Gravity as the Square of Gauge Theory

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Amplitudes 2010, Queen Mary University of London

based mainly on arXiv:1004.0693 with Zvi Bern, Tristan Dennen, Yu-tin Huang

How to Square Gauge Theory?

KLT relations

- 4-point:

$$\mathcal{M}_4(1,2,3,4) = -is_{12}A_4(1,2,3,4)\tilde{A}_4(1,2,4,3)\,.$$

- 5-point:

$$\begin{aligned}\mathcal{M}_5(1,2,3,4,5) =& is_{12}s_{34}A_5(1,2,3,4,5)\tilde{A}_5(2,1,4,3,5)\\ &+ is_{13}s_{24}A_5(1,3,2,4,5)\tilde{A}_5(3,1,4,2,5)\,.\end{aligned}$$

- *n*-point?

How to Square Gauge Theory?

KLT relations, *n*-point

$$\mathcal{M}_n(1,2,\ldots,n) = i(-)^{n+1}\Big[A_n(1,2,\ldots,n)\sum_{\text{perms}} f(i_1,\ldots,i_j)\bar{f}(l_1,\ldots,l_j)$$
$$\tilde{A}_n(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_j,n)\Big]$$
$$+\,\mathcal{P}(2,\ldots,n-2)\,.$$

with

$$\{i_1,\ldots,i_j\}\in\mathcal{P}(2,\ldots,\lfloor n/2\rfloor)\,,\quad \{l_1,\ldots,l_{j'}\}\in\mathcal{P}(\lfloor n/2\rfloor+1,\ldots,n-2)\,.$$

and

$$f(i_1,\ldots,i_j) = s_{1,i_j}\prod_{m=1}^{j-1}\left(s_{1,i_m}+\sum_{k=m+1}^{j} g(i_m,i_k)\right)\,,$$

where $g(i,j) = s_{ij}$ for $i > j$ and $g(i,j) = 0$ otherwise.

How to Square Gauge Theory?

Problem

KLT relations in this form express the unordered gravity amplitude in terms of color-ordered gauge theory amplitudes!

Possible Solutions

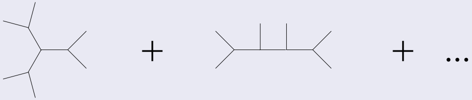
- Use “ordered” gravity amplitudes
  $\Rightarrow$ Drummond, Spradlin, Volovich, Wen [arXiv:0901.2363]
- Use full unordered gauge-theory amplitude
  $\Rightarrow$ Bern, Carrasco, Johansson [arXiv:0805.3993]

BCJ duality

The gauge theory amplitude can be written as

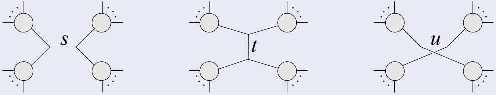
$$\mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i\, c_i}{\prod s_{\alpha_i}}\,,$$

- diagrams i only contain cubic vertices:



- numerators n_i satisfy Jacobi-like relations (“BCJ duality”):

$$c_i + c_j + c_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0\,.$$



How to Square Gauge Theory?

conjectured BCJ squaring relations

Gauge theory amplitudes

$$\mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i\, c_i}{\prod s_{\alpha_i}}\,, \qquad \tilde{\mathcal{A}}_n = \sum_{\text{diags. } i} \frac{\tilde{n}_i\, c_i}{\prod s_{\alpha_i}}$$

with numerators satisfying Jacobi-like relations:

$$c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0\,.$$

Gravity amplitude:

$$-i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

- Why do these squaring relations hold?
- What are the implications?

