = 0. \tag{2.96b}$$

The boundary conditions on X^{μ} are labelled as Dirichlet or Neumann boundary conditions according to whether the relative sign is positive or negative respectively, these names originating from their form when written in the (τ, σ)-coordinates. For the ψ and $\widetilde{\psi}$ they will be Ramond or Neveu-Schwarz boundary conditions. We must choose which of these to impose for each of the spacetime directions. We can implement these boundary conditions in such a way that we may again consider these fields to be defined over the entire complex plane by employing what is commonly referred to as the 'doubling trick' This simply comprises of defining that $X(z) \equiv \pm\tilde{X}(z)$ and $\psi(z) \equiv \pm\tilde{\psi}(z)$ for $\operatorname{Im} z < 0$ where the choice of sign is dependent upon the boundary condition, positive for Neumann boundary conditions and negative for Dirichlet boundary conditions. In this case if z is in the upper half-plane then we can consider $\bar{z}$ to be in the lower half-plane, thus we can replace $\tilde{X}^{\mu}(\bar{z})$ by $D^{\mu}{}_{\nu}X^{\nu}(\bar{z})$ and $\tilde{\psi}^{\mu}(\bar{z})$ by $D^{\mu}{}_{\nu}\psi^{\nu}(\bar{z})$ where $D^{\mu}{}_{\nu}$ is the diagonal matrix with $p+1$ entries equal to 1 and $9-p$ entries equal to -1,

$$D^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}. \tag{2.97}$$

Ultimately the doubling trick has the effect of reproducing the boundary conditions via the introduction of nonzero correlators between holomorphic and antiholomorphic fields thus rendering them no longer independent. This is akin to the image charges introduced in classical electrostatics in order to replicate boundary conditions on the electric field by their mutual interaction with the physical charge distribution. Given the prescription above we may replace all occurrences of $\widetilde{X}^\mu$ and $\widetilde{\psi}^\mu$ with their holomorphic counterparts and treat $\bar{z}$ as new holomorphic variables, then we only require the correlators of the holomorphic fields which for the chiral scalar field are

$$\langle X^\mu(z)X^\nu(w)\rangle = -2\alpha'\eta^{\mu\nu}\log(z-w), \tag{2.98a}$$

$$\langle \psi^\mu(z)\psi^\nu(w)\rangle = \frac{\eta^{\mu\nu}}{z-w}. \tag{2.98b}$$

As with the closed string, the Neveu-Schwarz and Ramond conditions will lead to different sectors of the open string with the bosonic spectrum arising from the NS states and fermionic states coming from the R sector. The interpretations of the Neumann and Dirichlet boundary conditions are quite different and well known; the Neumann boundary conditions describe an open string whose end-points move freely through those spacetime directions, whereas the Dirichlet boundary conditions fix the end-points of the string at certain points in those directions. In the past it was difficult to assign a sensible physical interpretation to open strings with end-points which are restricted to hypersurfaces in spacetime. However, it is now widely acknowledged that these hypersurfaces are physical objects with their own dynamics, these are *D-branes*. Their significance is emphasised by the fact that we cannot even exclude them if we desire to do so — T-duality forces us to confront them.

Consider once again an open string, if we have keep the time-like direction with Neumann boundary conditions (leaving aside more exotic objects) and then take p other directions to be Neumann then this open string is part of a p-dimensional D-brane, a Dp-brane, which sweeps out a $(p+1)$-dimensional world-volume. The presence of a massless gauge boson in the open string spectrum implies that this object can carry a gauge field, while the requirement that the open string be able to interact with closed strings (which must be the case if the world-sheet is described by a local theory) ensures that the D-brane must be coupled to the massless closed string states.

The introduction of D-branes actually revealed the true richness of string theory. A thorough analysis of their properties indicates that the type I and type II superstring theories are not in fact different possibilities but rather are different states of the same theory which are related by T-duality. The standard interactions of type II strings are described by world-sheets without boundaries but if we wish to consider the interactions of closed strings with D-branes then we must turn our considerations to world-sheets with one or more boundaries. The simplest configurations consist of simple Neumann or Dirichlet boundary conditions for each of the spacetime directions as in (2.96) and describes a stack of identical D-branes placed in parallel. There are also other states of the theory which contain more complicated configurations of D-branes which can be obtained by compactifying certain directions, by placing Dp-branes together with Dp'-branes for $p \neq p'$, placing D-branes at different angles such that they are no longer parallel or some combination of these.

2.6.1 Compactifications and T-duality

The essence of T-duality lies in the toroidal compactification of the spacetime of the string. As an example let us consider the X^9 coordinate to be compactified with radius R such that we can periodically identify the points

$$X^9 = X^9 + 2\pi R. \tag{2.99}$$

For conventional field theories a Kaluza-Klein type analysis leads one to conclude that the momentum p in the compact direction is quantised as

$$p = \frac{n}{R}, \qquad n \in \mathbb{Z}, \tag{2.100}$$

which in the uncompactified space appears to generate a family of increasingly massive states from each field the masses of which are determined by the $1/R$. This remains true for the string, but its extended nature leads to other interesting features.

The first of these new features is the concept of *winding*. A closed string may wrap itself around the compact dimension a number of times given by the *winding number* w which is expressed on the world-sheet by the coordinate field becoming multivalued,

$$X^9(\sigma + 2\pi) = X^9(\sigma) + 2\pi R w. \tag{2.101}$$

States of nonzero winding are topological solitons and winding number must be conserved by interactions. As a result of (2.101), whilst the expansions (2.24) remain valid, the computation of the zero mode operators must be modified from the argument of (2.25). Now the total change in traversing the length of the string is

$$2\pi R w = \oint \left(dz\partial X^9 + d\bar{z}\bar{\partial} X^9\right) = 2\pi \left(\frac{\alpha'}{2}\right)^{1/2} (\alpha_0^9 - \widetilde{\alpha}_0^9), \tag{2.102}$$

while the total Noether current for spacetime momentum in the compact direction is

$$p = \frac{1}{2\pi\alpha'} \oint \left(dz\partial X^9 - d\bar{z}\bar{\partial} X^9\right) = \left(\frac{1}{2\alpha'}\right)^{1/2} (\alpha_0^9 + \widetilde{\alpha}_0^9). \tag{2.103}$$

So for the periodic dimension we assign a momentum to the left-moving (holomorphic) oscillator modes and right-moving (antiholomorphic) oscillator modes.

$$p_L^9 \equiv \left(\frac{2}{\alpha'}\right)^{1/2} \alpha_0^9 = \frac{n}{R} + \frac{wR}{\alpha'}, \tag{2.104a}$$

$$p_R^9 \equiv \left(\frac{2}{\alpha'}\right)^{1/2} \widetilde{\alpha}_0^9 = \frac{n}{R} - \frac{wR}{\alpha'}. \tag{2.104b}$$

So the mode expansion for the closed string coordinate field is unchanged for the noncompact directions, whereas the compact direction has the following decomposition of the zero modes

$$X^9(z,\bar{z}) = x^9 - i\frac{\alpha' n}{2R}\ln|z|^2 - i\frac{wR}{2}\ln\frac{z}{\bar{z}} + i\Big(\frac{\alpha'}{2}\Big)^{1/2}\sum_{n\neq 0}\frac{1}{n}\Big(\frac{\alpha_n^9}{z^n} + \frac{\tilde{\alpha}_n^9}{\bar{z}^n}\Big). \tag{2.105}$$

Likewise, the α_0^9, $\widetilde{\alpha}_0^9$ operators may be substituted in this way throughout, in particular this modifies the mass-shell condition with contributions from the winding and momentum in the compact direction.

If we vary the radius of the compact dimension then as R becomes larger, the momentum states become lighter and winding states become heavier; if R is taken to be infinitely large then the winding states become infinitely massive and cease to be dynamical while the momentum states begins to resemble a continuum, therefore we appear to recover the uncompactified result. Conversely, as the radius is made smaller, the momentum states become heavier and the winding states become lighter; as the radius approaches zero we find that all momentum states such that $n \neq 0$ become infinitely massive while the pure winding states form a continuum. It appears that both limits yield an uncompactified theory. The spectrum of the compactified theory is invariant under the exchange $n \leftrightarrow w$ and $R \leftrightarrow \alpha'/R$ implying that the description of a compact dimension of radius R is physically equivalent to that of radius $R' = \alpha'/R$. This exchange is equivalent to sending $\alpha_0^9 \to \alpha_0^9$ and $\widetilde{\alpha}_0^9 \to -\widetilde{\alpha}_0^9$ and the equivalence can be extended to the full theory by sending

$$\begin{aligned} &\alpha_n^9 \to \alpha_n^9, \quad \widetilde{\alpha}_n^9 \to -\widetilde{\alpha}_n^9, \\ &\psi_r^9 \to \psi_r^9, \quad \widetilde{\psi}_t^9 \to -\widetilde{\psi}_r^9, \end{aligned} \tag{2.106}$$

where superconformal invariance ensures that the fermionic modes must transform in the same manner. In this way we can write the T-dual world-sheet fields for the compact dimension as

$$\begin{aligned} X'^9(z,\bar{z}) &= X^9(z) - \widetilde{X}^9(\bar{z}), \\ \psi'^9(z) &= \psi^9(z), \\ \widetilde{\psi}'^9(\bar{z}) &= -\widetilde{\psi}^9(\bar{z}), \end{aligned} \tag{2.107}$$

which is effectively a spacetime parity transformation applied to only the holomorphic fields. Doing so will invert the relative chiralities of the holomorphic and antiholomorphic R sectors and as such T-duality relates the type IIA string compactified on a small radius to the type IIB string on a large radius and vice versa. These arguments are easily extended to multiple compactified dimensions, with the result of an odd number of T-dualities will cause an interpolation between types IIA and IIB string theories, whilst an even number of T-dualities will return us to the same theory.

What then are the effects of T-duality on the open string? Then can be no invariant concept of winding for the open string, since it can always be unwound from the periodic direction; hence when taking the limit $R \to 0$, the states with nonzero momentum in the compact direction are removed from the dynamics but we have no new continuum to replace the lost dimension — the theory appears to be reduced to nine spacetime dimensions. This makes sense when we consider the action of T-duality on the world-sheet fields, (2.107), which must transform Neumann boundary conditions to Dirichlet boundary conditions. If we treat the original theory as one of open strings on a stack of D9-branes, then its T-dual is a theory of open strings on a stack of D8-branes localised in the X^9 direction. This can be reduced to a Dp-brane for $p < 8$ by performing more T-dualities.

2.6.2 Definition of the boundary state

Tree-level diagrams for open strings necessarily contain just one border on the world-sheet and so they can only depend upon a single D-brane. Interactions between D-branes must then be governed by loop diagrams. The lowest order contribution to these processes can be described by an open string stretched between two D-branes and propagating in a closed loop, giving a cylindrical world-sheet which is conformally equivalent to an annulus. If this world-sheet is parameterised by a pair of coordinates (τ, σ) then we may exchange these two via a conformal transformation, the result could be interpreted as a tree-level diagram for a closed string to propagate the finite distance between the D-branes. We refer to these two descriptions as the 'open string channel' and 'closed string channel' respectively. A priori there is no reason why these two processes should have the same amplitude, rather it is a consequence of world-sheet symmetry. Since the closed string channel describes a tree level process it is possible to factorise the amplitude into states describing the emission and absorption of closed strings by a D-brane glued together by a propagator. These boundary states $|B\rangle$ are BRST invariant and they enforce the appropriate boundary conditions for the D-branes associated with them on an amplitude. Boundary states appeared even in the original

S-matrix formulation of string theory [61–64] from the factorisation of open string loop diagrams in the closed string channel, but would later be understood in the context of boundary conditions in path integrals over world-sheets fields as laid out in [65–67]. Today they are commonly encountered as a tool for describing the world-sheet of strings interacting with D-branes and this is the main source of our interest in them here. A more general review of their modern uses may be found in the reviews [68, 69].

For both the NS-NS and R-R sectors of the type II superstring, $|B\rangle$ can be written as the product of a matter part and a ghost part

$$|B\rangle = |B_m\rangle\,|B_g\rangle\,, \tag{2.108}$$

where the matter part can be further decomposed into parts for the coordinate field X and its superpartner Ψ, while the ghost state decomposes into independent bc and $\beta\gamma$ ghost states,

$$|B_m\rangle = |B_X\rangle\,|B_\Psi\rangle\,, \qquad |B_g\rangle = |B_{bc}\rangle\,|B_{\beta\gamma}\rangle\,. \tag{2.109}$$

As an example which will allow us to lay down much of the technology which is required for Chapter 4, let us consider the case of flat D-branes without gauge fields as they were introduced at the beginning of this Section. The matter part of the boundary state is uniquely specified by stating that it must solve the boundary conditions given by equations (2.96) as operator equations written in the closed string channel. Written in terms of oscillator modes these conditions may be summarised as

$$\begin{aligned} &\left(\alpha_n^\mu + D^\mu_\nu \widetilde{\alpha}^\nu_{-n}\right)|B_X\rangle = 0, \quad n \neq 0, \\ &p^\alpha\,|B_X\rangle = (x^i - y^i)\,|B_X\rangle = 0\ , \end{aligned} \tag{2.110}$$

and

$$\left(\psi_r^\mu + i\eta D^\mu_\nu \widetilde{\psi}^\nu_{-r}\right)|B_\Psi, \eta\rangle = 0. \tag{2.111}$$

Here we have indices α to label the $(p+1)$ Neumann directions and i to label the $(9-p)$ Dirichlet directions of a Dp-brane located at y. The variable $\eta = \pm 1$ is used here to specify Ramond or Neveu-Schwarz boundary conditions, but to obtain the final BRST invariant boundary requires the GSO projection which will involve a linear combination of both $|B_\Psi, \eta\rangle$.

The solutions to these equations for the bosonic matter fields can be expressed as coherent states of the oscillators, they are

$$|B_X\rangle = \delta^{(9-p)}(x^i - y^i)\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\cdot D\cdot\widetilde{\alpha}_{-n}\right]|0; p^a = 0\rangle\,. \tag{2.112}$$

For the fermionic fields there are different solutions for the Neveu-Schwarz and Ramond sectors; for the NS sector the solution takes the form

$$|B_\Psi, \eta\rangle_{NS} = \exp\left[i\eta \sum_{r=1/2}^{\infty} \psi_{-r} \cdot D \cdot \widetilde{\psi}_{-r} \right] |0\rangle \,, \tag{2.113}$$

while in the R sector it is given by

$$|B_\Psi, \eta\rangle_{R} = \exp\left[i\eta \sum_{r=1}^{\infty} \psi_{-r} \cdot D \cdot \widetilde{\psi}_{-r} \right] |B_\Psi, \eta\rangle_R^{(0)} \,. \tag{2.114}$$

Here we have factorised the R sector ground state which lies in the representation of Majorana spinor biproducts for ten dimensional spacetime such that

$$|B_\Psi, \eta\rangle_R^{(0)} = \mathcal{M}_{AB}^{(\eta)} |A\rangle \left| \tilde{B} \right\rangle , \tag{2.115}$$

where

$$\mathcal{M}^{(\eta)} = C\Gamma^0\Gamma^1 \ldots \Gamma^p \left(\frac{1 + i\eta\Gamma_{11}}{1 + i\eta} \right) , \tag{2.116}$$

with C being the charge conjugation matrix.

Before using the boundary state to compute amplitudes involving D-branes, one must perform the GSO projection. In the NS-NS sector the projected state is

$$|B\rangle_{NS} = \frac{1}{2} \left(|B, +\rangle_{NS} - |B, -\rangle_{NS} \right) \tag{2.117}$$

and in the R-R sector the GSO projected boundary state is

$$|B\rangle_{R} = \frac{1}{2} \left(|B, +\rangle_{R} + |B, -\rangle_{R} \right) . \tag{2.118}$$

Chapter 3

String Theory at Low Energy – Supergravity

Perhaps one of the most obvious features of string theory for the active practitioner is the holistic approach required in the answering of certain questions; rather than a fully formed framework describing spacetime and matter at the smallest scales, we have instead inherited a wealth of tools with wildly different domains of applicability with which to tackle our problems. Nowhere is this more evident than when probing the low energy dynamics within string theory. Generally in such investigations there are three complimentary approaches which may be employed:

- *'Stringy' microscopic description* — The quantum description discussed in Chapter 2 can be used to compute exact amplitudes for many different phenomena. The utility of this approached is magnified by the existence of a microscopic description for non-perturbative objects of the string spectrum.

- *Semi-classical background field description* — commonly referred to as the non-linear sigma model, field dependent couplings are added to the world-sheet action of a string allowing the description of strings interacting with background fields.

- *Effective field theory* — A quantum field theory which reproduces the S-matrix elements of the full string theory for energies much smaller than the string scale may be used to facilitate the easy computation of the dominant term in amplitudes for processes involving the massless states of the string spectrum.

Having already introduced the first of these methods, we shall now proceed to familiarise ourselves with the latter two.

In Chapter 2 we considered the dynamics of quantum superstrings on a flat spacetime background. However, in principle nothing prevents us from generalising this theory to one describing propagation of strings on a curved background by introducing the metric $G_{\mu\nu}(X)$ to facilitate contractions between spacetime indices in the action described by both (2.8) and (2.13). Of course, the metric being solely a function of X, this leads to the spoiling of world-sheet supersymmetry and so additional terms containing the spacetime Riemann tensor, $R^{\mu}{}_{\nu\rho\sigma}$, and Cristoffel symbols, $\Gamma^{\mu}_{\nu\rho}$, are required to restore this [70]. The result is the following action which is quoted without proof

$$S_G = \frac{1}{2\pi}\int d^2z\left(G_{\mu\nu}\left[\frac{1}{\alpha'}\partial X^\mu\bar{\partial}X^\nu + \psi^\mu\bar{\partial}\psi^\nu + \widetilde{\psi}^\mu\partial\widetilde{\psi}^\nu + \psi^\mu\Gamma^\nu_{\rho\sigma}\bar{\partial}X^\rho\psi^\sigma \right.\right.$$
$$\left.\left. + \widetilde{\psi}^\mu\Gamma^\nu_{\rho\sigma}\partial X^\rho\widetilde{\psi}_\sigma\right] - \frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\widetilde{\psi}^\rho\widetilde{\psi}^s\right) \tag{3.1}$$

Though this seems to be a perfectly natural course of action, it might be argued that it has been poorly motivated and that we would prefer gravity to be a necessary part of the theory, not one that can be included on a whim. Yet this is not the case, for the introduction of the curved metric can be justified if one considers its interpretation as a quantum phenomenon; if the metric of a spacetime which is nearly flat is decomposed into a flat metric with a term describing the fluctuations, $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}$, then the partition function describing a string propagating in this background may be expressed schematically as

$$\int DXe^{-S_G} = \int DXe^{-S}\left[1 + \int\frac{d^{10}k}{(2\pi)^{10}}\int d^2z\, H_{\mu\nu}V_0^\mu\widetilde{V}_0^\nu e^{ik\cdot X} + \ldots\right] \tag{3.2}$$

where S is the flat space action for the superstring and H is the Fourier transform of the graviton field h. It can be seen here that the partition function contains the vertex operator $W_{(0,0)} = h_{\mu\nu}V_0^\mu\widetilde{V}_0^\nu e^{ik\cdot X}$ for the emission of a graviton with the wavefunction $h_{\mu\nu}(X)$. Written in this way, the amplitudes for a string within a background gravitational field resemble those of a string in flat space with additional insertions representing graviton interactions. These notions may be pursued further and one may show that the requirement of Weyl invariance for the action of a string in the background given by $G_{\mu\nu}$ yields a beta function, the condition that this vanish is equivalent to the Einstein equations. This would, however, take us too far afield and further details may be found in [71].

The approach taken above suggests further generalisations to the string action. One could include background fields representing the finite set of massless states from the string spectrum within the action. These too will have beta functions which must vanish if we are to maintain local Weyl invariance and the beta functions resemble spacetime equations of motion for the background fields. Now since physical configurations of phenomenological interest typically should have energies far lower than the string scale which parameterises the massive excitations, then such a model is of particular relevance since it can describe the interactions of strings with classical sources for these fields. Of course, these sources can also be non-perturbative objects from the spectrum of the theory and it is now well established that the D-branes of string theory are in fact the microscopic description of p-brane solutions in supergravity.

In this chapter we introduce a covariant action used to describe type IIB supergravity in Section 3.1 before moving on to discuss the simplest solutions to the equations of motion in Section 3.2. These solutions describe extremely massive objects extended in a subset of the spacetime directions. The extremal solutions which are BPS states are stable and can represent the low energy effective solutions describing the D-branes of the last chapter. Finally in Section 3.3 we discuss how T-duality and S-duality of the full string theory appear in the supergravity limit and what their effects on supergravity fields are.

3.1 Supergravity in ten dimensions

For this work we will be primarily interested in the low energy limit of the type IIB superstring theory for describing the spacetime bulk, which has the effective field theory known as *type IIB supergravity*. Recall that the type IIB superstring theory is obtained by performing the same GSO projection to both the holomorphic and antiholomorphic sectors of the closed superstring to obtain a chiral spectrum. The massless content of the spectrum can be represented as the product of these two sectors in terms of $SO(8)$ representations by

$$\begin{aligned}(\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{35}_{+} \\ \oplus \mathbf{8_c} \oplus \mathbf{8_c} \oplus \mathbf{56_s} \oplus \mathbf{56_s}.\end{aligned} \tag{3.3}$$

The bosonic degrees of freedom depicted on the first line contain the $\mathbf{35}$ representation of the graviton and the self-dual $\mathbf{35}_{+}$ representation of the gauge boson described by a 4-form potential $C_{(4)}$. The pair of $\mathbf{28}$ representations will generate a couple of 2-form potentials, that of the R-R sector shall be denoted $C_{(2)}$ and that of the NS-NS sector is the familiar Kalb-Ramond field $B_{(2)}$. Finally, the two singlets will become the dilaton field Φ and the R-R scalar $C_{(0)}$. The fermionic degrees of freedom on the second line contain a pair of $\mathbf{56_s}$ representations for two gravitinos whilst the $\mathbf{8_c}$'s describe a pair of Majorana-Weyl spinors

Here we will introduce the action for the bosonic fields of type IIB supergravity the equations of motion of which are consistent, to leading order in string scale, with those imposed as beta functions on the action of the string in the presence of background fields. We refer to this action as being in the *string frame*. In addition to the fields specified below equation (3.3) we include the metric $G_{\mu\nu}$, while for the Kalb-Ramond 2-form we define the standard field strength $H_{(3)} = dB_{(2)}$ and for the R-R p-form potentials we define the field strengths $F_{(p+1)} = dC_{(p)}$. Furthermore, as will be explained below, it will be convenient to make the following field redefinitions

$$\widetilde{F}_{(3)} = F_{(3)} - C_{(0)} \wedge H_{(3)}, \tag{3.4a}$$

$$\widetilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}. \tag{3.4b}$$

It should be noted that from here on in we shall neglect the fermionic degrees of freedom on the basis that we may force all fermionic fields to vanish without finding ourselves at odds with the bosonic equations of motion. This is essentially because the fermionic equations of motion must be 'odd' in the fermionic fields and so there cannot exist terms within them consisting solely of bosonic fields.

