PhD Thesis

String Techniques for the Computation of the Effective Action in Brane World Models

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ABSTRACT

The present thesis contains the development of new techniques to compute the Yukawa couplings for the fundamental matter in the context of string theoretic Brane World Models. Exploiting the unitarity of String Theory these couplings are derived by means of factorisation of the classical limit of the two-loop twisted partition function in generic toroidal compactifications. In fact by taking the Schwinger time of the strings propagating in the loops to infinity, the string theoretic diagram is reduced to a field theoretic interaction involving the low-lying states of oscillations of the so-called twisted strings. These are strings stretched between two D-Branes either intersecting at angles or with different magnetic fields turned on in their world-volumes and they reproduce four dimensional chiral fermions in the low energy limit of the theory. The models involving intersecting or magnetised D-Branes are related to one another by means of T-Duality. The factorisation of the resulting diagram yields three propagators and two complex conjugate copies of the sought Yukawa couplings, whose moduli dependence (both on the open and on the closed string moduli) is determined in full generality in toroidal compactifications of the models. The actual partition function is computed, by using the conformal properties of String Theory, in the closed string channel, in which it becomes a tree-level interaction between three closed strings ending on magnetised D-Branes. The computation performed is generalised to the case of $g+1$ external states and it corresponds to the g-loop twisted partition function in the open string channel. The fundamental ingredients for the calculation are the Reggeon vertex for the emission of $g+1$ off-shell closed strings and the boundary state for a wrapped magnetised D-Brane. A careful analysis of the phase factors involved in determining the form of these objects in toroidal compactifications leads also to a generalisation of the known T-Duality rules.

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1. INTRODUCTION

String Theory is today considered one of the most promising candidates to provide a framework for the embedding of all of the fundamental interactions in nature in a unified coherent quantum and relativistic description. As it is well-known its fundamental ingredients are one dimensional quantised objects (the strings) that must propagate in a ten dimensional target space-time in order for the theory to be consistent and well defined. Initially constructed to reproduce the results of the cross sections for scattering processes involving strongly interacting particles known as hadrons, string theory was only later discovered to have some features that could yield a new interpretation as a possible theory of quantum gravity. The main reason for this shift in the energy scale of application of the theory was the presence, in the low-energy spectrum of oscillations of closed strings, of a spin-2 particle, initially addressed to as the "pomeron", which was then identified with the graviton. Thus, while much of the theoretical effort to understand the physics of the high energy interactions between the known fundamental particles was leading to the construction of the so called Standard Model of Particle Physics, on the other hand String Theory was starting to be valued as a possible description of all of the forces of nature, including gravity, in particular after it was realised that in the known superstring theories, all anomalies were perfectly canceled. The presence of gauge symmetries in string theory arose as a natural consequence of the introduction in the open strings spectrum, which instead contains a massless mode of oscillation with spin 1, of the so called Chan-Paton factors. Initially introduced to recover the global symmetries of the hadron models by attaching to the open string endpoints gauge charges in the form of quark-antiquark monopoles, these factors were then reinterpreted as gauge symmetry charges and in theories containing open strings (traditionally only the type I superstring, which contains open unoriented strings) gauge anomalies were canceled by choosing the gauge group $SO(32)$. The other two types (IIA and IIB) of superstrings traditionally constructed admitted only closed strings and did not have a gauge sector in their spectrum. Finally one more exotic theory was built under the name of Heterotic string, consisting of closed strings formed by a particular combination of superstrings and bosonic strings. The latter in fact are fully consistent only if propagating in a 26-dimensional target space-time, while the former live in ten dimensions. By compactifying sixteen of the dimensions of the bosonic string a consistent supersymmetric theory of closed strings was obtained also accommodating for the presence of gauge symmetries. Indeed in the internal directions compactified on toroidal manifolds, the Kaluza-Klein quantised momenta and the winding numbers for the heterotic strings become vectors that span particular (Lorentzian and self-dual) lattices in order for the theory to be consistent. Such lattices can then be identified with the weight lattices of gauge symmetries under which the heterotic strings result to be charged. The mentioned constraints on these lattices yield either an $SO(32)$ or an $E_8 \times E_8$ symmetry and in both cases the cancellation of the gauge anomalies is ensured. In particular the $E_8 \times E_8$ group arisen with the discovery of the heterotic strings naturally seemed to contain its $SU(3) \times SU(2) \times U(1)$ structure. This is historically the reason why the first string-theoretic models reproducing the physics of the Standard Model were constructed in the context of the $E_8 \times E_8$ heterotic string theory. Of course similarly to the case of the type I superstring the amount of supersymmetry in the heterotic theory in ten dimensions is too high and has to be reduced to allow for the presence of chiral four-dimensional fermions to describe the fundamental matter. By compactifying the theory on suitable Calabi-Yau manifolds [1], it is possible to restrict to $\mathcal{N} = 1$ SUSY in four-dimension, which is desirable for instance in minimal supersymmetric generalisation of the physics of the Standard Model. The compactification of the extra-dimensions here provides also a means to easily reduce the gauge symmetry down to the gauge structure of the Standard Model. In fact by turning on a non-trivial background for the internal components of the gauge fields in a subgroup H of $E_8 \times E_8$, the four-dimensional gauge symmetry is given by the subgroup of $E_8 \times E_8$ commuting with H. For instance the choice $H = SU(3)$ leads to a four dimensional $E_6 \times E_8$ symmetry (E_6 actually being a possible gauge structure for a GUT, related to the remaining E_8 only via gravitational interactions), while $H = SU(3) \times \mathbb{Z}_2$ can directly yield $SU(3) \times SU(2) \times U(1)$ in four dimensions. Fermions in these backgrounds are naturally chiral and replicated a number of times determined by the topology of the compact part of the target space-time. Such number corresponds for instance to half of the Euler characteristic of the Calabi-Yau manifolds in backgrounds with $H = SU(3)$ [2]. A generic Calabi-Yau compactification of string theory is however of great complexity to be addressed, as it does not even lead to a fully solvable world-sheet sigma model for the strings. As a consequence much of the initial work to embed the Standard Model in heterotic string theories was devoted to compactifications involving particular flat limits of internal curved manifolds known as orbifolds [3] (see also for instance [4, 5] for explicit generalisations). Here the world-sheet theory for the strings is free, hence it is possible to perform some phenomenologically interesting string-theoretic calculations to go beyond the simple analysis of the spectrum of the constructed models and study the interactions. The resulting couplings arisen in this way can be compared to the action of the (minimal supersymmetric version of the) Standard Model [6]. In particular in the context of models with (spontaneously broken) $\mathcal{N} = 1$ SUSY many works were produced to determine the low-energy description of the interaction in the form of a super Yang-Mills action and to determine the background moduli dependence of its parameters [7, 8, 9, 10, 11, 12, 13, 14, 15].