Important related work

Stringy approach/generalizations of BCJ (see talk of Vanhove)

- Bjerrum-Bohr, Damgaard, Vanhove [arXiv:0907.1425]
- Tye, Zhang [arXiv:1003.1732]
- Bjerrum-Bohr, Vanhove [arXiv:1003.2396]
- Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove [arXiv:1003.2403]

Applications of Squaring Relations at loop level (see talk of Carrasco)

- Bern, Carrasco, Johansson [arXiv:1004.0476]
- Vanhove [arXiv:1004.1392]

Generalized Gauge Invariance

Generalized Gauge Transformations

Gauge theory amplitude

$$\mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i \, c_i}{\prod s_{\alpha_i}}$$

is invariant under

$$n_i \quad \rightarrow \quad n_i + \Delta_i$$

with

$$\sum_{\text{diags. } i} \frac{\Delta_i \, c_i}{\prod s_{\alpha_i}} \; = \; 0\,.$$

- Δ_i "move around" contact terms, can be local or non-local
- Preserves Jacobi-like relations if

$$\Delta_i + \Delta_j + \Delta_k \; = \; 0\,.$$

Generalized Gauge Invariance

The Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\,\tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Generalized Gauge Transformation of the Squaring Relations

$$n_i \quad \rightarrow \quad n_i + \Delta_i \quad \text{with} \quad \sum_{\text{diags. } i} \frac{\Delta_i\,c_i}{\prod s_{\alpha_i}} \;=\; 0\,, \quad \Delta_i + \Delta_j + \Delta_k = 0$$

Gravity amplitude transforms as

$$-i\mathcal{M}_n \quad \rightarrow \quad -i\mathcal{M}_n + \sum_{\text{diags. } i} \frac{\Delta_i\,\tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Generalized Gauge Invariance

The Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Consistency Requirement

If Δ_i, $\tilde{n}_i$ satisfy Jacobi-like relations:

$$\sum_{\text{diags. } i} \frac{\Delta_i\, c_i}{\prod s_{\alpha_i}} = 0 \qquad \Rightarrow \qquad \sum_{\text{diags. } i} \frac{\Delta_i\, \tilde{n}_i}{\prod s_{\alpha_i}} = 0\,.$$

Origin: c_i are color factors any gauge group

$\Rightarrow$ identity only relies on algebraic properties of c_i

$\Rightarrow$ must work for $c_i \to \tilde{n}_i$

$\Rightarrow$ Δ_i actually don't need to satisfy Jacobi-like relations!

Deriving the Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

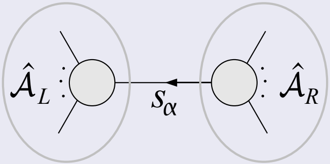
Strategy

- Squaring relations trivial at 3-point:

$$-i\,\mathcal{M}_3 = A_3 \times \tilde{A}_3\,.$$

- Proceed inductively, using on-shell recursion relations

$$\mathcal{A}_n = \sum_{\alpha} \hat{\mathcal{A}}_L \frac{i}{s_\alpha} \hat{\mathcal{A}}_R\,, \qquad \mathcal{M}_n = \sum_{\alpha} \hat{\mathcal{M}}_L \frac{i}{s_\alpha} \hat{\mathcal{M}}_R\,,$$



Deriving the Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Assumptions

- A local choice of n_i exists such that

$$\mathcal{A}_n = \sum_{\text{diags. } i} \frac{n_i\, c_i}{\prod s_{\alpha_i}}\,, \qquad n_i + n_j + n_k = 0\,.$$

- Complex on-shell deformations of momenta

$$p_a \to \hat{p}_a(z) = p_a + z q_a\,, \qquad p_a \cdot q_a = q_a^2 = 0$$

exist such that

$$\hat{\mathcal{M}}_n(z) \to 0\,, \qquad \hat{\mathcal{A}}_n(z) \to 0\,, \qquad \hat{\tilde{\mathcal{A}}}_n(z) \to 0 \qquad \text{as} \qquad z \to \infty\,.$$

(BCFW particularly suitable)

Deriving the Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\,\tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

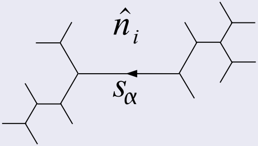
Gauge theory

Expand amplitude in terms of residues: $$\mathcal{A}_n = \sum_\alpha \frac{\hat{\mathcal{A}}_n^\alpha}{s_\alpha}\,.$$

Residues are well-defined, gauge-invariant. Two ways to compute them:

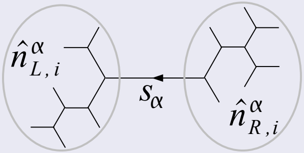
- directly from amplitude:

$$\hat{\mathcal{A}}_n^\alpha = \sum_{\alpha\text{-diags. } i} \frac{\hat{n}_i(z_\alpha)c_i}{\prod \hat{s}_{\alpha_i}(z_\alpha)} = \sum$$



- from the recursion relation:

$$\hat{\mathcal{A}}_n^\alpha = \sum_{\alpha\text{-diags. } i} \frac{i\,\hat{n}_{L,i}^\alpha\,\hat{n}_{R,i}^\alpha\,c_i}{\prod \hat{s}_{\alpha_i}(z_\alpha)} = \sum$$



Deriving the Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

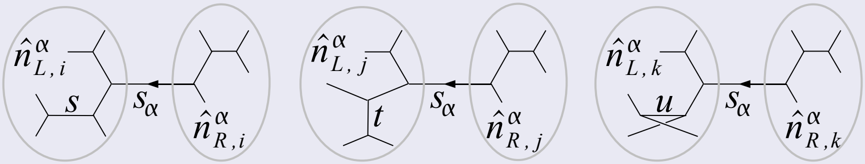
Gauge theory

Representations A_n^α can differ by a generalized gauge transformation:

$$\boxed{\hat{n}_i(z_\alpha) = i\, \hat{n}^\alpha_{L,i}\, \hat{n}^\alpha_{R,i} + \Delta^\alpha_i} \qquad \sum_{\alpha\text{-diags. } i} \frac{\Delta^\alpha_i c_i}{\prod \hat{s}_{\alpha_i}(z_\alpha)} = 0\,.$$

Generalized gauge transformation preserves Jacobi-like identities:

$$\Delta^\alpha_i + \Delta^\alpha_j + \Delta^\alpha_k = 0\,.$$



Deriving the Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Gravity

Expand gravity amplitude using recursion relation:

$$\mathcal{M} = \sum_\alpha \frac{i}{s_\alpha} \hat{\mathcal{M}}_L(z_\alpha) \hat{\mathcal{M}}_R(z_\alpha) = \sum_\alpha \frac{i}{s_\alpha} \sum_{\alpha\text{-diags. } i} \frac{\left[i\, \hat{n}^\alpha_{L,i}\, \hat{\tilde{n}}^\alpha_{L,i}\right]\left[i\, \hat{n}^\alpha_{R,i}\, \hat{\tilde{n}}^\alpha_{R,i}\right]}{\prod \hat{s}_{\alpha_i}(z_\alpha)}\,.$$

Recalling $\hat{n}_i(z_\alpha) = i\, \hat{n}^\alpha_{L,i}\, \hat{n}^\alpha_{R,i} + \Delta^\alpha_i$, we have

$$\mathcal{M}_n = \sum_\alpha \frac{i}{s_\alpha} \sum_{\alpha\text{-diags. } i} \left[\frac{\hat{n}_i(z_\alpha) \hat{\tilde{n}}_i(z_\alpha)}{\prod \hat{s}_{\alpha_i}(z_\alpha)} - \frac{\Delta^\alpha_i\, \hat{\tilde{n}}_i(z_\alpha) + \tilde{\Delta}^\alpha_i\, \hat{n}_i(z_\alpha)}{\prod \hat{s}_{\alpha_i}(z_\alpha)} + \frac{\Delta^\alpha_i \tilde{\Delta}^\alpha_i}{\prod \hat{s}_{\alpha_i}(z_\alpha)} \right].$$

Deriving the Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\,\tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Gravity

$$-i\mathcal{M}_n = \sum_{\alpha} \frac{i}{s_\alpha} \sum_{\alpha\text{-diags. } i} \frac{\hat{n}_i(z_\alpha)\hat{\tilde{n}}_i(z_\alpha)}{\prod \hat{s}_{\alpha_i}(z_\alpha)} \stackrel{?}{=} \sum_{\text{diags. } i} \frac{n_i\,\tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Could differ by a function that is