Thus, splitting the action up into terms containing only NS-NS fields, only R-R fields and the Chern-Simons action which contains both, we then write it as $S_{IIB} = S_{NS} + S_R + S_{CS}$ where

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H_{(3)}^2\right], \tag{3.5a}$$

$$S_R = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[F_{(1)}^2 + \widetilde{F}_{(3)}^2 + \frac{1}{2}\widetilde{F}_{(5)}^2\right], \tag{3.5b}$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}. \tag{3.5c}$$

Here we have written the kinetic terms for the p-forms in a compact form which is expanded as follows,

$$F_{(p)}^2 = \frac{1}{p!}F_{(p)\mu_1\ldots\mu_p}F_{(p)}^{\mu_1\ldots\mu_p}. \tag{3.6}$$

These terms have been canonically normalised except that of $F_{(4)}$ which has an additional factor of $1/2$. Strictly speaking, we cannot write a covariant action for the field strength $F_{(5)}$ as it is self-dual, so we must take the action S_{IIB} subject to the additional constraint that $\widetilde{F}_{(5)} = {}^\star\widetilde{F}_{(5)}$. As such, we can consider this factor to correct for our overcounting by twice the actual number of degrees of freedom.

The modified field strengths defined by equations (3.4) are the physical gauge invariant field strengths for type IIB supergravity, but instead of being closed they have the nonstandard Bianchi identities

$$d\widetilde{F}_{(3)} = -F_{(1)} \wedge H_{(3)}, \tag{3.7a}$$

$$d\widetilde{F}_{(5)} = H_{(3)} \wedge F_{(3)}. \tag{3.7b}$$

It should be noted that, due to the presence of the dilaton, the action S_{IIB} as written above does not have the standard form of the Einstein-Hilbert action for the first term. Such a form would be desirable if we wish to discuss the measurement of energy and charges in a meaningful way and to do so one should rescale the metric. If we do this rescaling with the inclusion of a factor dependent on the asymptotic value of the dilaton, Φ_0, in order to ensure that the metric is asymptotically that of Minkowski space then it can be written as

$$G_{\mu\nu} \to e^{\frac{\Phi-\Phi_0}{2}} G_{\mu\nu}. \tag{3.8}$$

Reformulating the action in terms of this rescaled metric results in the *Einstein frame action* given by the previous S_{IIB} with the following modified terms

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left[R - \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} e^{-(\Phi-\Phi_0)} H_{(3)}^2 \right], \tag{3.9a}$$

$$S_R = -\frac{g_s^2}{4\kappa^2} \int d^{10}x \sqrt{-G} \left[e^{2(\Phi-\Phi_0)} F_{(1)}^2 + e^{(\Phi-\Phi_0)} \widetilde{F}_{(3)}^2 + \frac{1}{2} \widetilde{F}_{(5)}^2 \right]. \tag{3.9b}$$

Notice that we now have an action with the physical gravitational coupling which has been expressed in terms of the string coupling, in turn determined by the asymptotic value of the dilaton $g_s \equiv e^{\Phi_0}$:

$$2\kappa^2 \equiv 2\kappa_{10}^2 g_s^2 = 16\pi G_N. \tag{3.10}$$

Finally, by direct computation of graviton scattering amplitudes one may verify that

$$2\kappa_{10}^2 = (2\pi)^7 \alpha'^4. \tag{3.11}$$

3.2 Extended objects in supergravity – p-brane solutions

Having now defined an action for a viable effective field theory we shall now proceed to examine its solutions and how these should be interpreted in the wider context of string theory. String theory is defined perturbatively via the partition function of the world-sheet action (2.13), as outlined in Chapter 2, but not only does this give us the perturbative spectrum of string theory, it also allows the extraction of information regarding non-perturbative objects. Traditionally, when one wishes to understand the non-perturbative aspects of a quantum theory it is common to study its associated solitons; that is, local finite energy solutions to the equations of motion. This is a result of these equations being non-linear and thus it is to be expected that the solutions are inaccessible to perturbations around solutions to the linearised equations. One might ask then whether there exist classical solutions for supergravity which represent such solitonic objects and it is in fact rich with such solutions. Furthermore, some of these solutions represent BPS states for which the classical relationship between their mass and charges are protected from quantum corrections by supersymmetry. Such solutions are therefore of great interest if one wishes to make statements about string theory at strong coupling.

Hence we shall be seeking supergravity solutions which describe objects extended in p spatial dimensions, these will sweep out a $(p+1)$-dimensional world-volume in spacetime whilst being localised in the remaining $9-p$ transverse directions. These are generically referred to as p-branes. Such objects can be coupled naturally to any $(p+1)$-form potential $A_{(p+1)}$ via the interaction

$$\int_{V_{p+1}} A_{(p+1)}, \tag{3.12}$$

where V_{p+1} is the world-volume, and therefore they can carry a charge for this gauge field which is determined by integrating the Hodge dual of the field-strength over a transverse hypersphere containing the p-brane,

$$Q = \int_{S_\infty^{8-p}} {}^\star dA_{(p+1)}. \tag{3.13}$$

This then limits the possible charged objects we may find since in type IIB supergravity there exist only the 0-, 2- and 4-form potentials and hence we only expect to see -1-, 1- and 3-branes carrying charge.

We would like to find the simplest possible examples of p-branes which intuitively would mean the vacuum solutions for a 'flat' brane. The spacetime of this solution should therefore be asymptotically flat in the transverse directions and translationally invariant in the longitudinal directions along the world-volume. In the next section we shall move on to review the equations of motion resulting from the action written in Section 3.1 before moving on to discuss these simple p-brane solutions. An important and useful simplification that should be noticed is that we can consistently set to zero all fermionic fields, the Kalb-Ramond field and all but one R-R gauge field in order to obtain a solution describing objects which couple only to the metric, dilaton and a single R-R gauge field.

3.2.1 General p-brane solutions

Most generally, varying the action $S_{IIB} = S_{NS} + S_R + S_{CS}$ in the Einstein frame with respect to the dynamical fields G, Φ^*, $B_{(2)}$, $C_{(0)}$, $C_{(2)}$, $C_{(4)}$, where S_{NS} and S_R are given by equation (3.9) and the Chern Simons action S_{CS} is given by (3.5c), results in the following equations of motion after judicious application of the self-duality condition and Bianchi identities;

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R - S_{\mu\nu} = 0, \tag{3.14a}$$

$$\nabla^2\Phi - \frac{1}{2}e^{-\Phi}H_{(3)}^2 - g_s^2 e^{2\Phi}F_{(1)}^2 - \frac{g_s^2}{2}e^{\Phi}\widetilde{F}_{(3)}^2 = 0, \tag{3.14b}$$

$$d\left(e^{-\Phi\star}H_{(3)}\right) + g_s^2\left(e^{\Phi}\,{}^\star\widetilde{F}_{(3)} \wedge F_{(1)} + {}^\star\widetilde{F}_{(5)} \wedge \widetilde{F}_{(3)}\right) = 0, \tag{3.14c}$$

$$d\left(e^{2\Phi\star}F_{(1)}\right) - e^{\Phi}\,{}^\star\widetilde{F}_{(3)} \wedge H_{(3)} = 0, \tag{3.14d}$$

$$d\left(e^{\Phi\star}\widetilde{F}_{(3)}\right) - \widetilde{F}_{(5)} \wedge H_{(3)} = 0, \tag{3.14e}$$

$$d^\star\widetilde{F}_{(5)} - H_{(3)} \wedge F_{(3)} = 0, \tag{3.14f}$$

for $p \in \{-1, 1, 3\}$. Here the result has been expressed in terms of a quantity $S_{\mu\nu}$ which is composed of the energy-momentum tensors for the canonically normalised scalar and the gauge fields,

$$S_{\mu\nu} = \frac{1}{2}\left(T_{\Phi\,\mu\nu} + e^{-\Phi}T_{H\,\mu\nu} + g_s^2\left[e^{2\Phi}T_{F_{(1)}\mu\nu} + e^{\Phi}T_{\widetilde{F}_{(3)}\mu\nu} + T_{\widetilde{F}_{(5)}\mu\nu}\right]\right), \tag{3.15a}$$

$$T_{\Phi\,\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}G_{\mu\nu}\partial_\rho\Phi\partial^\rho\Phi, \tag{3.15b}$$

$$T_{H\,\mu\nu} = \frac{1}{2}H_{(3)\,\mu}{}^{\rho\sigma}H_{(3)\,\nu\rho\sigma} - \frac{1}{2}G_{\mu\nu}H_{(3)}^2, \tag{3.15c}$$

$$T_{F_{(p+2)}\mu\nu} = \frac{1}{(p+1)!}F_{(p+2)\,\mu}{}^{\rho_1\ldots\rho_{p+1}}F_{(p+2)\,\nu\rho_1\ldots\rho_{p+1}} - \frac{1}{2}G_{\mu\nu}F_{(p+2)}^2. \tag{3.15d}$$

Examination of these equations makes it clear that there are several possibilities available to us when searching for simple solutions which may represent objects in supergravity. The simplest are of course vacuum solutions to Einstein's equations, since we may set all fields other than the metric to zero without introducing an inconsistency in (3.14); these solutions are generalisations of the Schwarzschild solution which will describe an extended massive object. Continuing with this analogy to black-holes, we can consider a charged p-brane solution coupled to a $(p+1)$-form R-R potential which should reduce to the Schwarzschild solution as the charge becomes vanishingly small. However, equation (3.14b) forbids us from introducing a gauge field $A_{(p+1)}$ for $p = -1$ or $p = 1$ without a non-trivial dilaton field; instead we must take the following truncated equations

*For convenience, the dilaton has been rescaled by its asymptotic value $\Phi \to \Phi - \Phi_0$

of motion

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R - \frac{1}{2}T_{\Phi\,\mu\nu} - \frac{g_s^2}{2}e^{\frac{3-p}{2}\Phi}T_{F_{(p+2)}\mu\nu} = 0, \tag{3.16a}$$

$$\nabla^2\Phi - g_s^2\frac{3-p}{4}e^{\frac{3-p}{2}\Phi}F_{(p+2)}^2 = 0, \tag{3.16b}$$

$$d\left(e^{\frac{3-p}{2}\Phi}\star F_{(p+2)}\right) = 0. \tag{3.16c}$$

The case in which we take a nonvanishing gauge field $A_{(4)}$ clearly differs from the case just considered since the dilaton and gauge fields will not be coupled.

To determine solutions for these fields requires that we be more specific about the properties we wish them to have. We are interested in the *flat* p-brane, a static object which is invariant under translations in its $p+1$ world-volume directions that we label as $(t, x^1, \ldots, x^p)$, while spherically symmetric in the transverse directions. In addition to this, we would like the p-brane to be an isolated object and should then impose asymptotic flatness at large distances from the brane, i.e. only in the transverse directions. This would suggest that we use an ansatz for the metric of the form

$$ds^2 = f(r)\left(-W(r)dt^2 + dx_i^2\right) + g(r)\left(W^{-1}(r)dr^2 + r^2 d\Omega_{8-p}^2\right) \tag{3.17}$$

where $d\Omega_{8-p}^2$ is the metric on the transverse unit $(8-p)$-sphere and the functions f, g, W are to be determined but can only depend upon the radial coordinate in the transverse space. Likewise we may suppose that the dilaton field is purely a function of r, while the ansatz we choose for the gauge field is dependent upon whether we are interested in the ‘electric’ solution or ‘magnetic’ solution. Here we shall consider the electric solution which we take to vanish asymptotically

$$C_{(p+1)\,01\ldots p} = h(r). \tag{3.18}$$

If, however, we wished to write an ansatz for a magnetic solution then the fact that the integral of $dC_{(p+1)}$ of the $(p+2)$-dimensional hypersphere at spatial infinity is non-trivial prevents us from writing a globally well-defined gauge potential. In this case it is more convenient to specify that the field strength $F_{(p+2)}$ be proportional to the volume form on $(p+2)$-sphere. Yet this is not strictly necessary in order to generate magnetic solutions, instead one may compute the electric solutions for the $(6-p)$-brane dual to the p-brane of interest and rewrite these solutions as magnetic solutions for this p-brane.

Thus, inserting these proposed solutions into the equations of motion (3.16) one may determine the unknown functions f, g, W, Φ and h. The solution for the metric is

$$ds^2 = H_p^{-\frac{p-7}{8}}(r)\left(-W(r)dt^2 + dx_i^2\right) + H_p^{\frac{p+1}{8}}(r)\left(W^{-1}(r)dr^2 + r^2 d\Omega_{8-p}^2\right), \tag{3.19}$$

$$H_p(r) = 1 + \alpha_p \left(\frac{r_p}{r}\right)^{7-p}, \qquad W(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p},$$
$$r_p^{7-p} = d_p (2\pi)^{p-2} g_s N \alpha'^{\frac{7-p}{2}i}, \qquad d_p = 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma\left(\frac{7-p}{2}\right), \tag{3.20}$$

$$\alpha_p = \sqrt{1 + \left(\frac{r_H^{7-p}}{2r_p^{7-p}}\right)^2} - \frac{r_H^{7-p}}{2r_p^{7-p}}. \tag{3.21}$$

While the solutions for the dilaton and the R-R potential is

$$e^{2\Phi} = g_s^2 H_p(r)^{\frac{3-p}{2}}, \qquad C_{(p+1)} = g_s^{-1}\left(H_p^{-1}(r) - 1\right) dt \wedge dx^1 \wedge \ldots \wedge dx^p. \tag{3.22}$$

We can see that indeed these solutions resemble a generalisation of the charged black hole solution to some extended charged configuration known as a black brane; they feature a number of similarities including a horizon at the radius $r = r_H$ together with a singularity at $r = 0$ and an extremality parameter α_p. In the limit that this parameter is taken to unity we find the extremal p-brane solutions which are the subject of Section 3.2.2. These are BPS solutions and hence their stability is assured by supersymmetry, making them worthwhile of study. In comparison, the non-extremal solutions are both classically unstable under linear perturbations along their world-volume and quantum-mechanically unstable due to the total area of the event horizon (and hence entropy) of several black holes is generally greater than that of a p-brane of equal mass.

3.2.2 The extremal limit

In the extremal limit we may use the solutions (3.19-3.22) but in this case we take $H_p(r) = 1 + (r_p/r)^{7-p}$ and $W(r) = 1$. As such, the solutions for the metric, dilaton and R-R potential are

$$ds^2 = H_p^{-\frac{1}{2}}(r)\eta_{\mu\nu}dx^\mu dx^\nu + H_p^{\frac{1}{2}}(r)dx^i dx^i, \tag{3.23a}$$
$$e^{2\Phi} = g_s^2 H_p(r)^{\frac{3-p}{2}}, \tag{3.23b}$$
$$C_{(p+1)} = -g_s^{-1}\left(H_p^{-1}(r) - 1\right) dt \wedge dx^1 \wedge \ldots \wedge dx^p, \tag{3.23c}$$

for $\mu = 0, 1, \ldots, p$ and $i = p+1, \ldots, 9$. The horizon for these solutions now resides at $r = 0$ and so it is both singular and occupies zero area as a result of the factor multiplying the metric of the transverse $(8-p)$-sphere vanishing. This is not the case in the instance of the 3-brane, in fact the geometry at the horizon is that of $AdS_5 \times S^5$, a matter of fundamental importance for the AdS/CFT correspondence. Of course, if one wishes to make use of p-brane solutions to investigate topics such as black hole thermodynamics then it is necessary to construct a black hole-like solution with a horizon of non-zero size; luckily this can be accomplished by manipulating these simple solutions to give others which preserve fewer supersymmetries, either by intersecting several p-branes, boosting them to non-zero momentum or wrapping them on some compact geometry.

Thus far we have taken the solutions (3.19), (3.22) and (3.23) to be indicative of the presence of an extended object, but as is often the case with classical descriptions of fields we find there are singularities at which the equations of motion are not satisfied. It is always tempting to then include sources in the theory to account for these singularities and complete the solutions, this could be done by including a delta function source term on the right hand side of each equation of motion in (3.16) of the form $J(x^i)\delta^{(9-p)}(x^I)$. This would represent the coupling of an external object, localised to the region of spacetime occupied by the world-volume, to the fields of supergravity in an analogous fashion to the way in which one may attempt to modify the classical equations of motion for electromagnetism by introducing point charge sources. However it is not always possible to do this consistently; if the self-interaction of the source is unchecked then the equations of motion remain unsatisfied in the presence of a source and the description in these terms is insufficient. This is the case for non-extremal p-branes, however, for the extremal p-brane we find ourselves saved by supersymmetry which results in a cancellation between the gravitational self-interaction and those of the dilaton and R-R gauge field. In doing so we are able to relate the parameters in the supergravity solution which arise as constants of integration to parameters of the world-volume description, namely the *brane tension*, τ_p and the *brane charge* μ_p.

To achieve this we must supplement the truncated action describing supergravity fields in the spacetime bulk with a world-volume action describing the p-brane coupling to these fields

$$S = S_{\text{bulk}} + S_{\text{wv}} \tag{3.24}$$

where for the bulk term we take the dilaton and R-R p-form together with the metric as should now be familiar in comparison with equation (3.9),

$$S_{\text{bulk}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left[R - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{g_s^2}{2}e^{\frac{3-p}{2}\Phi}F_{(p+2)}^2\right], \tag{3.25}$$

while the world-volume action contains the coupling of the D-string to gravity and the R-R gauge field via auxiliary fields on the world-volume X^μ, h_{ab}. In the Einstein frame this is written as

$$\begin{aligned} S_{\text{wv}} = -\frac{T_p}{2}\int d^{p+1}\sigma\sqrt{-h}\left[e^{\frac{p-3}{2(p+1)}\Phi}h^{ij}\partial_i X^\mu\partial_j X^\nu G_{\mu\nu} - (p-1)\right] \\ + \frac{\mu_p}{(p+1)!}\int d^{p+1}\sigma\epsilon^{i_1\ldots i_{p+1}}A_{(p+1)\mu_1\ldots\mu_{p+1}}\partial_{i_1}X^{\mu_1}\ldots\partial_{i_{p+1}}X^{\mu_{p+1}}. \end{aligned} \tag{3.26}$$

Spacetime indices will be Greek letters to denote all directions, i.e. $\mu \in \{0, 1, \ldots, 9\}$, while lower case Latin letters indicate the world-volume directions, $i \in \{0, 1, \ldots, p\}$ and upper case indicates the transverse directions, $I \in \{p+1, \ldots, 9\}$. The parameter T_p determines the physical tension of the p-brane via the $T_p = g_s \tau_p$ and the comparison between a one-loop vacuum amplitude on the world-sheet with the field theory amplitudes describing dilaton and graviton exchange between two p-branes allows us to relate both the tension and charge to the string scale,

$$\mu_p = T_p = (2\pi)^{-p} \alpha'^{-\frac{p+1}{2}}. \tag{3.27}$$

The equations of motion of the auxiliary fields provide constraints on the equations for the physical fields; for the world-volume metric we find

$$h_{ij} = e^{\frac{p-3}{2(p+1)}\Phi} \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}. \tag{3.28}$$

The constraint equation for the world-volume coordinate fields are

$$\begin{aligned} \nabla^2 X^\mu + h^{ij} \partial_i X^\rho \partial_j X^\sigma \left(\Gamma^\mu_{\rho\sigma} + \frac{p-3}{2(p+1)} \partial_{(\rho} \Phi \delta^\mu_{\sigma)} \right) \\ + \frac{1}{2\sqrt{-h}} \frac{\mu_p}{T_p} e^{-\frac{p-3}{2(p+1)}\Phi} \epsilon^{i_1 \ldots i_{p+1}} F^\mu_{(p+2)\nu_1 \ldots \nu_{p+1}} \partial_{i_1} X^{\nu_1} \ldots \partial_{i_{p+1}} X^{\nu_{p+1}} = 0. \end{aligned} \tag{3.29}$$

The constraint (3.28) can be used to eliminate the world-volume metric from the equations of motion for the bulk fields; furthermore, we can take static gauge for the world-volume coordinates and posit the ansatz that the p-brane is at rest at the origin in the transverse directions to satisfy the constraint (3.29),

$$X^a(\sigma) = \sigma^a \quad a = 0, 1, \ldots, p \tag{3.30a}$$

$$X^m(\sigma) = 0 \quad m = p+1, \ldots, 9. \tag{3.30b}$$

After varying the action with respect to the bulk fields and then applying the constraints from the world volume fields we obtain the equations (3.16) with source terms included. The resulting equations of motion for the metric, dilaton and R-R two-form respectively are

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - \frac{1}{2} T_{\Phi\,\mu\nu} - \frac{g_s^2}{2} e^{\frac{3-p}{2}\Phi} T_{F_{(p+2)}\mu\nu} = \frac{\kappa^2 T_p}{\sqrt{-G}} J_{G\,\mu\nu}(x^i) \delta^{(9-p)}\left(x^I\right), \tag{3.31a}$$

$$\nabla^2 \Phi - g_s^2 \frac{3-p}{4} e^{\frac{3-p}{2}\Phi} F^2_{(p+2)} = \frac{\kappa^2 T_p}{2\sqrt{-G}} J_\Phi(x^i) \delta^{(9-p)}\left(x^I\right), \tag{3.31b}$$

$$\nabla_\rho \left(e^{\frac{3-p}{2}\Phi} F^\rho_{(3)\mu\nu} \right) = \frac{2\kappa^2 \mu_p}{\sqrt{-G}} J_{F\,\mu\nu}(x^i) \delta^{(9-p)}\left(x^I\right). \tag{3.31c}$$

where the sources can be expressed as

$$J_{G\,\mu\nu}(x^i) = \begin{cases} e^{\frac{p-3}{4}\Phi}\sqrt{-\det G_{ij}}G_{\mu\nu} & \text{for } \mu,\nu \in \{0,1,\dots,p\} \\ 0 & \text{otherwise} \end{cases} \tag{3.32a}$$

$$J_\Phi(x^i) = \frac{p-3}{2}e^{\frac{p-3}{4}\Phi}\sqrt{-\det G_{ij}} \tag{3.32b}$$

$$J_{F\,\mu\nu}(x^i) = \begin{cases} \epsilon^{\mu\nu} & \text{for } \mu,\nu \in \{0,1,\dots,p\} \\ 0 & \text{otherwise} \end{cases} \tag{3.32c}$$

Using the above equations one may then obtain the extremal p-brane solutions by expanding the supergravity fields in $1/r$ and solving iteratively. Whereas in the absence of sources the function H_p could be any harmonic function of the transverse coordinates, there is now a requirement that any singularities in H_p must originate from the sources we include. These sources are $(9-p)$-dimensional Dirac delta functions located at the origin of the transverse space and so H_p must have a single pole there with its coefficient r_p equal to the coefficient of the delta function. It is in this way that one may determine the value of r_p which is quoted in (3.20).

3.3 String theory dualities in supergravity

In Section 2.6.1 it was noted that given a superstring theory defined on a flat ten-dimensional spacetime we could define its T-dual as the same theory after a parity operation on half of the world-sheet oscillators. Clearly such transformations will affect the spectrum of the string and therefore can act in a non-trivial way on the massless spacetime fields of supergravity. Here we briefly recount the standard arguments to determine how these fields are transformed which can be found in any modern text, *e.g.* [72, 73]. Consider, for example, the space $R^{8,1}\times S^1$ where we label the compact direction by the coordinate y. We may decompose the string-frame metric, Kalb-Ramond potential and RR potentials as

$$ds^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu + G_{yy}\left(dy + A_\mu dx^\mu\right)^2, \tag{3.33a}$$

$$B_{(2)} = \hat{B}_{(2)} + B_{\mu y}dx^\mu \wedge \left(dy + A_\nu dx^\nu\right), \tag{3.33b}$$

$$C_{(p)} = \hat{C}_{(p)} + C_{(p-1)} \wedge \left(dy + A_\mu dx^\mu\right), \tag{3.33c}$$

where the fields $\hat{B}_{(2)}$, $C_{(p-1)}$ and $\hat{F}_{(p+1)}$ only take components in the uncompact directions.

As mentioned in the introduction of this chapter, one can describe the propagation of a superstring in the background of the massless NS-NS fields by inserting these fields into the world-sheet action as nonlinear couplings for the world-sheet fields. We shall neglect the fermionic world-sheet field here noting that they can be reinstated by using supersymmetry and then we can write this action as $S = S_G + S_B + S_\Phi$ where

$$S_G = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{3.34a}$$

$$S_B = \frac{i}{4\pi\alpha'} \epsilon^{ab} \int d^2\sigma \sqrt{-g} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{3.34b}$$

$$S_\Phi = \frac{1}{4\pi} \int d^2\sigma \sqrt{-g} \Phi R. \tag{3.34c}$$

The first term obviously comes from equation (3.1), while the second term expresses the notion that the string carries a charge associated with the Kalb-Ramond potential. The last term is a generalisation of the term which generates powers of the string coupling constant when considering the S-matrix expression of Section 2.5, it is this term which links the value of the string coupling constant to the asymptotic value of the dilaton field. Assuming that the supergravity fields are independent of y we may write an action equivalent to this by the use of a Lagrange multiplier which we label X'^y, this is

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} \Bigg(g^{ab} \left[G_{yy} V_a V_b + 2 G_{y\mu} V_a \partial_b X^\mu + G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right] + i\epsilon^{ab} \left[2 B_{y\mu} V_a \partial_b X^\mu + B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + 2 X'^y \partial_a V_b \right] + \alpha' R \Phi \Bigg). \tag{3.35}$$

Solving the equation of motion for the Lagrange multiplier implies that $V_a = \partial_a X^y$ and we would recover the action (3.34). However, if we instead solve the equation of motion for the vector V_a then substitute this solution into the action then we also find an action of the form of (3.34) but with the metric $\widetilde{G}_{\mu\nu}$ and $\widetilde{B}_{\mu\nu}$ given by

$$d\widetilde{s}^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu + G_{yy}^{-1} \left(dy - B_{\mu y} dx^\mu \right)^2, \tag{3.36a}$$

$$\widetilde{B}_{(2)} = \hat{B}_{(2)} - A_\mu dx^\mu \wedge dy. \tag{3.36b}$$

These are the T-dual fields. Since the dilaton appears in (3.34) at a higher order in the α'-expansion, a one-loop world-sheet calculation is necessary to determine its relationship with its T-dual, however we can motivate the expression for this relationship by considering the supergravity action (3.5a). Because of (3.10), this action is quadratic in the string coupling constantg_s and after a dimensional reduction to nine-dimensional spacetime the reduced string frame action now has a coupling $g_s = e^\Phi (2\pi R)^{1/2}$. This action should describe the same physics as its T-dual which is compactified on a dimension of radius $R' = \alpha'/R$ and so g_s is equal to the dual coupling which has a similar relation, $\widetilde{g}_s = e^{\widetilde{\Phi}} (2\pi R')^{-1/2}$. Hence the T-dual dilaton is given by

$$e^{\widetilde{\Phi}} = e^\Phi \frac{\alpha'^{1/2}}{R}. \tag{3.37}$$

The T-dual fields of the RR gauge fields are more difficult to identify using the RNS formulation of the superstring so we simply quote the result which may be found in [74],

$$\widetilde{C}_{(p)} = \hat{C}_{(p)} + C_{(p-1)} \wedge (dy - B_{\mu y} dx^{\mu}) \,. \tag{3.38}$$

In addition to T-duality, there also exist various other dualities that relate different string theories. One we will have use for in the next chapter is S-duality which relates one string theory at weak coupling to another at strong coupling and so we will make some short comments regarding this. This duality makes itself manifest at the level of supergravity via an $SL(2,\mathbb{R})$ symmetry; if we take

$$\tau = C_{(0)} + ie^{-\Phi}, \tag{3.39a}$$

$$M = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} |\tau|^2 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & 1 \end{pmatrix}, \tag{3.39b}$$

$$F^i_{(3)} = \begin{pmatrix} H_{(3)} \\ F_{(3)} \end{pmatrix}, \tag{3.39c}$$

then we may rewrite the action of type IIB supergravity in the Einstein frame as

$$\frac{1}{2\kappa_{10}^2}\int d^10x\sqrt{-G}\left(R - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\operatorname{Im}\tau)^2} - \frac{1}{2}F_{(1)}^2 - \frac{M_{ij}}{2}F^i_{(3)}\cdot F^j_{(3)} - \frac{1}{2}\widetilde{F}_{(5)}^2\right) - \frac{\epsilon^{ij}}{4\kappa_{10}^2}\int C_{(4)}\wedge F^i_{(3)}\wedge F^j_{(3)}. \tag{3.40}$$

This is explicitly invariant under the following symmetry transformation

$$\begin{gathered} \tau' = \frac{a\tau + b}{c\tau + d}, \\ F'^i_{(3)} = \Lambda^i_j F^j_{(3)}, \qquad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \\ \widetilde{F}'_{(5)} = \widetilde{F}_{(5)}, \qquad G'_{\mu\nu} = G_{\mu\nu}. \end{gathered} \tag{3.41}$$

It should be noted here that the metric invariant under the symmetry above is the Einstein frame metric. The elements of $SL(2,\mathbb{R})$ are specified by four real numbers a, b, c, d such that $ad - bc = 1$. An important feature of this symmetry is that it mixes the two-form potentials and since the Kalb-Ramond two-form couples to the fundamental string (or ‘F-string’) while the RR two-form couples to the D1-brane (or ‘D-string’) it would imply that this symmetry may also exchange D- and F- strings. Indeed this is the case and this symmetry may be extended to the full string theory where it describes a duality between the weakly coupled string theory where F-strings are light and a strongly coupled theory where the D-strings are light.

Chapter 4

Supergravity Solutions from String Scattering Amplitudes

In the previous chapter we chiefly considered supergravity, the low energy limits of string theories, on spacetime backgrounds with ten uncompactified dimensions and briefly discussed some effects which result from compactifying some of these directions. In this chapter we shall review the two-charge supergravity solutions of the D1-P and D5-P brane systems on the spacetime $\mathbb{R}^{1,4} \times S^1 \times T^{4*}$ and expand them perturbatively in the charges. Next, the boundary states for these D-brane configurations are determined and these are used to compute the one-point functions of the massless closed string states which characterises the coupling of the D-brane to the supergravity fields. This allows us to compare with the perturbative expansion confirming that these stringy interactions reproduce the supergravity interactions, at weak coupling, of the D-brane configuration.

We shall examine the D1-P and D5-P systems together since their analysis is very much alike. In fact, all of the relevant physics can be seen in the D5-P case and then the appropriate expressions for the D1-P system can be written down straight away. Hence, when convenient we shall concentrate our efforts on the D5-P case and simply note the result relevant to D1-P.

The D-branes in these set-ups may be represented as in table (4.1) which highlights which directions in spacetime are associated with Neumann (N) and Dirichlet (D) boundary conditions. In the case of the D1-brane it is spatially extended only in the direction of the distinguished S^1, while the D5-brane is also wrapped around the T^4. It should be noted that both types of brane

*we make a distinction between one S^1 and the others that make up the toroidal space T^4 since the charges of the system will be associated with this direction

	R^4					T^4				S^1
	0	1	2	3	4	5	6	7	8	9
D1	N	D	D	D	D	D	D	D	D	N
D5	N	D	D	D	D	N	N	N	N	N

Table 4.1: Description of the spacetime configuration of D1- and D5-branes. Neumann (N) spacetime directions are those in which the brane is extended, whereas in the Dirichlet (D) directions the branes are point-like.

appear point-like in the uncompactified directions.

The 'P' associated with the D1-P and D5-P configurations refers to the momentum charge carried by both of these branes in the form of Kaluza-Klein modes propagating around the S^1 direction. Rather than being fixed at a constant displacement in the Dirichlet directions, the branes both have a displacement that depends on time and the position along the S^1 — it has a travelling wave profile which travels in on a lightlike trajectory. The branes may also be wrapped around the S^1 multiple times and this winding number represents the other charge possessed by these configurations. These concepts are illustrated in Figure 4.1

4.1 Two-charge system in the D1-P and D5-P duality frames

Consider type IIB string theory on $\mathbb{R}^{4,1} \times S^1 \times T^4$ for which we will use the light-cone coordinates $u = (t + y)$, $v = (t - y)$ that have been expressed in terms of the time coordinate t and a coordinate parameterising the S^1 direction, y. The indices $(I, J, \ldots)$ refer collectively to the other eight directions which may be further decomposed into the uncompact $\mathbb{R}^4$ directions labelled by $(i, j, \ldots)$ and those along the torus T^4 labelled by $(a, b, \ldots)$.

The family of classical supergravity solutions in which we are interested describe two-charge D-brane bound states [27–31] and are connected through T and S dualities to the solution describing a multi-wound fundamental string with a purely right (or left) moving wave [75, 76], smeared along the T^4 directions and along y [27, 28]. In the D1-P duality frame, we take the D1-brane to be wrapped n_w times around y; letting the length of the y direction be $2\pi R$, the brane then has overall extent $L_T = 2\pi n_w R$ and we use $\hat{v}$ for the corresponding world-volume coordinate on the D-brane, having periodicity L_T. The non-trivial fields are the metric, the dilaton and the R-R two-form gauge potential:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} dv \left(- du + K dv + 2A_I dx^I \right) + H^{\frac{1}{2}} dx^I dx^I \ , \\ e^{2\Phi} &= g_s^2 H \ , \quad C^{(2)}_{uv} \ = \ -\tfrac{1}{2}(H^{-1} - 1) \ , \quad C^{(2)}_{vI} \ = \ -H^{-1} A_I \ , \end{aligned} \tag{4.1}$$

where the harmonic functions take the form

$$H = 1 + \frac{Q_1}{L_T} \int_0^{L_T} \frac{d\hat{v}}{|x_i - f_i(\hat{v})|^2} \ , \quad A_I = -\frac{Q_1}{L_T} \int_0^{L_T} \frac{d\hat{v} \dot{f}_I(\hat{v})}{|x_i - f_i(\hat{v})|^2} \ , \quad K = \frac{Q_1}{L_T} \int_0^{L_T} \frac{d\hat{v} |\dot{f}_I(\hat{v})|^2}{|x_i - f_i(\hat{v})|^2} \ , \tag{4.2}$$

where $f_i(\hat{v} + L_T) = f_i(\hat{v})$ and where $\dot{f}$ denotes the derivative of f with respect to $\hat{v}$. The functions f_I describe classically the null travelling wave on the D-string. These harmonic functions are obtained via the 'smearing' process over the T^4 which results in the D1-brane being delocalised in these directions, and the effects of this are apparent in the power of the denominator in these harmonic functions; whereas this power would usually be related to the number $\widetilde{p} = d - 4 - p$ specifying the D$\widetilde{p}$-brane dual to the Dp-brane where $d = 10$, we have effectively removed the T^4 dimensions from this counting, hence reducing this figure by 4.

Q_1 is proportional to g_s and to the D-brane winding number n_w and is given by

$$Q_1 \ = \ \frac{(2\pi)^4 n_w g_s (\alpha')^3}{V_4} \ . \tag{4.3}$$

To obtain the supergravity solution in the D5-P duality frame we must perform T-duality transformations in the T^4 directions, then by using the symmetry of the IIB equations of motion to reverse the sign of B and $C^{(4)}$, we obtain the fields

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} dv \left(- du + \left(K - H^{-1}|A_a|^2\right) dv + 2A_i dx_i \right) + H^{\frac{1}{2}} dx^i dx^i + H^{-\frac{1}{2}} dx^a dx^a \ , \\ e^{2\Phi} &= (g_s')^2 H^{-1} \ , \qquad B_{va} \ = \ -H^{-1} A_a \ , \\ C^{(4)}_{vbcd} &= -H^{-1} A_a \epsilon_{abcd} \ , \quad C^{(6)}_{vi5678} \ = \ -H^{-1} A_i \ , \quad C^{(6)}_{uv5678} \ = \ -\tfrac{1}{2}\left(H^{-1} - 1\right) , \end{aligned} \tag{4.4}$$

where g_s' is the string coupling in the new duality frame and ϵ_{abcd} is the alternating symbol with $\epsilon_{5678} = 1$. The effect of rewriting the functions in (4.2) in terms of D5-P frame quantities is to substitute the D1 with the D5 charge, $Q_1 \to Q_5 = g_s' n_w \alpha'$. From now on, we drop the prime and refer to the D5-P frame string coupling as g_s.

To summarise, one may obtain the solutions for these different duality frames using just the solution for the fundamental string wrapped on the S^1 and carrying a null travelling wave by first carrying out an S-duality transformation to obtain the D-string wrapped on S^1, then by T-dualising in the T^4 directions. This may be represented by the following diagram

$$\begin{pmatrix} P_y \\ F1_y \end{pmatrix} \xrightarrow{S} \begin{pmatrix} P_y \\ D1_y \end{pmatrix} \xrightarrow{T_{5678}} \begin{pmatrix} P_y \\ D5_{5678y} \end{pmatrix}$$

where we denote the D-brane wrapped around spatial directions $i_1, \ldots, i_p$ by D$p_{i_1 \ldots i_p}$.

From the large distance behaviour of the g_{vv} component of the metrics above, one can read off how the momentum charge is related to the D-brane profile function f. For instance, in the D1-P frame we have

$$\frac{n_w}{L_T} \int_0^{L_T} |\dot{f}|^2 d\hat{v} = \frac{g_s n_p \alpha'}{R^2} \,, \tag{4.5}$$

where n_p is the Kaluza-Klein integer specifying the momentum along the compact y direction.

$$g = \eta + 2\kappa \hat{h} \,, \qquad B = \sqrt{2}\kappa \hat{b} \,, \qquad C = \sqrt{2}\kappa \hat{C} \tag{4.6}$$

where as usual, $\kappa = 2^3 \pi^{7/2} g_s (\alpha')^2$. We then expand the relevant components of (4.4) for small Q_5, keeping only linear order terms, which yields the field components that we shall reproduce from the disk amplitudes:

$$\hat{h}_{vi} = \frac{Q_5}{2\kappa L_T} \int_0^{L_T} \frac{-\dot{f}_i \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \,, \; \hat{h}_{vv} = \frac{Q_5}{2\kappa L_T} \int_0^{L_T} \frac{|\dot{f}|^2 \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \,, \; \hat{b}_{va} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} \frac{\dot{f}_a \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \,,$$

$$\hat{C}^{(4)}_{vbcd} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} d\hat{v} \, \frac{\dot{f}_a \epsilon_{abcd}}{|x_i - f_i(\hat{v})|^2} \,, \qquad \hat{C}^{(6)}_{vi5678} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} d\hat{v} \, \frac{\dot{f}_i}{|x_i - f_i(\hat{v})|^2} \,. \tag{4.7}$$

4.2 Boundary states

The key ingredients of our string computation are the boundary conditions which must be imposed upon the world-sheet fields of a string ending on a D-brane with a travelling wave, which we now review. We consider a Euclidean world-sheet with complex coordinate $z = \exp(\tau - i\sigma)$ such that $\tau \in \mathbb{R}$ and $\sigma \in [0, \pi]$. We first review the boundary conditions applicable for a D-brane wrapped only once around y and later account for higher wrapping numbers.

We begin with the following world-sheet action for the superstring coupled to a background gauge field A^μ on a D9-brane following [77, 78]:

$$S \;=\; S_0 + S_1 \,, \tag{4.8}$$

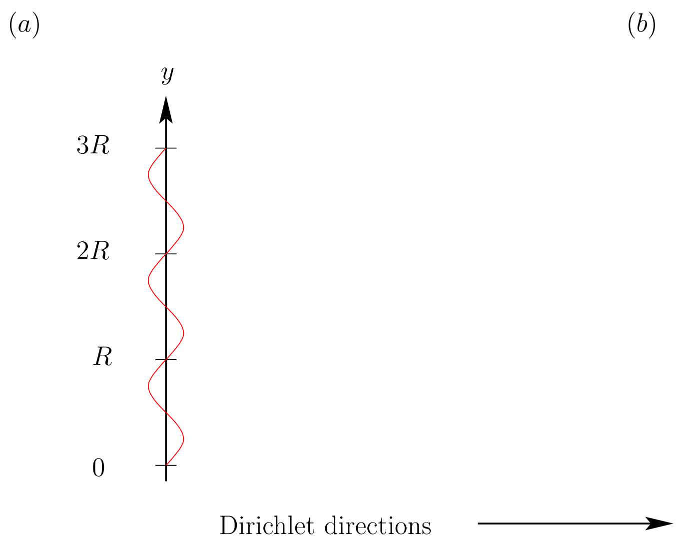

Figure 4.1: Schematic depicting the wave profile of the D-branes along the S^1 space and projected onto the transverse directions illustrating different winding states. Figure (a) shows a D-brane wrapped once around the compact direction ($w = 1$), whereas figure (b) shows a D-brane wrapped three times ($w = 3$).

where S_0 and S_1 are the world-sheet bulk and boundary actions respectively,

$$S_0 = \frac{1}{2\pi\alpha'} \int_M d^2z \, \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right) , \tag{4.9a}$$

$$S_1 = i \int_{\delta M} dz \, \left(A_\mu(X) \left(\partial X^\mu + \bar{\partial} X^\mu \right) - \frac{1}{2} (\psi^\mu + \widetilde{\psi}^\mu) F_{\mu\nu} (\psi^\nu - \widetilde{\psi}^\nu) \right) \tag{4.9b}$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the abelian field strength. In the absence of a boundary the action S_0 would be invariant under the supersymmetry transformations

$$\delta X^\mu = \varepsilon \psi^\mu + \tilde{\varepsilon} \widetilde{\psi}^\mu \; , \quad \delta \psi^\mu = -\varepsilon \partial X^\mu \; , \quad \delta \widetilde{\psi}^\mu = -\tilde{\varepsilon} \bar{\partial} X^\mu \tag{4.10}$$

however the presence of the boundary breaks the $\mathcal{N} = 2$ world-sheet supersymmetry to $\mathcal{N} = 1$ supersymmetry. When we include S_1, the total action $S_0 + S_1$ preserves $\mathcal{N} = 1$ supersymmetry only up to the boundary conditions [77], which we impose at $z = \bar{z}$. Defining

$$E_{\mu\nu} \; = \; \eta_{\mu\nu} + 2\pi\alpha F_{\mu\nu} \; , \tag{4.11}$$

varying the above action yields the boundary conditions [77]

$$\left[E_{\mu\nu} \widetilde{\psi}^\nu \; = \; \eta E_{\nu\mu} \psi^\nu \right]_{z=\bar{z}} \; , \tag{4.12a}$$

$$\left[E_{\mu\nu} \bar{\partial} X^\nu - E_{\nu\mu} \partial X^\nu - \eta E_{\nu\rho,\mu} \widetilde{\psi}^\nu \psi^\rho - E_{\mu\nu,\rho} \psi^\nu \psi^\rho + E_{\nu\mu,\rho} \widetilde{\psi}^\nu \widetilde{\psi}^\rho \right]_{z=\bar{z}} \; = \; 0 \, , \tag{4.12b}$$

where η takes the value 1 or -1 corresponding to the NS and R sectors respectively. By applying the supersymmetry transformations (4.10) to the action (4.8) and employing these boundary conditions, one finds that (4.8) is invariant under the $\mathcal{N} = 1$ supersymmetry generated by these transformations with the constraint $\varepsilon = \eta\tilde{\varepsilon}$.