- local
- gauge-invariant
- dimension $(\text{momentum})^2$

No such function can exist!

$$-\,i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\,\tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

The Squaring Relations from a Lagrangian Viewpoint

Motivation

- Amplitudes computed from the ordinary YM Lagrangian
  do not satisfy Jacobi-like relations!
- Can Jacobi-like relations

$$n_i + n_j + n_k = 0$$

  arise from a Lagrangian?
- In what sense is

$$\mathcal{L}_{\text{gravity}} \;=\; (\mathcal{L}_{\text{gauge}})^2 \quad ?$$

The Squaring Relations from a Lagrangian Viewpoint

Ordinary $\mathcal{L}_{\text{YM}}$ does not lead to BCJ-compatible amplitudes

Strategy

- Expand gauge theory Lagrangian as

$$\mathcal{L} \;=\; \mathcal{L}_{\text{YM}} + \mathcal{L}_5 + \mathcal{L}_6 + \ldots$$

- Determine $\mathcal{L}_n$, $n \geq 5$ to make Jacobi-like relations manifest
- $\mathcal{L}_n$, $n \geq 5$ must not alter amplitudes!
- Use auxiliary fields to turn Lagrangian cubic
- Square cubic interactions in momentum space $\Rightarrow \quad \mathcal{L}_{\text{gravity}}$

The Squaring Relations from a Lagrangian Viewpoint

5-point

- No covariant, local $\mathcal{L}_5$ can ensure Jacobi
- Instead:

$$\mathcal{L}_5 = -\tfrac{1}{2}g^3\left(f^{a_1a_2b}f^{ba_3c} + f^{a_2a_3b}f^{ba_1c} + f^{a_3a_1b}f^{ba_2c}\right)f^{ca_4a_5} \times \partial_{[\mu}A^{a_1}_{\nu]}A^{a_2}_{\rho}A^{a_3\mu}\frac{1}{\Box}(A^{a_4\nu}A^{a_5\rho})\,.$$

- non-local and vanishing by Jacobi-identity
- can add one "self-BCJ" term:

$$\Delta\mathcal{L}_5 \propto g^3 f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}\Big(\partial_{(\mu}A^{a_1}_{\nu)}A^{a_2}_{\rho}A^{a_3\mu} + \partial_{(\mu}A^{a_2}_{\nu)}A^{a_3}_{\rho}A^{a_1\mu} + \partial_{(\mu}A^{a_3}_{\nu)}A^{a_1}_{\rho}A^{a_2\mu}\Big)\frac{1}{\Box}(A^{a_4\nu}A^{a_5\rho})\,.$$

- introducing auxiliary fields $\Rightarrow$ local and cubic

The Squaring Relations from a Lagrangian Viewpoint

6-point

- $\mathcal{L}_5$ not sufficient to ensure Jacobi at 6-point
- $\mathcal{L}_6$ contains terms of the form

$$\frac{1}{\Box}(\partial A^{a_1} A^{a_2} A^{a_3})\frac{1}{\Box}(A^{a_4} A^{a_5})\partial A^{a_6}, \quad \frac{1}{\Box}(A^{a_1} A^{a_2})\partial A^{a_3}\frac{1}{\Box}(\partial A^{a_4} A^{a_5})A^{a_6}!, \ldots$$

- $\mathcal{L}_6$ vanishes by Jacobi-identity
- 30 different "self-BCJ" terms
  $\Rightarrow$ BCJ seems easy to satisfy at tree-level

n-point: General structure

- need to add new "vanishing" terms $\mathcal{L}_n$ for all n
- full local cubic Lagrangian $\Rightarrow$ infinitely many auxiliary fields
- Non-polynomial structure not surprising: gives covariant $\mathcal{L}_{\text{gravity}}$!
- To find general $\mathcal{L}_n$: systematic approach? symmetry principle?