For the systems under consideration the gauge field takes a plane-wave profile and so A^μ will be a function only of the bosonic field $V = (X^0 - X^9)$, where X^0 is the string coordinate along time and X^9 indicates the compact y direction. A physical gauge field can be written as $A^I(V)$, where we set to zero the light-cone components. Then the non-vanishing components of $E_{\mu\nu}$ take the form

$$E_{uv} = E_{vu} = -\frac{1}{2}\,, \qquad E_{IJ} = \delta_{IJ}\,, \qquad E_{Iv} = -E_{vI} = \dot{f}_I(V)\,, \tag{4.13}$$

where we have defined $f_I = -2\pi\alpha' A_I$.

We can rewrite the fields appearing (4.12a) and (4.12b) in modes by using the expansions

$$X^\mu(z,\bar{z}) \;=\; x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln z - i\sqrt{\frac{\alpha'}{2}}\widetilde{\alpha}_0^\mu \ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\widetilde{\alpha}_m^\mu}{\bar{z}^m}\right), \tag{4.14}$$

$$\psi^\mu(z) \;=\; \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\nu}\frac{\psi_r^\mu}{z^{r+\frac{1}{2}}}\,, \qquad \widetilde{\psi}^\mu(\bar{z}) \;=\; \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\nu}\frac{\widetilde{\psi}_r^\mu}{\bar{z}^{r+\frac{1}{2}}}\,, \tag{4.15}$$

where $\nu = 0$ and $\frac{1}{2}$ for R and NS respectively. Note however that in our case the presence of a non-constant field strength $F_{\mu\nu}$ makes the boundary conditions nonlinear in the oscillators. We will see that, for the amplitudes in which we are interested, only the linear terms contribute.

As usual, we can change from the open string picture to the closed string picture, and derive the boundary conditions describing a closed string emitted or absorbed by the D-brane. This has the effect of

$$\alpha_n^\mu \to -\alpha_{-n}^\mu\,, \qquad \psi_r^\mu \to i\psi_{-r}^\mu \qquad \forall\,\mu, n, r\,. \tag{4.16}$$

We can then obtain the boundary conditions for a lower dimensional D-brane by performing a series of T-dualities; after these transformations, the components of f along the dualised coordinates describe the profile of the brane. We perform four or eight T-dualities in order to obtain the boundary conditions appropriate for a D5 or a D1-brane, for instance in order to move from the D9 frame to the D5-P frame we T-dualise along each x^i which sends

$$\widetilde{\alpha}_n^i \to -\widetilde{\alpha}_n^i\,, \qquad \widetilde{\psi}_r^i \to -\widetilde{\psi}_r^i\,. \tag{4.17}$$

By following the procedure outlined above, we can summarise the boundary conditions for the closed string oscillators as follows

$$\widetilde{\psi}_r^\mu \;=\; i\eta\, R^\mu{}_\nu \psi_{-r}^\nu \;+\ldots\,, \qquad \widetilde{\alpha}_n^\mu \;=\; -\,R^\mu{}_\nu \alpha_{-n}^\nu \;+\ldots\,, \tag{4.18}$$

where '...' indicates that we ignore terms which are higher than linear order in the oscillator modes. We shall justify this below (4.35). The reflection matrix R is obtained from (4.12a) and (4.12b) by performing the transformations (4.16) and (4.17) and replacing V by its zero-mode v:

$$R^{\mu}{}_{\nu}(v) = T^{\mu}{}_{\rho}(E^{-1})^{\rho\sigma}E_{\nu\sigma} \ , \tag{4.19}$$

where the matrix T performs the T-duality (4.17), i.e. it is diagonal with values -1 in the x^i directions and 1 otherwise. R has the lowered-index form

$$R_{\mu\nu}(v) \ = \ \eta_{\mu\rho}R^{\rho}{}_{\nu}(v) \ = \ \begin{pmatrix} -2|\dot{f}(v)|^2 & -\frac{1}{2} & 2\dot{f}^i(v) & 2\dot{f}^a(v) \\ -\frac{1}{2} & 0 & 0 & 0 \\ 2\dot{f}^i(v) & 0 & -\mathbb{1} & 0 \\ -2\dot{f}^a(v) & 0 & 0 & \mathbb{1} \end{pmatrix} . \tag{4.20}$$

We refer the reader to [78–80] for a detailed discussion of the boundary state describing a D-brane with a travelling wave. For our purposes it is sufficient to know the linearised boundary conditions for the non-zero modes (4.18) that the boundary state must satisfy, and to construct explicitly only the zero-mode structure of the boundary state. Addressing firstly the bosonic sector, the boundary conditions on the zero modes are

$$p_v + \dot{f}^i(v)\, p_i = 0, \qquad p_u = 0, \qquad p_a = 0, \qquad x^i = f^i(v). \tag{4.21}$$

Here the first three equations follow directly from (4.18), the fourth equation must be included to account for the T-duality transformations but it also has the simple geometric interpretation as the requirement that the boundary of the closed string world-sheet be attached to the D-brane; since this D-brane has the form of a null wave in the transverse spacetime directions this requirement also states that the boundary of the world-sheet take on a null wave form. The first equation in (4.21) may be represented as $i\frac{\partial}{\partial v} = \dot{f}^i(v)p_i$ and similarly the last constraint may be represented as $i\frac{\partial}{\partial p_i} = f^i(v)$. Then the boundary state zero-mode structure in the t, y and x^i direction is

$$\int dv\, du \int \frac{d^4p_i}{(2\pi)^4} e^{-ip_i f^i(v)} \, |p_i\rangle \, |u\rangle \, |v\rangle \ . \tag{4.22}$$

So far we have essentially discussed a D-brane with a travelling wave in a noncompact space; we next generalise this description to the case of compact y and higher wrapping number. One may view a D-brane wrapped n_w times along the y-direction as a collection of n_w different D-brane strands with a non-trivial holonomy gluing these strands together. This approach was developed in [81, 82] for the case of branes with a constant magnetic field.

In the presence of a null travelling wave with arbitrary profile $f(V)$, the individual boundary states of each strand will differ in their oscillator part and not just in their zero-mode part described above. However, we are interested in the emission of massless closed string states, which have zero momentum and winding along all compact directions. In this sector the full boundary state is simply the sum of the boundary states for each constituent, along with the condition that the value of the function f at the end of one strand must equal the value of f at the beginning of the following strand. We label the strands of the wrapped D-brane with the integer s; then restricting to the sector of closed strings with trivial winding (m) and Kaluza-Klein momentum (k), the boundary state takes the following form:

$$|D5;P\rangle^{k,m=0} = -\frac{\kappa\,\tau_5}{2}\sum_{s=1}^{n_w}\int du \int_0^{2\pi R} dv \int \frac{d^4p_i}{(2\pi)^4} e^{-ip_i f^i_{(s)}(v)}\,|p_i\rangle\,|u\rangle\,|v\rangle\,\left|D5;f_{(s)}\right\rangle^{k,m=0}_{X,\psi}\,, \tag{4.23}$$

where $\tau_5 = [(2\pi\sqrt{\alpha'})^5\sqrt{\alpha'}g_s]^{-1}$ is the physical tension of a D5-brane. We have written explicitly only the bosonic zero-modes along t, y and the x^i directions and we denote by $\left|D5;f_{(s)}\right\rangle^{k,m=0}_{X,\psi}$ the remaining part of the boundary state. The range of integration over $v = t - y$ follows from the periodicity condition of the space-time coordinate y.

We next address the fermion zero modes in the R-R sector. Letting $A, B, \ldots$ be 32-dimensional indices for spinors in ten dimensions[†], and letting $|A\rangle|\widetilde{B}\rangle$ denote the ground state for the Ramond fields $\psi^\mu(z)$ and $\widetilde{\psi}^\mu(\bar{z})$ respectively, the R-R zero mode boundary state in the $(-\frac{1}{2},-\frac{3}{2})$ picture (before the GSO projection) takes the form

$$|D5;P\rangle^{(\eta)}_{\psi,0} = \mathcal{M}^{(\eta)}_{AB}\,|A\rangle_{-\frac{1}{2}}\left|\widetilde{B}\right\rangle_{-\frac{3}{2}} \tag{4.24}$$

where $\mathcal{M}$ satisfies the following equation [83],

$$\Gamma_{11}\,\mathcal{M}\,\Gamma^\mu - i\eta\,R^\mu_{\ \nu}\,(\Gamma^\nu)^T\,\mathcal{M} \;=\; 0\,. \tag{4.25}$$

A solution to this equation for the case of our reflection matrix R (4.20) is given by[‡]

$$M = i\,C\left(\frac{1}{2}\Gamma^{vu} + \dot{f}^I(v)\Gamma^{Iv}\right)\Gamma^{5678}\left(\frac{1\!\!1 - i\eta\Gamma_{11}}{1-i\eta}\right). \tag{4.26}$$

where C is the charge conjugation matrix. The GSO projection has the effect of

$$|D5;P\rangle_{\psi,0} = \frac{1}{2}\left(|D5;P\rangle^{(1)}_{\psi,0} + |D5;P\rangle^{(-1)}_{\psi,0}\right) \tag{4.27}$$

and so the zero mode part of the D5-P R-R boundary state for the strand with profile $f_{(s)}$ is

$$|D5;f_{(s)}\rangle_{\psi,0} = i\left[C\left(\frac{1}{2}\Gamma^{vu} + \dot{f}^I_{(s)}(v)\Gamma^{Iv}\right)\Gamma^{5678}\frac{1+\Gamma_{11}}{2}\right]|A\rangle_{-\frac{1}{2}}\left|\widetilde{B}\right\rangle_{-\frac{3}{2}} \tag{4.28}$$

which we can insert into the relevant part of the boundary state (4.23).

[†] For the spinors and the charge conjugation matrix, we use the conventions of [83].

[‡] The overall phase of M is a matter of convention; see also [84].

4.3 Disk amplitudes for the classical fields

We now calculate the fields sourced by the D5-P bound state by computing the disk one-point functions for emission of a massless state, starting with the NS-NS fields. Since the states are massless they have non-zero momentum only in the four noncompact directions of the $\mathbb{R}^4$, i.e. they have spacelike momentum (see also [83]). The NS-NS one-point function thus takes the form

$$\mathcal{A}^{(\eta)}_{\rm NS}(k) \;\equiv\; \langle p_i = k_i| \, \langle p_v = 0| \, \langle p_u = 0| \, \langle n_a = 0| \, \mathcal{G}_{\mu\nu}\psi^{\mu}_{\frac{1}{2}}\, \widetilde{\psi}^{\nu}_{\frac{1}{2}} \, |D5;P\rangle^{k,m=0} \tag{4.29}$$

where for an S^1 direction with radius R we normalize the momentum eigenstates as $\langle n|m\rangle = 2\pi R\,\delta_{nm}$ and the position eigenstates as $\langle x|y\rangle = \delta(x-y)$. The decomposition of $\mathcal{G}_{\mu\nu}$ into irreducible representations giving the metric, dilaton and RR two-form may be written as

$$\mathcal{G}_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{\sqrt{2}}\hat{b}_{\mu\nu} + \frac{\phi}{2\sqrt{2}}\left(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu\right) \,, \tag{4.30}$$

where k_μ and l_ν are null vectors, so that $k\cdot k = 0$ and $l\cdot l = 0$, as well as being mutually orthogonal which for null vectors takes the unusual form of the condition $k\cdot l = 1$ after appropriate normalisation. We have normalised the terms of this decomposition in such a way that we might be able to match them exactly to the canonically normalised fields seen in equation (4.6). The contribution to the zero mode part of the amplitude from a single strand with profile $f_{(s)}(v)$ is

$$V_4 V_u \frac{\kappa\,\tau_5}{2} \int_0^{2\pi R} dv\, e^{-ik_i f^i_{(s)}(v)} \,, \tag{4.31}$$

where V_u represents the infinite volume of the D-brane in the u direction. Since we have used a delocalised probe ($p_v = 0$), the string amplitude contains an integral over the length of the strand of the D-brane. In the classical limit n_w is very large, the typical wavelength of the profile is much bigger than R, and so f is almost constant over each strand [27, 28]. The contribution to the value of each supergravity field is thus (4.31) divided by the volume of the strand:

$$\mathcal{A}^{(s)}_0(k) \;=\; \frac{\kappa\,\tau_5}{2}\frac{1}{2\pi R}\int_0^{2\pi R} dv\, e^{-ik_i f^i_{(s)}(v)} \,. \tag{4.32}$$

The contribution from the n_w different strands of the brane is therefore

$$\mathcal{A}_0(k) \;=\; \frac{\kappa\,\tau_5}{2}\frac{1}{2\pi R}\sum_{s=1}^{n_w}\int_0^{2\pi R} dv\, e^{-ik_i f^i_{(s)}(v)} \,, \tag{4.33}$$

and we combine the integrals over each strand to give the integral over the full world-volume coordinate $\hat{v}$, giving

$$\mathcal{A}_0(k) \;=\; \frac{\kappa\,\tau_5}{2}\frac{n_w}{L_T}\int_0^{L_T} d\hat{v}\, e^{-ik_i f^i(\hat{v})} \,. \tag{4.34}$$

Adding in the non-zero modes, the coupling of the boundary state to the NS-NS fields is

$$\mathcal{A}^{(\eta)}_{\text{NS}}(k) \;=\; -\,i\eta\frac{\kappa\,\tau_5\,n_w}{2L_T}\int_0^{L_T} d\hat{v}\,e^{-ik_i f^i(\hat{v})}\mathcal{G}_{\mu\nu}R^{\nu\mu}(\hat{v}) \tag{4.35}$$

where $R(\hat{v})$ is the obvious strand-by-strand extension of the reflection matrix (4.20).

We can now observe why we were justified in ignoring terms higher than linear order in the oscillator boundary conditions (4.18). To arrive at the above result we substitute $\widetilde{\psi}^{\nu}_{\frac{1}{2}}$ for an expression involving only creation modes using (4.18), and only the linear term can contract with the remaining annihilation mode to give a non-zero result. A similar argument holds for the R-R amplitude.

The GSO projection has the effect of

$$\mathcal{A}_{\text{NS}}(k) \;=\; \frac{1}{2}\left(\mathcal{A}^{(1)}_{\text{NS}}(k) - \mathcal{A}^{(-1)}_{\text{NS}}(k)\right) \tag{4.36}$$

and we read off the canonically normalized fields of interest via

$$\hat{h}_{vi}(k) \;=\; \frac{1}{2}\frac{\delta\mathcal{A}_{\text{NS}}}{\delta\hat{h}^{vi}}\,, \qquad \hat{h}_{vv}(k) \;=\; \frac{\delta\mathcal{A}_{\text{NS}}}{\delta\hat{h}^{vv}}\,, \qquad \hat{b}_{va}(k) \;=\; \frac{\delta\mathcal{A}_{\text{NS}}}{\delta\hat{b}^{va}}\,. \tag{4.37}$$

The space-time configuration associated with a closed string emission amplitude is obtained by multiplying the derivative of the amplitude with respect to the closed string field by a free propagator and taking the Fourier transform [83]. In general for a field $a_{\mu_1\ldots\mu_n}$ we have

$$a_{\mu_1\ldots\mu_n}(x) = \int\frac{d^4k}{(2\pi)^4}\left(-\frac{i}{k^2}\right)a_{\mu_1\ldots\mu_n}(k)\,e^{ikx}\,, \tag{4.38}$$

with $a_{\mu_1\ldots\mu_n}(k)$ given in terms of derivatives of $\mathcal{A}$ as in (4.37). Using the identity

$$\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik^i(x^i-f^i)}}{k^2} \;=\; \frac{1}{4\pi^2}\frac{1}{|x^i-f^i|^2} \tag{4.39}$$

and the relation

$$Q_5 \;=\; \frac{2\kappa^2\,\tau_5\,n_w}{4\pi^2}\,, \tag{4.40}$$

we obtain

$$\hat{h}_{vi} \;=\; \frac{Q_5}{2\kappa L_T}\int_0^{L_T}\frac{-\dot{f}_i\,d\hat{v}}{|x^i-f^i|^2},\quad \hat{h}_{vv} \;=\; \frac{Q_5}{2\kappa L_T}\int_0^{L_T}\frac{|\dot{f}|^2\,d\hat{v}}{|x^i-f^i|^2},\quad \hat{b}_{va} \;=\; \frac{Q_5}{\sqrt{2}\kappa L_T}\int_0^{L_T}\frac{\dot{f}_a\,d\hat{v}}{|x^i-f^i|^2}$$

in agreement with (4.7).

We next calculate the coupling between the R-R zero mode boundary state and the on-shell R-R potential state [83–85]:

$$\left\langle\hat{C}_{(n)}\right| \;=\; {}_{-\frac{1}{2}}\left\langle\widetilde{B},\tfrac{k}{2}\right|\,{}_{-\frac{3}{2}}\left\langle A,\tfrac{k}{2}\right|\left[C\Gamma^{\mu_1\ldots\mu_n}\frac{1\!\!1-\Gamma_{11}}{2}\right]_{AB}\frac{(-1)^n}{4\sqrt{2}\,n!}\,\hat{C}_{\mu_1\ldots\mu_n} \tag{4.41}$$

where the numerical factor contains an extra factor of $\frac{1}{2}$ to account for the fact that we are not using the full superghost expression. Using the fact (see e.g. [85]) that

$$\left(\langle A| \left\langle \tilde{B}\right|\right) \left(|D\rangle \left|\tilde{E}\right\rangle\right) = -\langle A|D\rangle \, \langle \tilde{B}|\tilde{E}\rangle = -(C^{-1})^{AD}(C^{-1})^{BE} \,, \tag{4.42}$$

we find the coupling of the R-R potential to the (already GSO projected) boundary state for an individual strand (4.28) to be

$$\begin{aligned} \mathcal{A}^{(s)}_{\mathrm{R},\psi} &= \langle \hat{C}_{(n)}|D5; f_{(s)}\rangle_{\psi,0} \\ &= \frac{-i}{4\sqrt{2}\,n!}\,\mathrm{Tr}\left[\Gamma_{\mu_n\cdots\mu_1}\left(\frac{1}{2}\Gamma^{vu} + \dot{f}^I_{(s)}(v)\Gamma^{Iv}\right)\Gamma^{5678}\frac{1+\Gamma_{11}}{2}\right]_{AB}\hat{C}^{\mu_1\cdots\mu_n}\,. \end{aligned} \tag{4.43}$$

This then combines with the bosonic zero mode part of the amplitude $\mathcal{A}_0^{(s)}$ given in (4.32) and we sum over strands to obtain the full R-R amplitude $\mathcal{A}_{\mathrm{R}}$. We then extract the gauge field profile via

$$\hat{C}^{(n)}_{\mu_1\ldots\mu_n}(k) \;=\; \frac{\delta\mathcal{A}_{\mathrm{R}}}{\delta\hat{C}^{(n)\mu_1\ldots\mu_n}} \quad (\mu_1<\mu_2\ldots<\mu_n)\,, \tag{4.44}$$

and as for the NS-NS calculation we insert the propagator and perform the Fourier transform. The fields which are non-trivial only in the presence of a travelling wave are then

$$\hat{C}^{(4)}_{vbcd} \;=\; \frac{Q_5}{\sqrt{2}\kappa L_T}\int_0^{L_T} d\hat{v}\,\frac{\dot{f}_a\epsilon_{abcd}}{|x_i - f_i(\hat{v})|^2}\,, \qquad \hat{C}^{(6)}_{vi5678} \;=\; \frac{Q_5}{\sqrt{2}\kappa L_T}\int_0^{L_T} d\hat{v}\,\frac{\dot{f}_i}{|x_i - f_i(\hat{v})|^2} \tag{4.45}$$

which agrees with (4.7). This completes the link between the microscopic and macroscopic descriptions of a D5-brane with a travelling wave.

Chapter 5

String Theory at High Energy – The Eikonal Approximation

In this chapter we will examine the scattering of strings from D-branes at large energies, placing particular emphasis on the utility of the eikonal operator in reproducing the effects of small-angle scattering which has seen significant interest recently [2, 35]. Historically, physics at the Planck scale has always been an important aspect of string theory for study; at Planckian energies we expect gravitational interactions to become significant to microscopic phenomena and therefore it is here that string theory has the opportunity to offer fresh insights into their fundamental nature. Of particular interest are string-based microscopic descriptions of black-hole like objects such as those employed in Chapter 4, which could provide answers regarding the processes of black-hole formation and their absorption of falling matter. Equally important are questions regarding the interpretation of spacetime itself at the Planck scale — is there a natural cutoff on the distances we are able to probe provided by some physical mechanism? Should spacetime be noncommutative?

Here we wish to address the issue of massive states in the scattering of strings from D-branes, the motivation behind this is threefold. First of all, such scattering amplitudes provide an explicit factoring for higher order amplitudes which must feature a sum over intermediate (and generally massive) states. The next reason is that, as extended objects, strings moving through a generic spacetime background will be subject to forces which can result in mass level transitions for the string; since many backgrounds are known to be generated by D-brane configurations then these scattering amplitudes yield a microscopic description of such interactions between the string and its background [86]. Finally, they will also provide a nontrivial check on the validity of the eikonal operator.

In Section 5.1 we briefly review the kinematics appropriate to a string probe interacting with a D-brane source. Useful identities and notation are introduced before discussing the different regimes that exist at high energy. In Section 5.2 we review the motivation and derivation of the eikonal operator as applied to string-brane scattering in [35]. In Section 5.3 we compute the high energy limit of the exact scattering amplitude for a string state from the leading Regge trajectory interacting with a Dp-brane using effective operator methods. In Section 5.4 this result is compared with the analogous quantity calculated from matrix elements of the eikonal operator and it is concluded that there is perfect agreement and the eikonal operator captures all of the relevant physics. Finally, the general features of the small-angle scattering examined throughout this chapter are then compared and contrasted with another important regime in high energy scattering — fixed-angle scattering.

5.1 Parameterisation of high energy kinematics

In the following computations we will consider some initial state with momentum p_1 such that $\alpha' M_1^2 = -\alpha' p_1^2 = 4n$; after interacting with the Dp-brane we are left with another state with momentum p_2 which in general can have a different mass, $\alpha' M_2^2 = -\alpha' p_2^2 = 4n'$. We can decompose these momenta into vectors which are parallel and orthogonal to the directions in which the D-brane is extended, that is

$$p_i^\mu = p_{i\parallel}^\mu + p_{i\perp}^\mu . \tag{5.1}$$

The mass of a D-brane scales as $1/g_s$ and so to leading order in the perturbative expansion it is infinitely massive. Consequently we may neglect its recoil and momentum is only conserved in directions parallel to the D-brane,

$$p_{1\parallel}^\mu + p_{2\parallel}^\mu = 0. \tag{5.2}$$

For brevity we denote $p_{1\parallel}^\mu = -p_{2\parallel}^\mu = p_\parallel^\mu$. Using the mass-shell condition we may relate this quantity to the orthogonal components of momentum

$$p_{1\perp}^2 = -\, p_\parallel^2 - \frac{4n}{\alpha'}, \tag{5.3a}$$

$$p_{2\perp}^2 = -\, p_\parallel^2 - \frac{4n'}{\alpha'}. \tag{5.3b}$$

The vectors $p_{i\perp}$ are by definition space-like* so we can infer that $p_\parallel$ is necessarily time-like. Considering this we define the following kinematic invariants for use as the parameters in our

*ignoring the specific case of a D-instanton.

computations

$$s \equiv -p_{\|}^2, \tag{5.4a}$$

$$t \equiv -(p_1 + p_2)^2, \tag{5.4b}$$

$$E \equiv \sqrt{s}. \tag{5.4c}$$

After a Lorentz transformation to a frame in which the spatial components of p_1 are nonzero only in the orthogonal directions the quantity E will be equal to the energy of the string. The momentum exchanged between the string and the brane may be written as $q = p_1 + p_2 = 2(k_1 + k_2)$, hence $-q^2 = t$, and it is also useful to define a vector $\tilde{q} = 2(k_1 - k_2)$.

By definition, s and t have the following physical boundaries,

$$\max\{M_1^2,\, M_2^2\} \leq s < \infty, \tag{5.5}$$

$$\left(\sqrt{s - M_1^2} - \sqrt{s - M_2^2}\right)^2 \leq |t| \leq \left(\sqrt{s - M_1^2} + \sqrt{s - M_2^2}\right)^2. \tag{5.6}$$

It is convenient to include the additional assumption that M_1^2 and M_2^2 are fixed and much smaller than s, this renders much of our calculations amenable to analytic solution and the above can be reduced to $0 \leq |t| \leq 4s$. Physically this requirement simply states that we will not be considering extremely massive slow moving strings.

From the definition of $D^\mu{}_\nu$ in equation (2.97) it can be seen that the momenta satisfy $(D{\cdot}k)^\mu = k_{\|}^\mu - k_\perp^\mu$ and so, with the aid of the conservation of momentum, one can deduce the following identities,

$$k_1 \cdot k_2 = \frac{n}{2\alpha'} + \frac{n'}{2\alpha'} - \frac{t}{8}, \tag{5.7}$$

$$k_1 \cdot D \cdot k_1 = -\frac{s}{2} + \frac{n}{\alpha'}, \tag{5.8}$$

$$k_2 \cdot D \cdot k_2 = -\frac{s}{2} + \frac{n'}{\alpha'}, \tag{5.9}$$

$$k_1 \cdot D \cdot k_2 = \frac{s}{2} + \frac{t}{8} - \frac{n}{2\alpha'} - \frac{n'}{2\alpha'}. \tag{5.10}$$

For most of this chapter we shall be interested in the Regge limit, in which we take $\alpha' s \to \infty$ while keeping $\alpha' t$ fixed. These processes occur for large impact parameters and so are relatively long distance. In Section 5.5 we will consider a regime known as fixed angle scattering, for this we again take the limit $\alpha' s \to \infty$ while keeping t/s fixed. This is 'hard scattering' which has the benefit of probing small distances.

5.2 Defining the eikonal operator

The eikonal operator was first introduced to string theory in a series of papers [22–24] in which it was shown that the $2 \to 2$ graviton scattering amplitude failed to be unitary order-by-order in the perturbative expansion. This was resolved by a resummation of the leading divergences which are reduced to a harmless phase of the S-matrix

In [35] this was also shown to be the case for $1 \to 1$ graviton-brane scattering. The full amplitude for this is well known [87] and from this one can obtain the Regge limit expression for which we suppress the polarisation dependence,

$$A_1(s, \mathbf{q}) = \frac{\kappa T_p}{2} \Gamma\left(\alpha' \frac{q^2}{4}\right) e^{i\pi\alpha' \frac{q^2}{4}} \left(\alpha' s\right)^{-\alpha' \frac{q^2}{4}+1}, \tag{5.11}$$

This demonstrates that the tree-level amplitude can grow without bound with $\alpha' s$. If one simplifies the analysis and considers only the first term in an α'-expansion to obtain the field theory expressions then equation (5.11) becomes

$$A_1(E, \mathbf{q}) \to 2\kappa T_p \frac{E^2}{q^2}. \tag{5.12}$$

Schematically we can write down the S-matrix for these processes as an expression of the form

$$S(E, \mathbf{b}) = \mathbb{1} + i\frac{\mathcal{A}(E, \mathbf{b})}{2E} \tag{5.13}$$

where $\mathcal{A}$ is the impact parameter expression for the amplitude given by the perturbative expansion over world-sheets with the number of boundaries h,

$$A(s, \mathbf{q}) = \sum_{h=1}^{\infty} A_h(s, \mathbf{q}) \tag{5.14}$$

which can be obtained by performing a Fourier transformation from the space of transverse momenta q to that of impact parameter b,

$$\mathcal{A}(s, \mathbf{b}) = \int \frac{d^{8-p}q}{(2\pi)^{8-p}} A(s, \mathbf{q}) e^{i\mathbf{b}\cdot\mathbf{q}}. \tag{5.15}$$

If one then performs the same operation for the amplitude with two boundaries (the annulus diagram), then it is found that the leading divergence in energy for the field theory limit of this function can be written as the convolution in momentum space of two tree-level amplitudes,

$$A_2(E, \mathbf{q}) \to \frac{i}{4E} \int \frac{d^{8-p}k}{(2\pi)^{8-p}} \left(2E^2 \kappa T_p\right)^2 \frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}. \tag{5.16}$$

This relationship between the tree-level and one-loop amplitudes can be further simplified by expressing it in impact parameter space where this convolution becomes a product. Expressing the normalisation of the amplitude in terms of the 'size' of the Dp-brane solution R_p, which will be formally introduced in (5.27), the first two diagrams of (5.14) become

$$i\frac{\mathcal{A}_1}{2E} = i\left(\frac{R_p^{7-p}\sqrt{\pi}E}{2b^{6-p}}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\right), \tag{5.17}$$

$$i\frac{\mathcal{A}_2}{2E} = -\frac{1}{2}\left(\frac{R_p^{7-p}\sqrt{\pi}E}{2b^{6-p}}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\right) = \frac{1}{2}\left(i\frac{\mathcal{A}_1}{2E}\right)^2. \tag{5.18}$$

This is consistent with the suggestion that the perturbative expansion can be exponentiated to contribute just a phase to the S-matrix:

$$S(E,\mathbf{b}) = e^{i\frac{\mathcal{A}_1(E,\mathbf{b})}{2E}}. \tag{5.19}$$

The form of this S-matrix obviously preserves unitarity, despite the fact that none of the individual terms do. Of course, though we have stated all of the terms divergent in the energy for the tree-level amplitude in (5.17), the higher order terms in (5.14) will generally have subleading divergences which add corrections to the eikonal phase $\delta = i\mathcal{A}_1/2E$; these corrections are of higher order in R_P/b and represent classical corrections to the eikonal phase arising from the gravitational background of the Dp-brane.

All of this extends to the full string theory amplitudes, but the exponentiation requires a more subtle treatment since one must determine all of the string corrections to the field theory result. The leading divergences to the full string amplitudes come from the various critical points in the region of integration over the moduli and vertex operator insertions for the world-sheet. We shall see in Section 5.3 that for the disc amplitude the only such critical point arises when the two vertices on the disk come close together.

The annulus amplitude, like its field theory counterpart, can also be written as a convolution where now the two disc amplitudes are convoluted together with the reggeised graviton vertex V_2. This introduces poles to the amplitude in addition to those of the gravitons exchanged with the D-branes in the field theory case, these represent the exchange of all states in the leading Regge trajectory between the string probe and D-brane target. Hence we express this as

$$\frac{A_2(E,\mathbf{q})}{2E} = \frac{i}{2}\int\frac{d^{8-p}k}{(2\pi)^{8-p}}\frac{A_1(E,t_1)}{2E}\frac{A_1(E,t_2)}{2E}V_2(t_1,t_2,t), \tag{5.20}$$

where $t = -\mathbf{q}^2$, $t_1 = -\mathbf{k}^2$, $t_2 = -(\mathbf{q}-\mathbf{k})^2$ and

$$V_2(t_1,t_2,t) = \frac{\Gamma\left(1+\frac{\alpha'}{2}(t_1+t_2-t)\right)}{\Gamma^2\left(1+\frac{\alpha'}{4}(t_1+t_2-t)\right)}. \tag{5.21}$$

This doesn't appear to naturally factorise into a square of the tree-level amplitudes as occurred with the field theory limit, but we can rewrite the vertex V_2 as the vacuum expectation value of two string vertices carrying momenta $\mathbf{k}_1$, $\mathbf{k}_2$,

$$V_2(t_1, t_2, t) = \langle 0| \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{(2\pi)^2} : e^{i\mathbf{k}_1 \cdot \hat{\mathbf{X}}(\sigma_1)} :: e^{i\mathbf{k}_2 \cdot \hat{\mathbf{X}}(\sigma_2)} : |0\rangle \, , \tag{5.22}$$

where $\hat{\mathbf{X}}(\sigma)$ is the string coordinate with only a dependence on the periodic world-sheet coordinate σ and defined without zero-modes:

$$\hat{\mathbf{X}}(\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n}{n} e^{in\sigma} + \frac{\widetilde{\alpha}_n}{n} e^{-in\sigma} \right) . \tag{5.23}$$

As a result, when taking the Fourier transform of (5.20) it will now factorise into a product of two tree-level amplitudes, but they are promoted to operators,

$$\frac{i\mathcal{A}_2(E, \mathbf{b})}{2E} = -\frac{1}{2} \langle 0| \left(\int_0^{2\pi} \frac{d\sigma}{2\pi} : \frac{\mathcal{A}_1\left(E, \mathbf{b} + \hat{\mathbf{X}}(\sigma)\right)}{2E} \right)^2 |0\rangle \, . \tag{5.24}$$

This is easily generalised to any term of order h and the series (5.14) exponentiates for the full string S-matrix. The S-matrix is now defined in terms of the eikonal operator

$$S(E, \mathbf{b}) = e^{i2\hat{\delta}(E, \mathbf{b})}, \qquad 2\hat{\delta}(E, \mathbf{b}) = \frac{1}{2E} \int_0^{2\pi} \frac{d\sigma}{2\pi} : \mathcal{A}_1\left(E, \mathbf{b} + \hat{\mathbf{X}}(\sigma)\right) : . \tag{5.25}$$

In shifting the impact parameter by the string position operator $\hat{\mathbf{X}}$, as in [22], one can take into account the finite size of the string and it is in this respect that the eikonal operator of string theory differs from the field theory eikonal. It should be possible to derive the amplitudes evaluated by direct computation in the following section by using the eikonal operator. One should note that in the analysis of Section 5.4 neither $A_1(s, \mathbf{q})$ nor its Fourier transform $\mathcal{A}_1(s, \mathbf{b})$ need be specified and so these arguments are not restricted to any particular kinematic regime beyond that already assumed for the Regge limit.

The eikonal operator depends only on the bosonic modes associated to the directions transverse to the brane and therefore its action on the bosonic modes associated to the directions parallel to the brane and on the fermionic modes is trivial. This is true also for the high energy limit of the tree-level amplitudes, as emphasized in the previous sections. In this chapter we shall show that the matrix elements of the eikonal operator coincides precisely with the high energy limit of the tree-level string amplitudes. Once we are satisfied that this is the case we will consider the impact parameter space representation of the result (5.69) in the limit of large impact parameter, equivalent to small momentum transfer.

As in the field theory limit, the leading eikonal operator specified in (5.25) is the first term in a series expansion in R_p/b representing the classical corrections from the background produced by the stack of Dp-branes. The detailed structure of the S-matrix beyond this leading term is still not well understood and remains an interesting area of development. We can also consider quantum corrections to the S-matrix which arise from adding handles to the world-sheet diagrams, however, often in practice one is interested in the 't Hooft limit in which the 't Hooft coupling $\lambda = g_s N$ is very large and since we need g_s to be small in order to use string amplitudes meaningfully then we must take the number of D-branes N to be very large. The result of this is that the planar diagrams, those of genus zero, will come to dominate the S-matrix and so we can neglect these corrections.

5.3 The scattering of massive strings from a D-brane

The scattering amplitude for the interaction of two closed string states with a stack of N Dp-branes at tree level is given by the insertion of two closed string vertex operators onto the upper-half of the complex plane $\mathbb{H}$,

$$A_{n,n'} = \mathcal{N} \int_{\mathbb{H}_+} \frac{dz_1^2 dz_2^2}{V_{CKG}} \left\langle W^{(n)}_{(0,0)}(k_1, z_1, \bar{z}_1) W^{(n')}_{(-1,-1)}(k_2, z_2, \bar{z}_2) \right\rangle_{\mathbb{H}_+} . \tag{5.26}$$

Here we have the vertex operator (2.87) for a state with momentum $p_1 = 2k_1$ and mass $M^2 = 4n/\alpha'$ carrying a superghost charge $(0,0)$ and the vertex operator (2.85) for a state with momentum $p_2 = 2k_2$ and mass $M^2 = 4n'/\alpha'$ carrying a superghost charge $(-1,-1)$. To begin with we evaluate the Regge limit of the amplitude involving one graviton and one massive symmetric state at the level $n' = 1$ by first computing the full amplitude and then applying the limit to the resulting function. By doing this we will identify the leading contribution to come from the region of the world-sheet integrals in which the vertex operators approach one another and so uncover a general procedure for efficiently calculating such amplitudes using the OPE of these operators.

The normalization constant, $\mathcal{N}$, in (5.26) is the same for all of these amplitudes and is formed from the product of the normalizations of the vertices, $(\kappa/2\pi)^2$, and the topological factor for a disc amplitude $C_{D_2} = 2\pi^2\, T_p/\kappa$, where κ is the gravitational coupling constant in ten dimensions and T_p is the coupling for closed string states to a Dp-brane. Overall this gives the normalization

$$\mathcal{N} = \frac{\kappa T_p}{2} = \frac{R_p^{7-p}\, \pi^{\frac{9-p}{2}}}{\Gamma\left(\frac{7-p}{2}\right)}, \tag{5.27}$$

where R_p represents a characteristic size for the stack of D-branes and is related to the t'Hooft coupling, $\lambda = gN$ as follows,

$$R_p^{7-p} = gN\frac{(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p}}, \qquad \Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{5.28}$$

When calculating amplitudes on world-sheets with a boundary it is convenient to employ the doubling trick as described in Section 2.6 and so we send $X^\mu(\bar{z}) \to D^\mu{}_\nu X^\nu(\bar{z})$, $\tilde{\psi}^\mu(\bar{z}) \to D^\mu{}_\nu\psi^\nu(\bar{z})$ and $\widetilde{\varphi}(\bar{z}) \to \varphi(\bar{z})$ with D defined in equation (2.97). With this prescription we need only compute correlators between the holomorphic fields which define the world-sheet dynamics and in this case they are

$$\begin{aligned}\langle X^\mu(z)X^\nu(w)\rangle &= -2\alpha'\eta^{\mu\nu}\log(z-w),\\ \langle\psi^\mu(z)\psi^\nu(w)\rangle &= \frac{\eta^{\mu\nu}}{z-w},\\ \langle\phi(z)\phi(w)\rangle &= -\log(z-w).\end{aligned} \tag{5.29}$$

Using equations (2.86–2.87) it is easily verified that the vertex for the graviton in the $(0,0)$ picture is

$$W^{(0)}_{(0,0)}(k_1,z_1,\bar{z_1}) = -\epsilon_{\mu\nu}V_0^\mu(k_1,z_1)\widetilde{V}_0^\nu(k_1,\bar{z}_1) \tag{5.30}$$

with

$$V_0^\mu(k_1,z_1) = \frac{1}{\sqrt{2\alpha'}}\left(i\partial X^\mu(z_1) + 2\alpha' k_1\cdot\psi(z_1)\psi^\mu(z_1)\right)e^{ik_1\cdot X(z_1)}, \tag{5.31}$$

and the vertex operator in the $(-1,-1)$ picture for the state at the level $n=1$ is

$$W^{(1)}_{(-1,-1)}(k_2,z_2,\bar{z_2}) = G_{\rho\sigma\tau\zeta}V_{-1}^{\rho\sigma}(k_2,z_2)\widetilde{V}_{-1}^{\tau\zeta}(k_2,\bar{z}_2) \tag{5.32}$$

with

$$V_{-1}^{\rho\sigma}(k_2,z_2) = \frac{i}{\sqrt{2\alpha'}}e^{-\varphi(z_2)}\partial X^\rho(z_2)\psi^\sigma(z_2)e^{ik_2\cdot X(z_2)}. \tag{5.33}$$

Inserting the vertex operators (5.30-5.33) and the normalization (5.27) into the integral (5.26) we obtain

$$\begin{aligned}A_{0,1} =&\frac{\kappa T_p}{2}\int_{\mathbb{H}_+}\frac{d^2z_1d^2z_2}{V_{CKG}}\left\langle W^{(0)}_{(0,0)}(k_1,z_1,\bar{z}_1)W^{(1)}_{(-1,-1)}(k_2,z_2,\bar{z}_2)\right\rangle_{\mathbb{H}_+}\\ =&\frac{\kappa T_p}{8\alpha'^2}\,\epsilon_{\mu\lambda}D^\lambda{}_\nu G_{\rho\sigma\alpha\beta}D^\alpha{}_\tau D^\beta{}_\zeta\int_{\mathbb{H}_+}\frac{d^2z_1d^2z_2}{V_{CKG}}\\ &\Big\langle :\big(i\partial X^\mu(z_1) + 2\alpha' k_1\cdot\psi(z_1)\psi^\mu(z_1)\big)e^{ik_1\cdot X(z_1)}:\\ &:\big(i\bar{\partial}X^\nu(\bar{z}_1) + 2\alpha' k_1\cdot D\cdot\psi(\bar{z}_1)\psi^\nu(\bar{z}_1)\big)e^{ik_1\cdot D\cdot X(\bar{z}_1)}:\\ &:e^{-\varphi(z_2)}\partial X^\rho(z_2)\psi^\sigma(z_2)e^{ik_2\cdot X(z_2)}::e^{-\varphi(\bar{z}_2)}\bar{\partial}X^\tau(\bar{z}_2)\psi^\zeta(\bar{z}_2)e^{ik_2\cdot D\cdot X(\bar{z}_2)}:\Big\rangle.\end{aligned} \tag{5.34}$$

In evaluating the above correlator it can be shown that the leading term in the high energy Regge limit is given by the contraction of the two operators quadratic in the fermionic fields. This term is proportional to $2\alpha' \, k_1 \cdot D \cdot k_1 = \alpha' s + 2$ and the overall s factor assures that this term is dominant with respect to all the other contractions in the amplitude.

In order to see that this is the case it is important to note the following fact. If we define an $SL(2, \mathbb{R})$ invariant variable

$$\omega = \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(z_1 - \bar{z}_2)(\bar{z}_1 - z_2)} \tag{5.35}$$

and use this in writing the factor in the correlation function which results from the contraction of the $e^{ik\cdot X}$ operators

$$\left\langle e^{ik_1\cdot X(z_1)} e^{ik_1\cdot D\cdot X(\bar{z}_1)} e^{ik_2\cdot X(z_2)} e^{ik_2\cdot D\cdot X(\bar{z}_2)} \right\rangle = \omega^{-\alpha' \frac{t}{4}+1} (\omega - 1)^{-\alpha' s} (z_2 - \bar{z}_2)^2, \tag{5.36}$$

then the remaining explicit z-dependence can be combined with that from the other possible contractions in (5.34), together with the appropriate measure $\frac{d^2 z_1 d^2 z_2}{V_{CKG}} \mapsto d\omega (z_1 - \bar{z}_2)^2 (\bar{z}_1 - z_2)^2$, to give some multiplicative $SL(2, \mathbb{R})$ invariant function $F(\omega)$. The amplitude then takes the following schematic form

$$A_{0,1} = \frac{\kappa T_p}{2} \int_0^1 d\omega \, \omega^{-\alpha' \frac{t}{4}+1} (\omega - 1)^{-\alpha' s} F(\omega). \tag{5.37}$$

The behaviour of this integral when we take the limit $\alpha' s \to \infty$ is controlled by the integrand in the neighbourhood of the point $\omega = 0$ and since $F(\omega)$ is, in general, a sum of terms composed of powers of ω, $(1 - \omega)$ and their inverse quantities, $A_{0,1}$ itself consists of a sum of integrals of the form shown below,

$$\int_0^1 d\omega \, \omega^{-\alpha' \frac{t}{4}+a} (1 - \omega)^{-\alpha' s + b} = \frac{\Gamma\left(-\alpha' \frac{t}{4} + a + 1\right) \Gamma\left(-\alpha' s + b + 1\right)}{\Gamma\left(-\alpha' \frac{t}{4} - \alpha' s + a + b + 2\right)}. \tag{5.38}$$

It can be seen that in the high energy limit this quantity will scale with s as $(\alpha' s)^{-a-1}$ and therefore the dominant contribution to $A_{0,1}$ will come from the term with the lowest value of a. The definition of ω in equation (5.35) implies that these both point to large s behaviour being governed by the region of integration over the world sheet in which the two vertex operators are brought together, that is to say when $z_1 \to z_2$ and $\bar{z}_1 \to \bar{z}_2$. The analysis of this process will be expanded upon and made more systematic in the next Section.

Having learnt this, one can quickly deduce that we are interested in the terms of $F(\omega)$ which are obtained from the maximum possible number of contractions between the holomorphic and antiholomorphic fields in equation (5.34). In general there are many such terms but, as mentioned below (5.34), those which contain the contraction $k_1 \cdot \overbrace{\psi \, k_1 \cdot D \cdot \psi}$ will bring an additional factor $\alpha' s$ and so will ultimately be the leading terms at high energy.