Asymmetric Squaring Relations

BCJ Squaring Relations

$$n_i + n_j + n_k = 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

Gauge Transformation

$$n_i \to n_i + \Delta_i\,, \qquad \Delta_i + \Delta_j + \Delta_k \neq 0 \qquad \Rightarrow \qquad \sum_{\text{diags. } i} \frac{\Delta_i\, \tilde{n}_i}{\prod s_{\alpha_i}} = 0\,.$$

Generalized "Asymmetric" Squaring Relations

$$n_i + n_j + n_k \neq 0\,, \quad \tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0 \qquad \Rightarrow \qquad -i\mathcal{M}_n = \sum_{\text{diags. } i} \frac{n_i\, \tilde{n}_i}{\prod s_{\alpha_i}}\,.$$

see also: BCJ [arXiv:1004.0476]; other 5-point generalizations: BDSV [arXiv:1003.2403]

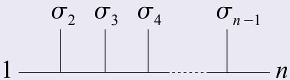
New Representations of Gauge and Gravity Amplitudes

Color-decomposition of Gauge Theory Amplitudes

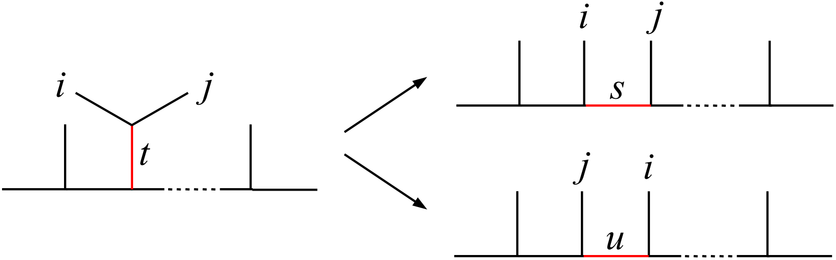
Del Duca, Dixon, and Maltoni [hep-ph/9910563]

$$\mathcal{A}_n(1,2,\ldots,n) = \sum_{\sigma\in S_{n-2}} c_{1,\sigma_2,\ldots,\sigma_{n-1},n} A_n(1,\sigma_2,\ldots,\sigma_{n-1},n)$$

$c_{1,\sigma_2,\ldots,\sigma_{n-1},n} \quad \longleftrightarrow$



Only relies on algebraic properties of color factors!



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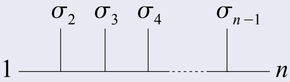
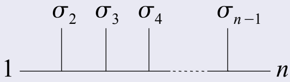


Only relies on algebraic properties of color factors!

Dual Decomposition of Gauge Theory Amplitudes

$$\mathcal{A}_n(1,2,\ldots,n) = \sum_{\sigma\in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} A_n^{\text{scalar}}(1,\sigma_2,\ldots,\sigma_{n-1},n)$$

$n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \quad \longleftrightarrow$



New Representations of Gauge and Gravity Amplitudes

Color-decomposition of Gauge Theory Amplitudes

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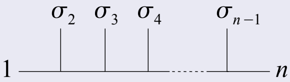
$$\mathcal{A}_n(1,2,\ldots,n) = \sum_{\sigma \in S_{n-2}} c_{1,\sigma_2,\ldots,\sigma_{n-1},n} A_n(1,\sigma_2,\ldots,\sigma_{n-1},n)$$

$$c_{1,\sigma_2,\ldots,\sigma_{n-1},n} \quad \longleftrightarrow$$

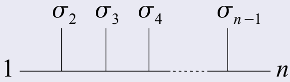


Only relies on algebraic properties of color factors!

New Representation of Gravity Amplitudes

$$\mathcal{M}_n(1,2,..,n) = \sum_{\sigma \in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \tilde{A}_n(1,\sigma_2,\ldots,\sigma_{n-1},n)$$

$$n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \quad \longleftrightarrow$$



Explicit Expressions for Numerators

- Numerators n_i can be explicitly constructed for 5-points, 6-points,...
  $\Rightarrow$ Brute force construction, still rather mysterious
- Is there an explicit, theory-independent, all-order expression for n_i?