By evaluating the correlation function in equation (5.34), then employing the physical state conditions $\epsilon_{\mu\nu}k_1^\mu = G_{\rho\sigma\tau\zeta}k_2^\rho = 0$ and momentum conservation $k_1^\mu+(D\cdot k_1)^\mu+k_2^\mu+(D\cdot k_2)^\mu = 0$, we can determine the function $F(\omega)$ and perform the integral in equation (5.37). If we take the limit $\alpha' s \to \infty$ we find that only one term dominates in this case, which is proportional to $k_1^\rho G_{\rho\sigma\tau\zeta}\, \epsilon^{\sigma\tau} k_1^\zeta$. Subsequent application of Stirling's approximation for the Gamma function yields the following form for the amplitude in the high energy limit

$$A_{0,1} \sim \frac{\kappa T_p}{2} e^{-i\pi\alpha' \frac{t}{4}} \Gamma\left(-\tfrac{\alpha' t}{4}\right) \left(\alpha' s\right)^{\frac{\alpha' t}{4}+1} \frac{\alpha'}{2} q^\rho G_{\rho\sigma\tau\zeta}\, \epsilon^{\sigma\tau} q^\zeta, \tag{5.39}$$

where we have replaced k_1 and k_2 with the physical momentum transferred $q = 2(k_1 + k_2)$ by virtue of the physical state conditions.

It is instructive to compare this result with the analogous result for the elastic scattering of a graviton by a D-brane,

$$A_{0,0} \sim \frac{\kappa T_p}{2} e^{-i\pi\alpha' \frac{t}{4}} \Gamma\left(-\tfrac{\alpha' t}{4}\right) \left(\alpha' s\right)^{\frac{\alpha' t}{4}+1} \epsilon_{1\sigma\tau}\, \epsilon_2^{\sigma\tau}. \tag{5.40}$$

Considered purely as functions of the complex variables s and t, these two equations demonstrate identical behaviour; the physical amplitude is obtained by taking s, t real with large positive s and negative t, but for positive t we see that there are poles corresponding to the exchange of a string of mass $M^2 = t$ with the D-brane and, furthermore, the s-dependence indicates that the exchanged string has spin $J = (\alpha' M^2 + 4)/2$ — it belongs to the Regge trajectory of the graviton. As the energy becomes very large this description of the exchange of a single string breaks down due to the violation of unitarity, however, as demonstrated in [35] unitarity may be recovered by taking into account loop effects. Examining the two amplitudes above we note that the elastic and the inelastic amplitude only differ in the multiplicative factor containing the initial and final polarisation tensors and the exchanged momentum.

It has just been shown that the usual methods used for calculating scattering amplitudes in string theory can result in the generation of a vast number of terms which transpire to be subleading in energy after integration over the world-sheet. If our interests lie only in the leading terms then we would like to be able to distinguish these subleading terms and discard them at the very beginning of our calculation; with OPE methods we can do this quite easily and, furthermore, they highlight the simple structure present in the class of amplitudes we consider here.

To compute the leading terms in the amplitude given by equation (5.26) in the Regge limit, $\alpha' s \to \infty$ and $\alpha' t$ fixed, it is key to note that the integral over the world-sheet in (5.26) is dominated at large $\alpha' s$ by the behaviour of the vertex operators as $z_1 \to z_2$, as exemplified in equation (5.37). Specifically, when working in the Regge regime for scattering processes with two external string states the leading contribution to the integral over the world-sheet can be taken from the OPE of the vertex operators. Subsequent integration over their separation $w = z_1 - z_2$ will give an amplitude with the expected Regge behaviour; however, care must be taken in this process, as we will see, since there exist terms subleading in w which will contribute factors of $\alpha' s$ and by doing so prevent us from neglecting them.

Rather than evaluating the correlator in (5.26), integrating the result and then computing the asymptotic form for large $\alpha' s$ we may instead determine the OPE of the two vertex operators and integrate out the dependence on the separation $w = z_1 - z_2$. This process yields a quasi-local operator referred to as the pomeron vertex operator, first introduced in [88] as the reggeon and recently used in [89, 90]. As illustrated in figure 5.1(a), the process of constructing this operator involves taking the limit in which two physical vertices approach one another on a surface topologically equivalent to the infinite cylinder, hence it can be considered to properly describe the t-channel exchange of a closed string. The pomeron vertex operator itself, up to subleading term in s, also satisfies the physical state conditions for any $\alpha' t$, and for $\alpha' t = 4n$ ($n = 0, 1, \ldots$) we can think of it as describing the exchange of a string with mass given by $\alpha' M^2 = 4n$. As explained in [89] the importance of single pomeron exchange lies in the fact that it dominates scattering amplitudes in the Regge regime both in QCD for $N_c \to \infty$ and in string theory.

The information contained within the pomeron vertex is only that of the two physical states which produce the exchanged pomeron and the restriction of our kinematic invariants to the Regge limit, as such this vertex may be used to generate many amplitudes of interest by insertion onto the appropriate world-sheet. In this instance we are interested in a world-sheet with a single boundary and boundary conditions which correspond to the presence of a Dp-brane, indicated in figure 5.1(b). This can be easily done using boundary state methods which were introduced in [65] and are reviewed in [68]. These methods are extremely powerful for dealing with problems involving string interactions with D-branes [1, 81] and for our purposes we can emulate the effects of the boundary state using the doubling trick, but further applications using the same vertex can be carried out systematically by bearing these facts in mind.

We will illustrate the use of the pomeron vertex operator with some specific examples before proceeding to the general case. If we label by (n, n') the amplitude in which a state of mass $\alpha' M_1^2 = 4n$ undergoes a transition to a state of mass $\alpha' M_2^2 = 4n'$ then the examples we shall consider will be $(0, n')$ and $(1, n')$. At the end of this section we will generalize our results to the case (n, n').

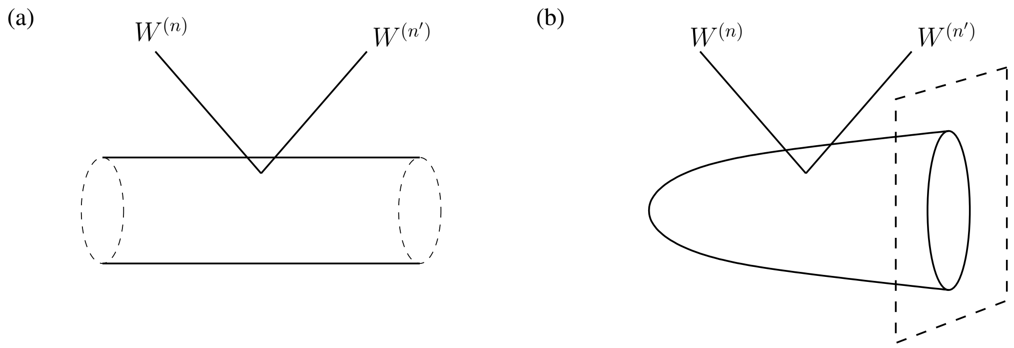

Figure 5.1: (a) The pomeron vertex operator given by two physical vertices W^n and $W^{n'}$ on the infinite cylinder. (b) The addition of a boundary gives the process in which these states interact with a D-brane.

5.3.1 Graviton-massive state transitions

In this case we examine the processes in which a massless state is excited to another of arbitrary mass. The physical polarizations for massive states will be written as tensor products of their holomorphic and antiholomorphic components,

$$\epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta} = \varepsilon_{\mu_1\ldots\mu_n\alpha} \otimes \widetilde{\varepsilon}_{\nu_1\ldots\nu_n\beta}, \tag{5.41}$$

$$G_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta} = \mathcal{G}_{\mu_1\ldots\mu_n\alpha} \otimes \widetilde{\mathcal{G}}_{\nu_1\ldots\nu_n\beta}. \tag{5.42}$$

Our first task will be to determine the OPE of the vertex operators $W^{(0)}_{(0,0)}(k_1, z_1, \bar{z}_1)$ and $W^{(n)}_{(-1,-1)}(k_2, z_2, \bar{z}_2)$ as $z_1 \to z_2$ and $\bar{z}_1 \to \bar{z}_2$; this task may seem daunting at first due to the large number of possible contractions, but as we shall see it is possible to immediately identify which contractions will end up giving the leading order contribution to this amplitude. In this particular computation our vertex operators are

$$\begin{aligned} W^{(0)}_{(0,0)}(k_1, z_1, \bar{z}_1) &= -\epsilon_{\mu\nu} V^{\mu}_0(k_1, z_1) \widetilde{V}^{\nu}_0(k_1, \bar{z}_1), \\ V^{\mu}_0(k_1, z_1) &= \frac{1}{\sqrt{2\alpha'}} \left(i\partial X^{\mu}(z_1) + 2\alpha' k_1 \cdot \psi(z_1) \psi^{\mu}(z_1) \right) e^{ik_1 \cdot X(z_1)}, \end{aligned} \tag{5.43}$$

and

$$\begin{aligned} W^{(n')}_{(-1,-1)}(k_2, z_2, \bar{z}_2) &= \mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma} \widetilde{\mathcal{G}}_{\lambda_1\ldots\lambda_{n'}\gamma} V^{\rho_1\ldots\rho_{n'}\sigma}_{-1}(k_2, z_2) \widetilde{V}^{\lambda_1\ldots\lambda_{n'}\gamma}_{-1}(k_2, \bar{z}_2), \\ V^{\rho_1\ldots\rho_{n'}\sigma}_{-1}(k_2, z_2) &= \frac{1}{\sqrt{n'!}} \left(\frac{i}{\sqrt{2\alpha'}} \right)^{n'} e^{-\varphi(z_2)} \prod_{i=1}^{n'} \partial X^{\rho_i} \psi^{\sigma} e^{ik_2 \cdot X(z_2)}. \end{aligned} \tag{5.44}$$

In deriving the pomeron vertex operator we consider first the insertion of the above two closed string vertices onto a world-sheet with the topology of the Riemann sphere. We need not consider the contractions between holomorphic and antiholomorphic operators until the resulting effective vertex is inserted onto a world-sheet with the topology of the disc. As such we can write the resulting OPE in terms of the world-sheet separation $w = z_1 - z_2$ and the point $z = \frac{z_1+z_2}{2}$ which then takes the following form

$$W^{(0)}_{(0,0)}\left(k_1, z+\tfrac{w}{2}, \bar{z}+\tfrac{\bar{w}}{2}\right)W^{(n')}_{(-1,-1)}\left(k_2, z-\tfrac{w}{2}, \bar{z}-\tfrac{\bar{w}}{2}\right) \sim -|w|^{-\alpha'\frac{t}{2}+2n'}\mathcal{O}(z,w)\widetilde{\mathcal{O}}(\bar{z},\bar{w}). \quad (5.45)$$

It is simple to check that the operators $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are polynomials of at most degree $(n'+1)$ in w^{-1} and $\bar{w}^{-1}$ respectively, with an exponential factor contributing terms subleading in the small w limit, that is

$$\mathcal{O}(z,w) = e^{i\frac{1}{2}(k_1-k_2)\cdot\partial X(z)w}\sum_{p=1}^{n'+1}\frac{\mathcal{O}_p(z)}{w^p}, \quad \widetilde{\mathcal{O}}(\bar{z},\bar{w}) = e^{i\frac{1}{2}(k_1-k_2)\cdot\bar{\partial}X(\bar{z})\bar{w}}\sum_{q=1}^{n'+1}\frac{\widetilde{\mathcal{O}}_q(\bar{z})}{\bar{w}^q}. \quad (5.46)$$

In the high energy limit it is necessary to retain these particular subleading terms in the exponential, as we will see, because contractions between $k_1 \cdot \partial X$ and $k_1 \cdot \bar{\partial}X$ will generate factors of s meaning that these terms cannot be neglected for $|w|^2 \sim (\alpha' s)^{-1}$. It is in fact these terms which will generate the Regge behaviour that we expect.

In the expansions given by equation (5.46) it will be the most singular terms which will dominate in the pomeron vertex operator, this operator being obtained by the integration of w in the OPE (5.45) over the complex plane, and this procedure will in general result in an integral for each of these terms of the form

$$\int_{\mathbb{C}} d^2w\,|w|^{-\alpha'\frac{t}{2}-2}e^{i\frac{\tilde{q}}{4}\cdot\partial X(z)w}e^{i\frac{\tilde{q}}{4}\cdot\bar{\partial}X(\bar{z})\bar{w}}. \quad (5.47)$$

The integration of (5.47) can be done by introducing new variables $u = \frac{\tilde{q}}{4}\cdot\partial X(z)\,w$, $\bar{u} = \frac{\tilde{q}}{4}\cdot\bar{\partial}X(\bar{z})\,\bar{w}$,

$$e^{-i\pi\alpha'\frac{t}{4}}\int_{\mathbb{C}} d^2u\,|u|^{-\alpha'\frac{t}{2}-2}\,e^{i(u+\bar{u})}\left(i\frac{\tilde{q}}{4}\cdot\partial X(z)\right)^{\alpha'\frac{t}{4}}\left(i\frac{\tilde{q}}{4}\cdot\bar{\partial}X(\bar{z})\right)^{\alpha'\frac{t}{4}}. \quad (5.48)$$

Then we integrate over the positions $u = re^{i\theta}$ using the following integrals:

$$\int_0^{2\pi} d\theta\, e^{-2ir\cos\theta} = 2\pi\, J_0(2r), \quad (5.49)$$

$$\int_0^{\infty} dr\, r^a\, J_0(2r) = \frac{1}{2}\frac{\Gamma\left(\frac{1+a}{2}\right)}{\Gamma\left(\frac{1-a}{2}\right)}, \quad (5.50)$$

and we get

$$\int_{\mathbb{C}} d^2w\,|w|^{-\alpha'\frac{t}{2}-2}e^{i\frac{\tilde{q}}{4}\cdot\partial X(z)w}e^{i\frac{\tilde{q}}{4}\cdot\bar{\partial}X(\bar{z})\bar{w}} = \Pi(t)\left(i\frac{\tilde{q}}{4}\cdot\partial X(z)\right)^{\alpha'\frac{t}{4}}\left(i\frac{\tilde{q}}{4}\cdot\bar{\partial}X(\bar{z})\right)^{\alpha'\frac{t}{4}}, \quad (5.51)$$

where $\Pi(t)$ is commonly referred to as the pomeron propagator [89, 90] and is given by the following

$$\Pi(t) = 2\pi \frac{\Gamma\left(-\alpha' \frac{t}{4}\right)}{\Gamma\left(1+\alpha' \frac{t}{4}\right)}\, e^{-i\pi\alpha' \frac{t}{4}}. \tag{5.52}$$

The two-point function is reduced to a one-point function of the effective pomeron vertex on the disc.

From what we have discussed so far, one can conclude that to leading order in energy the pomeron vertex operator should take the following form

$$\int_{\mathbb{C}} d^2w\, W^{(0)}_{(0,0)}(k_1, z+\tfrac{w}{2}, \bar{z}+\tfrac{\bar{w}}{2}) W^{(n')}_{(-1,-1)}(k_2, z-\tfrac{w}{2}, \bar{z}-\tfrac{\bar{w}}{2}) \sim -K_{0,n'}(q,\epsilon,G)\Pi(t)\mathcal{O}(z)\widetilde{\mathcal{O}}(\bar{z}), \tag{5.53}$$

where we have the pomeron propagator, the normal-ordered operators

$$\mathcal{O}(z) = \sqrt{2\alpha'}\left(i\frac{\tilde{q}}{4}\cdot\partial X\right)^{\alpha' \frac{t}{4}} k_1\cdot\psi e^{-\varphi} e^{i\frac{q}{2}\cdot X}, \tag{5.54a}$$

$$\widetilde{\mathcal{O}}(\bar{z}) = \sqrt{2\alpha'}\left(i\frac{\tilde{q}}{4}\cdot\bar{\partial} X\right)^{\alpha' \frac{t}{4}} k_1\cdot\widetilde{\psi} e^{-\widetilde{\varphi}} e^{i\frac{q}{2}\cdot X}, \tag{5.54b}$$

and a kinematic function, dependent upon the polarisation tensors and the momentum transferred from the string to the brane,

$$K_{0,n'}(q,\epsilon,G) = \frac{1}{n'!}\left(\frac{\alpha'}{2}\right)^{n'} (k_1)^{n'}\cdot\mathcal{G}_0\cdot\varepsilon_0\cdot(k_2)^n\ (k_1)^{n'}\cdot\widetilde{\mathcal{G}}_0\cdot\widetilde{\varepsilon}_0\cdot(k_2)^0. \tag{5.55}$$

Here we have introduced for later convenience the notation

$$(k_1)^{n'}\cdot\mathcal{G}_a\cdot\varepsilon_a\cdot(k_2)^n = 2^{n+n'-a-b}\left(\prod_{i=1}^{a}\eta^{\rho_i\mu_i}\right)\left(\prod_{j=a+1}^{n'} k_1^{\rho_j}\right)\left(\prod_{k=a+1}^{n} k_2^{\mu_k}\right)\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma}, \eta^{\sigma\alpha}\varepsilon_{\mu_1\ldots\mu_n\alpha} \tag{5.56}$$

where $a \in \{0,\ldots,\min\{n,n'\}\}$ will count the number of contractions between $\mathcal{G}$ and ε in addition to that arising from the fermionic fields, and an analogous expression holds for the polarisation of the antiholomorphic components. In (5.56) products of the form $\prod_{i=1}^{0}$ should be replaced by unity. This notation will prove useful since the polarisation tensors for all states on the leading Regge trajectory are symmetric in all holomorphic indices and symmetric in all antiholomorphic indices, as a result the order of contractions with these indices is immaterial; all we need do is keep track of how many factors of k_1 are contracted with $\mathcal{G}$, $\widetilde{\mathcal{G}}$ and how many factors of k_2 are contracted with ε, $\widetilde{\varepsilon}$. Furthermore, due to the requirement that longitudinal polarisations vanish we find that we may replace all occurrences of k_1 and k_2 in equation (5.56) and its antiholomorphic partner with the transferred momentum $q = 2(k_1 + k_2)$. In this case we can substitute the expression in equation (5.56) with

$$q^{n'} \cdot \mathcal{G}_a \cdot \varepsilon_a \cdot q^n = \left(\prod_{i=1}^{a} \eta^{\rho_i \mu_i}\right) \left(\prod_{j=a+1}^{n'} q^{\rho_j}\right) \left(\prod_{k=a+1}^{n} q^{\mu_k}\right) \mathcal{G}_{\rho_1 \ldots \rho_{n'} \sigma} \eta^{\sigma\alpha} \varepsilon_{\mu_1 \ldots \mu_n \alpha}. \quad (5.57)$$

The one-point function of the pomeron vertex is given by the contraction of the operators $\mathcal{O}(z)$, $\widetilde{\mathcal{O}}(\bar{z})$ in (5.54). To leading order in s, discarding the masses, we obtain

$$K_{0,n'}(q,\epsilon,G) \int_{D_2} \frac{dz d\bar{z}}{V_{CKG}} \Pi(t) \langle : \mathcal{O}(z) :: \widetilde{\mathcal{O}}(\bar{z}) : \rangle_{D_2} \sim K_{0,n'}(q,\epsilon,G)(\alpha' s)^{\alpha' \frac{t}{4}+1} \times \frac{\Pi(t)}{2\pi} \Gamma\Big(1 + \alpha' \tfrac{t}{4}\Big). \quad (5.58)$$

The factor $\frac{1}{2\pi}$ arises from the ratio between the integration over the insertion point of the vertex operator and the volume of the Conformal Killing Group of the disc, $SL(2,\mathbb{R})$.