Recall: New Representation of Gravity Amplitudes

$$\mathcal{M}_n = i \sum_{\sigma \in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \times \tilde{A}_n(1,\sigma_2,\ldots,\sigma_{n-1},n)$$

Recall: KLT

$$\mathcal{M}_n = i\Big[(-)^{n+1}\sum_{\text{perms}} f(i_1,..i_j)\bar{f}(l_1,..l_j)A_n(i_1,..i_j,1,n\text{-}1,l_1,\ldots,l_j,n)\Big]\tilde{A}_n(1,..n\text{-}1,n) + \mathcal{P}(2,..,n\text{-}2)$$

Explicit Expressions for Numerators

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Recall: New Representation of Gravity Amplitudes

$$\mathcal{M}_n = i \sum_{\sigma \in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \times \tilde{A}_n(1,\sigma_2,\ldots,\sigma_{n-1},n)$$

Recall: KLT

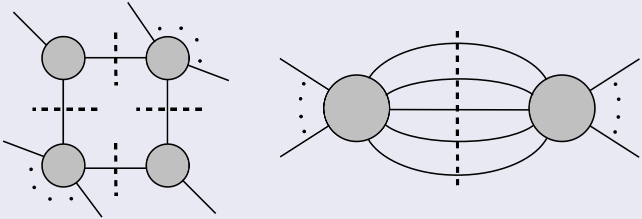
$$\mathcal{M}_n = i \underbrace{\left[(-)^{n+1} \sum_{\text{perms}} f(i_1,..i_j)\bar{f}(l_1,..l_j) A_n(i_1,..i_j,1,n\text{-}1,l_1,\ldots,l_j,n)\right]}_{n_{1,\ldots,n\text{-}1,n}} \tilde{A}_n(1,..n\text{-}1,n) + \mathcal{P}(2,..,n\text{-}2)$$

This representation is non-local and ideal: $n_{1,\sigma_1,\ldots,\sigma_{n\text{-}1},n} = 0$ for $\sigma_{n\text{-}1} \neq n\text{-}1$

The Squaring Relations at Loop Level

KLT at loop level

- KLT used in unitarity cuts for tree subamplitudes:



- Only applicable on the cut, and different for each cut
- Of practical importance, but no loop-level KLT relation

The Squaring Relations at Loop Level

Squaring relations at loop level

see also: Bern, Carrasco, Johansson [arXiv:1004.0476], and Talk by J. J. Carrasco

- through the unitarity method, tree derivation generalizes to loop level
- large z behavior $\leftrightarrow$ cut-constructability
  $\Rightarrow$ No issue if cuts are carried out in D dimensions
- assumption: numerators arranged to satisfy Jacobi-like relations:

$$(-i)^L \mathcal{A}_n^{L\text{-loop}} = \sum_{\text{diags. } i} \int \prod_{a=1}^{L} \frac{d^D l_a}{(2\pi)^D} \frac{n_i(l_1, \ldots, l_L)\, c_i}{\prod s_{\alpha_i}(l_1, \ldots, l_L)}, \quad n_i + n_j + n_k = 0\,.$$

Then:

$$(-i)^{L+1} \mathcal{M}_n^{L\text{-loop}} = \sum_{\text{diags. } i} \int \prod_{a=1}^{L} \frac{d^D l_a}{(2\pi)^D} \frac{n_i(l_1, \ldots, l_L)\, \tilde{n}_i(l_1, \ldots, l_L)}{\prod s_{\alpha_i}(l_1, \ldots, l_L)},$$

- holds for arbitrary loop momenta (with internal lines off-shell)
- A universal relation, not a different one for each cut

Summary and Outlook

Summary

- Origin of Squaring Relations understood from a QFT perspective
- Squaring implemented at a Lagrangian level
- Better understanding of "gravity=(gauge)^2" (for trees and loops)
- Various useful new expressions for gauge and gravity amplitudes
- Explicit (but non-local!) Jacobi-satisfying numerators

Open problems

- Simple, explicit expression for local, Jacobi-satisfying numerators
- Better understanding of BCJ at loop level
- Can we see BCJ in the Grassmannian for planar $\mathcal{N} = 4$ SYM? (reconcile manifest locality with manifest planarity)
- Implications for the UV finiteness of $\mathcal{N} = 8$ supergravity
- Non-perturbative analog of BCJ?