We obtain an $SL(2,\mathbb{R})$invariant function which gives the leading high energy behaviour of the amplitude for a graviton scattering from a D-brane into a state on the leading Regge trajectory at level n',

$$A_{0,n'}(s,t) = \frac{\kappa T_p}{2} K_{0,n'}(q,\epsilon,G) \Gamma(-\alpha' \tfrac{t}{4}) e^{-i\pi\alpha' \frac{t}{4}} (\alpha' s)^{\alpha' \frac{t}{4}+1}. \quad (5.59)$$

5.3.2 Transitions from the lowest massive state

We next move on to the case of a string with mass $M_1^2 = 4/\alpha'$ interacting with a D-brane to leave a string of mass $M_2^2 = 4n'/\alpha'$. The vertex operators are now given by

$$\begin{aligned} W^{(1)}_{(0,0)}(k_1, z_1, \bar{z}_1) &= -\varepsilon_{\mu\alpha} \widetilde{\varepsilon}_{\nu\beta} V_0^{\mu\alpha}(k_1, z_1) \widetilde{V}_0^{\nu\beta}(k_1, \bar{z}_1), \\ V_0^{\mu\alpha}(k_1, z_1) &= \frac{i}{2\alpha'} \left(i\partial X^\mu \partial X^\alpha + 2\alpha' k_1 \cdot \psi \psi^\alpha \partial X^\mu - i 2\alpha' \partial\psi^\mu \psi^\alpha \right) e^{ik_1 \cdot X}, \end{aligned} \quad (5.60)$$

and

$$\begin{aligned} W^{(n')}_{(-1,-1)}(k_2, z_2, \bar{z}_2) &= \mathcal{G}_{\rho_1 \ldots \rho_{n'} \sigma} \widetilde{\mathcal{G}}_{\lambda_1 \ldots \lambda_{n'} \gamma} V_{-1}^{\rho_1 \ldots \rho_{n'} \sigma}(k_2, z_2) \widetilde{V}_{-1}^{\lambda_1 \ldots \lambda_{n'} \gamma}(k_2, \bar{z}_2), \\ V_{-1}^{\rho_1 \ldots \rho_{n_2} \sigma}(k_2, z_2) &= \frac{1}{\sqrt{n'!}} \left(\frac{i}{\sqrt{2\alpha'}} \right)^{n'} e^{-\varphi(z_2)} \prod_{i=1}^{n'} \partial X^{\rho_i} \psi^\sigma e^{ik_2 \cdot X(z_2)}. \end{aligned} \quad (5.61)$$

The methods developed in the previous example need little modification in order to deal with this problem. With them we can easily determine the form of the pomeron vertex operator and it can be written in the same manner as in the $(0, n')$ case,

$$\int_{\mathbb{C}} d^2 w \, W^{(1)}_{(0,0)}(k_1, z + \tfrac{w}{2}, \bar{z} + \tfrac{\bar{w}}{2}) W^{(n')}_{(-1,-1)}(k_2, z - \tfrac{w}{2}, \bar{z} - \tfrac{\bar{w}}{2}) \sim -K_{1,n'}(q,\epsilon,G) \Pi(t) \mathcal{O}(z) \widetilde{\mathcal{O}}(\bar{z}), \quad (5.62)$$

where this time

$$K_{1,n'}(q,\epsilon,G) = \frac{1}{n'!}\left(\frac{\alpha'}{2}\right)^{n'+1}\Big[q^{n'}\cdot\mathcal{G}_0\cdot\varepsilon_0\cdot q^1\, q^1\cdot\widetilde{\varepsilon}_0\cdot\widetilde{\mathcal{G}}_0\cdot q^{n'} - \frac{2n'}{\alpha'}q^{n'-1}\cdot\mathcal{G}_1\cdot\varepsilon_1\cdot q^0\, q^1\cdot\widetilde{\varepsilon}_0\cdot\widetilde{\mathcal{G}}_0\cdot q^{n'}$$
$$- \frac{2n'}{\alpha'}q^{n'}\cdot\mathcal{G}_0\cdot\varepsilon_0\cdot q^1\, q^0\cdot\widetilde{\varepsilon}_1\cdot\widetilde{\mathcal{G}}_1\cdot q^{n'-1} + \left(\frac{n'}{2\alpha'}\right)^2 q^{n'-1}\cdot\mathcal{G}_1\cdot\varepsilon_1\cdot q^0\, q^0\cdot\widetilde{\varepsilon}_1\cdot\widetilde{\mathcal{G}}_1\cdot q^{n'-1}\Big] \tag{5.63}$$

and all other quantities remain as previously defined. Because of this, the resulting amplitude will be identical to that of equation (5.59) other than the form of the kinematic function which is given in (5.63),

$$A_{1,n'}(s,t) = \frac{\kappa T_p}{2}K_{1,n'}(q,\epsilon,G)\,\Gamma(-\alpha'\tfrac{t}{4})e^{-i\pi\alpha'\frac{t}{4}}\,(\alpha' s)^{\alpha'\frac{t}{4}+1}. \tag{5.64}$$

Note that we have taken $2\alpha' k_1\cdot D\cdot k_1 \sim -\alpha' s$ in the large s limit, neglecting a mass term of the order of unity. In the next case we will move on to masses which contribute terms of order n, n' and we reiterate that we shall be considering only those states for which the rest mass contribution to the total energy is negligible.

5.3.3 Transitions within the leading Regge trajectory

Here we return to our original consideration, the process in which a string in a state on the leading Regge trajectory is scattered from a Dp-brane into some other state on the leading Regge trajectory. The vertex operators are those given by equations (2.87) and (2.85), with polarisation tensors $\varepsilon\otimes\widetilde{\varepsilon}$ and $\mathcal{G}\otimes\widetilde{\mathcal{G}}$, respectively, and our methods will imitate those we have seen already. If we initially suppose n and n' to be fixed at some finite values, then by identifying the terms in the OPE of these vertices which will give the leading high energy contributions and integrating out the dependence on their separation we obtain the same general form for the pomeron vertex operator as in section 5.3.2,

$$\frac{1}{n!n'!}\int_{\mathbb{C}} d^2w\, W^{(n)}_{(0,0)}(k_1, z+\tfrac{w}{2}, \bar{z}+\tfrac{\bar{w}}{2})W^{(n')}_{(-1,-1)}(k_2, z-\tfrac{w}{2}, \bar{z}-\tfrac{\bar{w}}{2}) \sim -K_{n,n'}(q,\epsilon,G)\Pi(t)\mathcal{O}(z)\widetilde{\mathcal{O}}(\bar{z}), \tag{5.65}$$

where now we can see the full structure of the kinematic function, which may be written as the product of a contribution from the holomorphic operators with a contribution from the antiholomorphic operators,

$$K_{n,n'}(q,\epsilon,G) = \frac{1}{n!n'!}\left(\frac{\alpha'}{2}\right)^{n+n'}\sum_{a,b=0}^{\min\{n,n'\}}\left(-\frac{\alpha'}{2}\right)^{-a-b}C_{n,n'}(a)C_{n,n'}(b)$$
$$\times\, q^{n'-a}\cdot\mathcal{G}_a\cdot\varepsilon_a\cdot q^{n-a}\, q^{n-b}\cdot\widetilde{\varepsilon}_b\cdot\widetilde{\mathcal{G}}_b\cdot q^{n'-b}. \tag{5.66}$$

Here $\min\{n, n'\}$ indicates the smallest value from the set $\{n, n'\}$, and the combinatorial factors $C_{n,n'}$ (one each coming from the holomorphic and antiholomorphic contractions) come from the large number of possible contractions in the OPE which lead to the same operator after taking into account the symmetry of the polarisation tensors. These functions are given by

$$C_{n,n'}(p) = \frac{n!n'!}{p!(n-p)!(n'-p)!} \tag{5.67}$$

and this can be deduced in the following manner. If we consider the contribution from the holomorphic operators then $K_{n,n'}$ is determined by all possible contractions amongst

$$\varepsilon_{\mu_1\ldots\mu_n\alpha}\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma} : \prod_{i=1}^{n} \partial X^{\mu_i} e^{ik_1\cdot X} :: \prod_{i=1}^{n'} \partial X^{\rho_i} e^{ik_2\cdot X} : . \tag{5.68}$$

Since ε and $\mathcal{G}$ are symmetric in all indices these contractions can simply be labelled by the number of contractions between ∂X^{μ_i} and ∂X^{ρ_i}, let this be a. This being the case we must count how many ways one can generate a such contractions, first one must choose a operators from a total of n possibilities for which there are $\binom{n}{a} = n!/a!(n-a)!$ different choices. Similarly we must choose a further a operators from a set of n' possibilities giving another factor of $\binom{n'}{a}$. Finally, from this set of $2a$ operators there $a!$ possible ways to contract them in pairs. The product of these numbers gives the total number of different contractions which result in a factor $\mathcal{G}_a \cdot \varepsilon_a$ and is equal to the function $C_{n,n'}(a)$.

With this new form for the kinematic factor K we are finished, the rest of the computation having already been solved in the previous two examples. The final result for the amplitude of a finite mass string scattering from a D-brane is

$$A_{n,n'}(s,t) = \frac{\kappa T_p}{2} K_{n,n'}(q,\epsilon,G)\,\Gamma\left(-\alpha'\tfrac{t}{4}\right) e^{-i\pi\alpha'\frac{t}{4}} \left(\alpha' s\right)^{\alpha'\frac{t}{4}+1}. \tag{5.69}$$

5.4 Comparison with the eikonal operator

It should be possible to derive the amplitudes evaluated by direct computation in the previous sections by using the eikonal operator. One should note that in the following analysis neither $A_1(s,\mathbf{q})$ nor its Fourier transform $\mathcal{A}_1(s,\mathbf{b})$ need be specified and so these arguments are not restricted to any particular kinematic regime beyond that already assumed for the Regge limit.

The eikonal operator depends only on the bosonic modes associated to the directions transverse to the brane and therefore its action on the bosonic modes associated to the directions parallel to the brane and on the fermionic modes is trivial. This is true also for the high energy limit of the tree-level amplitudes, as emphasized in the previous sections. In this section we shall show that the matrix elements of the eikonal operator coincides precisely with the high energy limit of the tree-level string amplitudes. Once we are satisfied that this is the case we will consider the impact parameter space representation of the result (5.69) in the limit of large impact parameter, equivalent to small momentum transfer.

The exponential in (5.25) represents a resummation of the perturbative expansion and we expect that tree level diagrams arise from the linear term in this exponential; that is to say we should be able to obtain amplitudes for the high energy limit of tree level processes from matrix elements of the operator $\bar{\mathcal{A}}$ defined to be

$$\bar{\mathcal{A}} \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} : \mathcal{A}_1\left(s, \mathbf{b}+\mathbf{X}(\sigma)\right) := \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k \mathcal{A}_1(s,\mathbf{b})}{\partial b^{\mu_1}\ldots\partial b^{\mu_k}} \overline{X^{\mu_1}\ldots X^{\mu_k}}. \tag{5.70}$$

To demonstrate this technique we will first reproduce the tree-level graviton to graviton scattering amplitude, before moving on to the more general case. The initial and final states representing the graviton are:

$$|i\rangle = \epsilon_{1\mu\nu}\psi^{\mu}_{-\frac{1}{2}}\widetilde{\psi}^{\nu}_{-\frac{1}{2}}\,|0;0\rangle\,, \tag{5.71a}$$

$$|f\rangle = \epsilon_{2\rho\sigma}\psi^{\rho}_{-\frac{1}{2}}\widetilde{\psi}^{\sigma}_{-\frac{1}{2}}\,|0;0\rangle\,. \tag{5.71b}$$

Using the commutation relations of the fermionic modes and the fact that $\bar{\mathcal{A}}$ contains only the bosonic modes, the tree-level graviton to graviton amplitude is given by

$$\langle f|\,\bar{\mathcal{A}}\,|i\rangle = \epsilon_{1\mu\nu}\,\epsilon_{2\rho\sigma}\,\eta^{\mu\rho}\,\eta^{\nu\sigma}\,\langle 0;0|\,\bar{\mathcal{A}}\,|0;0\rangle = \mathrm{Tr}(\epsilon_1^T\,\epsilon_2)\,\mathcal{A}_1(s,\mathbf{b}). \tag{5.72}$$

It is clear from the definition of $\mathcal{A}_1(s,\mathbf{b})$ that this will reproduce the expected result in q-space upon a Fourier transformation.

Now we will show that the tree-level amplitude for a state of mass $\alpha' M_1^2 = 4n$ to become a state of mass $\alpha' M_2^2 = 4n'$ after scattering from a D-brane in the high energy limit is given by the corresponding matrix element of the operator $\bar{\mathcal{A}}$ between the initial and final states. To do this we must show that

$$\left\langle n'\right|\,\bar{\mathcal{A}}\,|n\rangle = \mathcal{A}_{n,n'}\left(s,\mathbf{b}\right) \tag{5.73}$$

where $A_{n,n'}(s,\mathbf{b})$ is the Fourier transform of equation (5.69). We will use the standard oscillator modes representation of the initial and final states

$$|n\rangle = \varepsilon_{\mu_1\ldots\mu_n\alpha}\,\widetilde{\varepsilon}_{\nu_1\ldots\nu_n\beta}\,\frac{1}{n!}\prod_{i=1}^{n}\alpha^{\mu_i}_{-1}\widetilde{\alpha}^{\nu_i}_{-1}\;\psi^{\alpha}_{-\frac{1}{2}}\widetilde{\psi}^{\beta}_{-\frac{1}{2}}\,|0;0\rangle\,, \tag{5.74a}$$

$$\left|n'\right\rangle = \mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma}\,\widetilde{\mathcal{G}}_{\lambda_1\ldots\lambda_{n'}\gamma}\,\frac{1}{n'!}\prod_{j=1}^{n'}\alpha^{\rho_j}_{-1}\widetilde{\alpha}^{\lambda_j}_{-1}\psi^{\sigma}_{-\frac{1}{2}}\widetilde{\psi}^{\gamma}_{-\frac{1}{2}}\,|0;0\rangle\,. \tag{5.74b}$$

Proceeding as before we obtain:

$$\langle n'|\,\bar{\mathcal{A}}\,|n\rangle = K_{\rho_1\ldots\rho_{n'}\lambda_1\ldots\lambda_{n'}\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\,\langle 0;0|\prod_{i=1}^{n'}\alpha_1^{\rho_i}\widetilde{\alpha}_1^{\lambda_i}\bar{\mathcal{A}}\prod_{j=1}^{n}\alpha_{-1}^{\mu_j}\widetilde{\alpha}_{-1}^{\nu_j}\,|0;0\rangle\,, \tag{5.75}$$

where the polarisations are contained within the tensor

$$K_{\rho_1\ldots\rho_{n'}\lambda_1\ldots\lambda_{n'}\mu_1\ldots\mu_n\nu_1\ldots\nu_n} = \frac{1}{n!n'!}\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma}\eta^{\sigma\alpha}\varepsilon_{\mu_1\ldots\mu_n\alpha}\,\widetilde{\mathcal{G}}_{\lambda_1\ldots\lambda_{n'}\gamma}\eta^{\gamma\beta}\widetilde{\varepsilon}_{\nu_1\ldots\nu_n\beta}. \tag{5.76}$$

To prove that the amplitudes derived in Section 5.3 are well described by the eikonal, we compute the values of equation (5.75) and compare them with results stated in equations (5.66) and (5.69).

In expanding the operator $\bar{\mathcal{A}}$ in oscillator modes it can be seen that very few terms will contribute a nonzero value to this matrix element. Since we are considering states belonging to the leading Regge trajectory, these terms can only be composed of the modes $\alpha_{\pm 1}$, $\widetilde{\alpha}_{\pm 1}$, and since they must be normal ordered there can be at most n occurrence of the operators α_1, $\widetilde{\alpha}_1$ and n' occurrences of the operators α_{-1}, $\widetilde{\alpha}_{-1}$. As a result we only expect nonzero contributions from terms of order $2|n'-n|$ through to $2(n'+n)$. Furthermore, the terms in $\bar{\mathcal{A}}$ containing k oscillators are generated by

$$\frac{1}{k!}\frac{\partial^k \mathcal{A}_1(s,\mathbf{b})}{\partial b^{\mu_1}\ldots\partial b^{\mu_k}}\overline{X^{\mu_1}\ldots X^{\mu_k}}, \tag{5.77}$$

and the invariance of the partial derivative under a change in order of differentiation will result in the oscillators being symmetric under exchange of their indices, so rather than keeping track of these indices we need only count how many ways we can generate oscillator terms of the form given above.

Using this one can deduce that we can substitute for $\bar{\mathcal{A}}$ in eq. (5.75) the following quantity

$$\sum_{a,b=0}^{\min\{n,n'\}}\left(\frac{\alpha'}{2}\right)^{n+n'-a-b}\frac{(-1)^{n+n'}}{(n-a)!(n-b)!(n'-a)!(n'-b)!}\frac{\partial^{2(n+n'-a-b)}\mathcal{A}_1(s,\mathbf{b})}{\partial b^{i_1}\ldots\partial b^{\ell_{n-b}}}$$
$$\times\,\alpha_{-1}^{i_1}\ldots\alpha_{-1}^{i_{n'-a}}\widetilde{\alpha}_{-1}^{j_1}\ldots\widetilde{\alpha}_{-1}^{j_{n'-b}}\alpha_1^{k_1}\ldots\alpha_1^{k_{n-a}}\widetilde{\alpha}_1^{\ell_1}\ldots\widetilde{\alpha}_1^{\ell_{n-b}}. \tag{5.78}$$

This has been written such that a and b will be seen to count the number of contractions between the polarisation tensors and they take on the range of values $a,b = 0,1,\ldots,\min\{n,n'\}$; naturally we must count how many ways these terms may be generated using the symmetry of the partial derivatives in the expansion of $\bar{\mathcal{A}}$. The result of this substitution is the amplitude shown below

$$\mathcal{A}_{n,n'}(s,\mathbf{b}) = K_{\rho_1\ldots\rho_{n'}\lambda_1\ldots\lambda_{n'}\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\sum_{a,b=0}^{\min\{n,n'\}}(-1)^{n+n'}\left(\frac{\alpha'}{2}\right)^{n+n'-a-b}C_{n,n'}(a)C_{n,n'}(b)$$
$$\times\,\frac{\partial^{2(n+n'-a-b)}\mathcal{A}_1(s,\mathbf{b})}{\partial b^{i_1}\ldots\partial b^{\ell_{n-b}}}\delta^{\rho_1 i_1}\ldots\delta^{\rho_{n'-a}i_{n'-a}}\delta^{k_1\mu_1}\ldots\delta^{k_{n-a}\mu_{n-a}}\delta^{\rho_{n'-a+1}\mu_{n'-a+1}}\ldots\delta^{\rho_{n'}\mu_n}$$
$$\times\,\delta^{\lambda_1 j_1}\ldots\delta^{\lambda_{n'-b}j_{n'-b}}\delta^{\ell_1\nu_1}\ldots\delta^{\ell_{n-b}\nu_{n-b}}\delta^{\lambda_{n'-b+1}\nu_{n'-b+1}}\ldots\delta^{\lambda_{n'}\nu_n}. \tag{5.79}$$

To see that this is indeed equivalent to equation (5.69), one can perform a Fourier transform on equation (5.79) to arrive at the expected result

$$A_{n,n'}(s,\mathbf{q}) = K_{n,n'}(\mathbf{q},\epsilon,G)A_1(s,\mathbf{q}), \tag{5.80}$$

where now the kinematic function coincides with the one given in equation (5.66) and $A_1(s,\mathbf{q})$ is as written in (5.11).

The amplitude written in momentum space has a simple structure; it is neatly factored into the graviton amplitude and some modifying kinematic function which is itself simply composed of a sum over all the ways one may saturate various contractions of the tensor (5.76) with the momentum $\mathbf{q}$. The amplitude written in impact parameter space, as in (5.79), has an index structure that is much more complicated than its Fourier transform in momentum space. However, this intricate structure is greatly simplified in the limit of very large impact parameters; in such a limit the function $\mathcal{A}_1$ takes the form

$$\mathcal{A}_1(s,\mathbf{b}) \sim s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\frac{R_p^{7-p}}{b^{6-p}} + i\frac{s\pi}{\Gamma\left(\frac{7-p}{2}\right)}\sqrt{\frac{\pi\alpha' s}{\ln\alpha' s}}\left(\frac{R_p}{\sqrt{2\alpha'\ln\alpha' s}}\right)^{7-p} e^{-\frac{b^2}{2\alpha'\ln\alpha' s}}, \tag{5.81}$$

where to reflect the change to coordinate space we choose to express the normalisation of the amplitude in terms of the scale R_p. To be more specific we will examine impact parameters for which $b \gg R_p \gg \sqrt{2\alpha'\ln\alpha' s}$, that is, those much larger than both the effective string length and the characteristic size of the Dp-branes; this being the case, we shall ignore the imaginary part of $\mathcal{A}_1(s,\mathbf{b})$ and focus on the real part. The significance of the imaginary part is that it describes amplitudes containing open strings as intermediate states (as illustrated in figure 5.2), the energy of which scales as b/α'. Provided that b is sufficiently large these processes won't contribute significantly and we can neglect the possibility of the production of open strings as external states. The result of these considerations is that the dominant contribution to $\mathcal{A}_{n,n'}$ comes from the term with the least number of derivatives in b. Without loss of generality let us assume that $n \le n'$ and denote the difference between the two by $\Delta = n' - n$, then this term is given by

$$\begin{aligned}\mathcal{A}_{n,n+\Delta}(s,\mathbf{b}) \sim\; & s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R_p^{7-p} K_{\rho_1\ldots\rho_{n+\Delta}\lambda_1\ldots\lambda_{n+\Delta}\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\left(-\frac{\alpha'}{2}\right)^{\Delta}\left(\frac{(n+\Delta)!}{\Delta!}\right)^2 \\ & \times \frac{\partial^{2\Delta} b^{-(6-p)}}{\partial b^{i_1}\ldots\partial b^{j_\Delta}}\delta^{\rho_1 i_1}\ldots\delta^{\rho_\Delta i_\Delta}\delta^{\rho_{\Delta+1}\mu_1}\ldots\delta^{\rho_{n+\Delta}\mu_n}\delta^{\lambda_1 j_1}\ldots\delta^{\lambda_\Delta j_\Delta}\delta^{\lambda_{\Delta+1}\nu_1}\ldots\delta^{\lambda_{n+\Delta}\nu_n}.\end{aligned} \tag{5.82}$$

The derivative above can be decomposed into a product consisting of an overall factor $\Delta!\, b^{-(6+2\Delta-p)}$ multiplied by some tensor which may be determined from the Gegenbauer polynomials in the

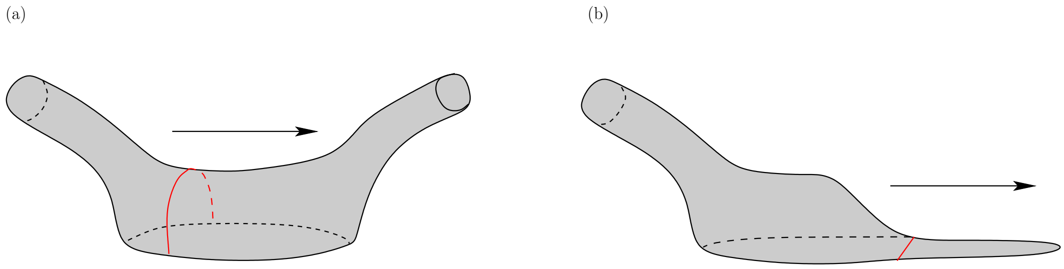

Figure 5.2: (a) The world-sheet describing the elastic scattering of a closed string from a D-brane can be sliced to indicate processes in which an internal open string propagates along the brane (b) These processes can be related to that in which the closed string is absorbed by the D-brane, leaving an open string excitation on the brane.

following way. The Gegenbauer polynomial $C_m^{(\lambda)}(x)$ is given in terms of the hypergeometric functions by

$$C_m^{(\lambda)}(x) = \binom{m+2\lambda-1}{m} {}_2F_1(-m, m+2\lambda; l+\tfrac{1}{2}; \tfrac{1}{2}(1-x)); \tag{5.83}$$

the tensor we are interested in is obtained by taking $C_{2\Delta}^{(6-p)/2}(x)$ and for each term in this polynomial we can attach the appropriate index structure by substituting $b_{i_r}/|b|$ for each factor of x, pairing the remaining indices up with Kronecker deltas and symmetrising the result. Since it will not be required for this analysis we shall make no attempt to write the generic form for this tensor and we shall instead write it as $K(\mathbf{b}, \epsilon, G)/(n!(n+\Delta)!)$ after taking the contractions with $K_{\rho_1\ldots\rho_{n'}\lambda_1\ldots\lambda_{n'}\mu_1\ldots\mu_n\nu_1\ldots\nu_n}$, this function can be thought of as a hyperspherical harmonic. Thus equation (5.82) becomes

$$\mathcal{A}_{n,n+\Delta}(s, \mathbf{b}) \sim s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R_p^{7-p} K(\mathbf{b}, \epsilon, G)\left(-\frac{\alpha'}{2}\right)^{\Delta} \frac{(n+\Delta)!}{n!\Delta!}\frac{1}{b^{6+2\Delta-p}}. \tag{5.84}$$

From this expression it is seen that these kinds of inelastic excitations of the string begin to contribute significantly to scattering processes for impact parameters $b \leq b_\Delta$ where

$$b_\Delta^{6+2\Delta-p} = \sqrt{\pi s}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R_p^{7-p}\left(\frac{\alpha'}{2}\right)^{\Delta}\frac{(n+\Delta)!}{n!\Delta!}. \tag{5.85}$$

The indication is that if one first considers scattering at impact parameters so large that only elastic scattering is relevant and then begins to reduce this impact parameter then the elastic channel will be gradually absorbed, first by small string transitions and by larger transitions as b is further decreased in size. However, care should be taken in drawing quantitative conclusions from the above result since it examines only a single particular inelastic channel, whereas any physical process would have many others which aren't considered by this analysis. These issues are discussed in more depth in terms of the full S-matrix in [35].

To give us some sort of quantitative notion of when these inelastic contributions should begin to be relevant we can determine the impact parameter b_D at which the work done by tidal forces acting on the string become of the same order as the energy required to excite oscillations on the string. These forces arises because points along the string follow neighbouring geodesics and there is a relative acceleration between two points on the string. For example, if we consider the string to be a rod extended along the direction X^μ then this acceleration is given by

$$a^\mu = -R^\mu_{\nu\rho\sigma} T^\nu T^\sigma X^\rho, \tag{5.86}$$

for the background with Riemann tensor $R^\mu_{\nu\rho\sigma}$ and vector T^μ tangent to the geodesic of the string. Provided our probe string remains at large distances from the brane relative to the geometry, $r \sim b_D \gg R_p$, the components of the Riemann tensor can be taken to be $c_p R_p^{7-p}/b^{9-p}$ since the Riemann tensor has units of inverse length squared and the first corrections to flat space begin at order R_p^{7-p}. Assuming that our string has a typical length of $\sqrt{\alpha'}$ then we find

$$a \sim c_p \sqrt{\alpha'} \frac{R_P^{7-p}}{b^{9-p}}. \tag{5.87}$$

For a very energetic string we may then write the tidal force acting on the string as

$$F \sim c_p \sqrt{\alpha'} E \frac{R_P^{7-p}}{b^{9-p}}, \tag{5.88}$$

or equally it could be expressed in terms of the potential energy

$$U \sim \frac{c_p}{8-p} \sqrt{\alpha'} E \frac{R_P^{7-p}}{b^{9-p}}. \tag{5.89}$$

When this energy is of the same order as the tension of the string, $U \sim \sqrt{\alpha'}$, then the oscillators of the string will become excited. This occurs at the impact parameter

$$b_D \sim \frac{c_p}{8-p} \alpha' E R^{7-p}. \tag{5.90}$$

5.5 High energy scattering at fixed angle

Having examined small-angle scattering at high energy in detail, it is illuminating to compare these results with an analysis of a different kinematic regime, that of fixed-angle scattering. A systematic method for studying high-energy string scattering amplitudes at fixed angles was developed by

Gross *et al.* towards the end of the 1980s [18, 19, 91] which was used to show that for these kinematics the behaviour of string scattering amplitudes is predominantly independent of both the theory and the particular particles involved. Furthermore this approach was then used [20] to derive a infinite number of linear relations between scattering amplitudes of string states with different spins and masses suggesting a broken symmetry which is restored in the high energy limit. The true nature of this symmetry remains poorly understood but it appears to be connected to the large number of zero-norm physical states in the string Hilbert space [92, 93].

The premise of the method employed by Gross is that each term in the perturbative expansion (2.92) is dominated by a saddlepoint solution at large energy. Each of these terms is a path integral over world-sheets of a fixed Euler characteristic and can be reduced to integrating the correlation function of n vertex operators over a slice of the finite dimensional moduli space of Riemann surfaces of the same topology. Now, common to all vertex operators is the factor $\exp[ik\cdot X]$ which, heuristically, we can consider to arise if we treat the amplitude as a wavefunction and require that this factor be present in order that it transform correctly under translation in spacetime. Because of this factor we can then write for the amplitude A of a particular world-sheet topology that

$$A(k_1,\ldots,k_n) \sim \prod_m \int d^2 z_m \sqrt{-g}\, f(z) \exp[-\sum_{i<j} k_i \cdot k_j G(z_i,z_j)], \tag{5.91}$$

where $G(z_i,z_j)$ is the scalar Green's function on a Riemann surface of this topology and $f(z)$ is generally some function of the world-sheet coordinates, particle momenta and polarisation. In the limit that all momenta become large the form of this exponential factor will allow us to perform the integrals by using the saddlepoint method, giving us the 'classical' solution. To do this we find the values of z_i and the moduli, $(\hat{z}_i, \hat{m})$, for which the exponent $\mathcal{E}$ takes an extremal value about which we may expand the exponent into a constant factor and a Gaussian factor. The integral is then easily solved to yield an amplitude of the form

$$A(k_1,\ldots,k_n) = F(\hat{z}_i,\hat{m}) \sqrt{\frac{1}{\mathcal{E}''}} e^{-\mathcal{E}(k_i,\hat{z}_i,\hat{m})} \tag{5.92}$$

where $\mathcal{E}''$ is the determinant of the matrix of second derivatives of $\mathcal{E}$ at the saddlepoint.

This is easily applied to the case of graviton $\to$ graviton scattering on the upper half of the complex plane [94], together with the inelastic amplitudes we have been considering in this chapter, where $G(z_i,z_j) = -2\alpha' \ln(z_i - z_j)$. All of these amplitudes may be written schematically as

$$A_{n,n'}(s,t) = \frac{\kappa T_p}{2} \int_0^1 d\omega\ (\omega-1)^{-\alpha' s}\, \omega^{-\alpha' \frac{t}{4}} F_{n,n'}(\omega), \tag{5.93}$$

which is just as in equation (5.37) but we have absorbed all dependency on the mass-level into the unspecified function $F_{n,n'}$ which is polynomial in the momenta s and t. Those parts of the integrand that have been specified explicitly may be expressed as an exponential $e^{\mathcal{E}}$ where

$$\mathcal{E}(\omega) = -\alpha' s \ln(\omega - 1) - \alpha' \frac{t}{4} \ln \omega. \tag{5.94}$$

In the limit that both $\alpha' s$ and $\alpha' t$ become extremely large, the integral in (5.93) is dominated by a local maximum in $\mathcal{E}$ located at

$$\omega_0 = \frac{\alpha' \frac{t}{4}}{\alpha' s + \alpha' \frac{t}{4}}, \tag{5.95}$$

as a result we can approximate this behaviour by expanding $\mathcal{E}$ about this point. Up to quadratic order in ω this expansion is

$$\mathcal{E}(\omega) \approx -\alpha' s \ln(\alpha' s) - \alpha' \frac{t}{4} \ln\left(\alpha' \frac{t}{4}\right) + \ln\left(\alpha' s + \alpha' \frac{t}{4}\right) + \frac{1}{2} \frac{\left(\alpha' s + \alpha' \frac{t}{4}\right)^3}{\alpha' s\, \alpha' \frac{t}{4}} (\omega - \omega_0)^2\,, \tag{5.96}$$

Since the integrand contains a Gaussian function which is strongly peaked around ω_0 we may approximate the integral by take its range to be over all real ω. As a result we find the high energy amplitude (5.93) may be written as

$$A_{n,n'}(s,t) \approx \frac{\kappa T_p}{2} e^{-\alpha' s \ln(\alpha' s) - \alpha' \frac{t}{4} \ln\left(\alpha' \frac{t}{4}\right) + \ln\left(\alpha' s + \alpha' \frac{t}{4}\right)} \times \int_{-\infty}^{+\infty} d\omega\, e^{\frac{1}{2} \frac{(\alpha' s + \alpha' t/4)^3}{\alpha' s\, \alpha' t/4} \omega^2} F_{n,n'}(\omega). \tag{5.97}$$

We can see that there is a soft exponential fall-off of the amplitude with energy, in stark contrast to the power law behaviour of point particle amplitudes. What's more, since this calculation represents closed strings being used as microscopic probes on the D-brane we can infer the 'thickness' of the D-brane as seen by the string from the decay length in the exponent, indicating that this thickness is of the order of the string length $\sqrt{\alpha'}$.

Chapter 6

Conclusions and Outlook

In this thesis two examples have been presented demonstrating the ways in which the scattering amplitudes of string theory can be used to compute quantities relating to gravitational phenomena. The first of these examined these phenomena at energies low relative to the string tension and was concerned with the description of objects resembling black holes in terms of the string world-sheet. The sense in which they resemble black holes is that these objects are D-brane configurations which are extremely massive and there should exist an entropy associated with them. In the second example we turn to phenomena with energies high relative to the string scale; for this we have examined the $1 \to 1$ scattering amplitudes of a class of massive closed strings which interacts with a D-brane.

In Chapter 4 we computed the tree-level amplitudes describing the emission of massless closed strings from a wrapped D-brane carrying a travelling wave, specifically we considered the D1-P and D5-P brane configurations. It was then shown that these amplitudes exactly reproduce the dominant contributions to the supergravity fields sourced by these D-brane systems at large distances. The CFT description treats the wave profile function f^i, which encodes the momentum of the D/P bound state, exactly. Thus the comparison between the string theory and supergravity results can be done at finite f^i in [33]. Together with the similar analysis of the bound state of D1-D5 branes [33], this represents further evidence that the connection between the microscopic and macroscopic descriptions of D-branes is retained by two-charge configurations.

Of course, given this association between solutions describing a two-charge solutions and their microscopic constituents, one might then ask whether the entropy of the two-charge solution can be obtained by the counting of microstates. According to the canonical Bekenstein-Hawking entropy this should vanish on account of the vanishing of the horizon area in the classical solution and this is clearly at odds with the world-sheet description which tells us that the phase space is nontrivial. It is possible that this can be reconciled by the proposal of [32] which states that the macroscopic entropy of a two-charge system should be defined by the sum of contributions of small black hole solutions and horizonless smooth solutions. The application of the scaling arguments of [32] to the duality frames we have considered determines that α'-corrections to the supergravity action do not yield a small black hole with nonzero horizon. In addition, we note that if we take the asymptotic values of the fields sourced by these brane configurations and then use the supergravity equations of motion to determine their values in the interior we obtain singular solutions; however, it is quite possible that one would find that if the equations of motion are modified by their α'-corrections then the resulting field configurations obtained from the same boundary values would be smooth throughout.

To further test the relationship between microstate counting and entropy in the context of black hole-like objects one could also extend our interest to systems with three charges, for these we expect to find nonzero horizons and so these can be used to represent macroscopic black holes. The analysis of [1] has in fact already been extended to include the bound system consisting of a D1- and D5-brane both carrying momentum in the form of a travelling null wave; these configurations present more of a challenge since the exact nature of the relationship between microstates and the classical supergravity solutions are not well understood, in this case the tree-level amplitudes were actually used to generate solutions to the linearised equations of motion for a simple class of microstates. These solutions demonstrated multipole moments not present in the black hole solution, lending credence to the proposal that individual microstates source fields with non-trivial multipole moments while the black hole solution is given by taking an average over an appropriate thermodynamic ensemble, thereby eliminating these moments [95–98]. In the future it would be useful to understand more about typical microstates of the D1-D5-P systems such that we can see what sort of backreactions they generate.

The second example concerning high energy scattering was discussed in Chapter 5. The dominant contribution to the tree-level scattering amplitude describing the scattering of a string of mass M_i from a D-brane into a string of mass M_f was calculated. This result was obtained with the leading Regge trajectory of states which are NS-NS states with spin $J = (\alpha' M^2 + 4)/2$ in mind, however, the result (5.69) actually holds for a broader class of states with spins which may be smaller than J and are described by the vertices (2.85) if we relax the condition that the polarisation ϵ be symmetric in all indices, instead we keep the lesser constraint that they be symmetric in holomorphic and antiholomorphic indices. This result was compared with the same quantity calculated as as a matrix element of the eikonal operator and they were proven to be identical.

This provides a nontrivial confirmation that the leading eikonal operator provides a complete semi-classical description of the leading energy contributions to string-brane scattering for small angles. The usefulness of this quantity in checking the accuracy of string diagrams with loops should be quite general since in the factorisation of these diagrams into tree-level diagrams these must then be ‘glued’ together using the sum over intermediate states, which necessarily must include many massive states and thus tree-level amplitudes of this kind.

The results of Chapter 5 have also found use recently in the analysis of singularities in the acceleration of neighbouring geodesics, as defined by (5.86), arising from tidal forces acting on strings propagating in a background given by the Lifshitz metric [86]. In that work the proposal that this singularity could be resolved by taking into account inelastic excitations of the string as it moves towards the IR was tested. This was accomplished using the fact that the Lifshitz metric can be considered to arise from a density of D0-branes and thus a string travelling through this spacetime could be described as interacting with these D0-branes by the cross-section obtained from (5.69) provided that that the density was sufficiently small. This revealed that these interactions cause excitations of the string, increasing its mass and slowing it sufficiently to avoid the divergent tidal forces.

It would be interesting to extend the techniques presented in Chapter 5 to the case of world-sheets with two boundaries. In principle we already have a simple prescription for using the eikonal operator that would allow us to compute the leading one-loop contribution to this process which is given by the quadratic term arising from the exponential in (5.25). If pomeron vertex operator methods can be extended to generate such corrections as well, then it would be interesting to see whether the equivalence between these two approaches continues to hold. However, since the pomeron vertex operator method generates its $\alpha' s$ dependency purely from the OPE of the original two string vertices, it is reasonable to believe that for world-sheets with higher numbers of borders they will generate subleading divergences of the same order of the tree-level amplitude. Therefore it seems likely that the pomeron vertex operator may tell us something nontrivial about the corrections to the eikonal operator; since in [35] all expected divergences were accounted for in the elastic amplitude it may be that the OPE methods will give contributions from the inelastic processes and that these contributions vanish after the sum over spin structure in the case of elastic amplitudes. If it is possible to extract the sub-leading divergences in energy at this level then one could extract the first classical correction to the eikonal operator, a vital step if we wish to understand what happens as the impact parameter of the string probe is reduced. If this could be extended further it may even be possible to describe processes in which the string is captured by the D-brane target, offering the tools to consider the microscopic description of matter absorption by a black hole.

Appendix A

Differential forms

A.1 Basic definitions

Antisymmetric tensor fields are found extensively throughout both string theory and supergravity and so it is worth investing time in outlining a slick notation to allow us to right down relations describing them in a compact manner. A *differential form of rank* k, or k*-form*, is a totally antisymmetric tensor field $T_{\mu_1\ldots\mu_k} = T_{[\mu_1\ldots\mu_k]}$. We can write this by using an algebra of anticommutating coordinate differentials dx^μ as

$$T = \frac{1}{k!} T_{\mu_1\ldots\mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}. \tag{A.1}$$

We define the *exterior derivative* of a k-form as being the $(k+1)$-form with components

$$(dT)_{\mu_1\ldots\mu_{k+1}} = (k+1)\partial_{[\mu_1} T_{\mu_2\ldots\mu_{k+1}]}; \tag{A.2}$$

while the product of a p-form A_p and a q-form B_q is a $(p+q)$-form denoted by $A_p \wedge B_q$ with the components

$$(A_p \wedge B_q)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1\ldots\mu_p} B_{\mu_{p+1}\ldots\mu_{p+q}}. \tag{A.3}$$

Owing to the antisymmetry is easy to convince ourselves that these have the properties $d^2 = 0$ and $A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p$.

Differential forms are the natural objects to integrate over differential manifolds; since the transformation of the tensor will cancel that of the measure, upon specification of an orientation we can write the coordinate invariant integral of a k-form T_k over a k-dimensional manifold,

$$\int d^k x T_{01\ldots k-1} \equiv \int T_k. \tag{A.4}$$

For a manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$ we have an extremely powerful tool in Stokes' theorem which relates the integral of the exterior derivative of a $(k-1)$-form over the k-dimensional space to the integral of form over the boundary,

$$\int_{\mathcal{M}} dT_{k-1} = \int_{\partial\mathcal{M}} T_{k-1}. \tag{A.5}$$

Note that these integrals are well defined without any reference to a metric. This property is one aspect of differential forms which makes them so useful, since they can encode very basic information about the topology of the manifold on which they are defined.

A.2 Hodge duality

The algebra of coordinate differentials is such that the space of independent k-forms on a n-dimensional manifold is equal to that of the $(n-k)$-forms. Given a metric on the manifold we may define a map called *Hodge duality* which takes us between these spaces. This then allows the integration of k-forms on manifolds of any dimension since the product of a k-form with its dual will yield a n-form.

To achieve this consider a Lorentzian manifold $\mathcal{M}$ of dimension n with a metric $g_{\mu\nu}$. We choose a basis of vectors, $\{e^\mu_a\}$, at each point in $\mathcal{M}$ to define a local Lorentz frame*; this will satisfy

$$e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab}. \tag{A.6}$$

There will also exist a dual basis of one-forms, which we denote e^a_μ, such that $e^a_\mu e^\mu_b = \delta^a_b$. In addition to these we define the Levi-Civita tensor with local Lorentz indices in n dimensions, $\epsilon^{a_1\ldots a_n}$ by

$$\epsilon^{01\ldots(n-1)} = 1 \tag{A.7}$$

with all other components determined by total antisymmetry in its indices. Similarly, with all indices lowered we have

$$\epsilon_{01\ldots(n-1)} = \eta_{0a_1}\cdots\eta_{(n-1)a_n}\epsilon^{a_1\ldots a_n} = -1. \tag{A.8}$$

Hence, with this we can write

$$\epsilon^{\mu_1\ldots\mu_n} = \sqrt{-g}\epsilon^{a_1\ldots a_n} e^{\mu_1}_{a_1}\cdots e^{\mu_n}_{a_n} \tag{A.9}$$

*We will take the signature such that $\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$

Defined in this way, the coordinate component expression for the Levi-Civita tensor will take the same values as its Lorentz frame counterpart and will be independent of the metric. This is not true when the indices are lowered for which we find that $\epsilon_{01\ldots(n-1)} = g$. Products of the Levi-Civita tensor are frequently encountered when dealing with k-forms and so the following property is indispensable,

$$\epsilon^{\mu_1\ldots\mu_k\rho_1\ldots\rho_{n-k}}\epsilon_{\mu_1\ldots\mu_k\sigma_1\ldots\sigma_{n-k}} = k!(n-k)!g\,\delta^{\rho_1}_{[\sigma_1}\ldots\delta^{\rho_{n-k}}_{\sigma_{n-k}]}. \tag{A.10}$$

With the technology noted above we can define the Hodge dual using the Levi-Civita tensor, on an n-dimensional manifold the Hodge dual of a k-form is an $(n-k)$-form with components given by

$$({}^\star T)_{\mu_1\ldots\mu_{n-k}} = \frac{1}{k!\sqrt{-g}}\epsilon_{\mu_1\ldots\mu_{n-k}\nu_1\ldots\nu_k}T^{\nu_1\ldots\nu_k}. \tag{A.11}$$

Using this we can further define the inner product between two k-forms as

$$(A_p, B_p) \equiv \int A_k \wedge {}^\star B_k = \int d^n x\sqrt{-g}\frac{1}{k!}A_{\mu_1\ldots\mu_k}B^{\mu_1\ldots\mu_k}. \tag{A.12}$$

A.3 Integration on manifolds and submanifolds

To integrate over a manifold and its submanifolds we must define their volume forms. For a n-dimensional manifold we define for the $(n-k)$-dimensional submanifold the volume form

$$d^{n-k}\Sigma_{\mu_1\ldots\mu_k} = dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_{n-k}}\frac{1}{(n-k)!\sqrt{-g}}\epsilon_{\nu_1\ldots\nu_{n-k}\mu_1\ldots\mu_k}. \tag{A.13}$$

Up to a sign this reproduces the standard invariant volume element for the manifold,

$$d^n\Sigma = -d^n x\sqrt{-g}. \tag{A.14}$$

By contracting the $(n-k)$-volume form with a totally antisymmetric rank k tensor $T^{\mu_1\ldots\mu_k}$ we can construct an $(n-k)$-form which can be integrated over an $(n-k)$-dimensional submanifold. This $(n-k)$-form is given by

$$\frac{1}{k!}d^{n-k}\Sigma_{\mu_1\ldots\mu_k}T^{\mu_1\ldots\mu_k} = {}^\star T. \tag{A.15}$$

We can also contract the $(n-k+1)$-volume form with the divergence of this tensor, which gives

$$\frac{1}{(k-1)!}d^{n-k+1}\Sigma_{\mu_1\ldots\mu_{k-1}}\nabla_\rho T^{\rho\mu_1\ldots\mu_{k-1}} = (-1)^{n-k}d^\star T. \tag{A.16}$$

We may further generalise Stokes' theorem from earlier so that it states that if we are integrating over a volume M_{n-k+1} on an $(n-k+1)$-dimensional submanifold with an $(n-k)$-dimensional boundary ∂M_{n-k+1}, then

$$\int_{M_{n-k+1}} d^\star T = \int_{\partial M_{n-k+1}} {}^\star T. \tag{A.17}$$

Writing this in terms of the volume forms this becomes

$$\int_{M_{n-k+1}} d^{n-k+1}\Sigma_{\mu_1\ldots\mu_{k-1}}\nabla_\rho T^{\rho\mu_1\ldots\mu_{k-1}} = \frac{(-1)^{n-k}}{k}\int_{\partial M_{n-k+1}} d^{n-k}\Sigma_{\mu_1\ldots\mu_k}T^{\mu_1\ldots\mu_k}. \quad \text{(A.18)}$$

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