The discovery of dualities provided some first insight in the non-perturbative regime of string theory through the introduction of D-Branes. Initially discovered as solitonic solutions [16], of spatial dimension p, of the Supergravity actions associated to the low-energy effective descriptions of the known supersymmetric string theories, Dp-Branes, D standing for Dirichlet, were subsequently understood to have a microscopic description in terms of open strings which localise their endpoints on these solitons [17, 18, 19, 20, 21]. For instance D-Branes naturally emerge when considering toroidal compactifications of type I theory and performing T-dualities along the compactified directions. The dimensionality of the world-volume spanned by the brane can be related to the number of dualities performed starting from the type I theory and corresponds to the number of directions where open strings have Neumann boundary conditions (in the other directions the boundary conditions are Dirichlet). Since the closed string sector of type I theory is the unoriented version of type IIB theory, by exploiting the web of T-Dualities which also relate the type IIB and type IIA theories, it is possible to see that Dp-Branes actually arise in type II theories as well, p being even for type IIA and odd for type IIB. Indeed while open strings must be attached to the branes with their endpoints, closed strings in these backgrounds still propagate in the ten-dimensional bulk and give rise to a physics of oriented closed strings belonging either to type IIA or type IIB theories. Basically stated, type II theories must also contain open strings attached to D-Branes and the latter can actually be interpreted as the Chan-Paton factors for such strings. In fact one of the main features of these solitonic objects is that the supersymmetric effective theory arising from their world-volumes is endowed with a $U(1)$ gauge group, coming from the massless longitudinal modes of oscillation of the open strings attached to them. The orthogonal modes instead intuitively contribute to pull and change the shape of the D-Brane which as a consequence is a dynamic rather than a rigid object in the target space-time. Since the gauge symmetry is an effect of the longitudinal oscillations of the open strings attached to the brane, it is manifest that, even before any compactification of the target space-time, it must already be realised as a lower-dimensional symmetry, whose gauge fields propagate only in $p+1$ dimensions. Furthermore considering the possibility of having N coincident D-Branes on the top of each other, then the gauge symmetry becomes non abelian and generates a $U(N)$ group. The intuitive picture is quite clear again, as the N^2 generators of $U(N)$ are directly related to the massless modes of oscillation of the N^2 strings stretched between any pair of branes in the stack. Thus the great novelty of the introduction of D-Branes in string theory is the possibility to "geometrise" the gauge theories, making also the quest for the Standard Model much easier in this context than in any previously explored one.

Ever since the discovery of D-Branes as a very attractive means to reproduce gauge theories in the context of string theory, a great deal of work has been produced to construct models involving these extended objects and which could lead to the physics of the Standard Model or its minimal supersymmetric generalisation in the low-energy effective action. Many of the models analysed in details are in type II (A and B) theories and involve open strings stretched between D-Branes. In particular one of the most successful attempts to first of all determine the spectrum of particles in the Standard Model has been the use of D-Branes intersecting at angles [22]. In this framework different sectors of the Standard Model Lagrangian arise from the low-lying states of the spectrum of different open strings. As already mentioned earlier the gauge bosons necessary to reproduce the gauge sector of the Lagrangian of the Standard Model arise from the massless longitudinal modes of oscillation of the strings attached with both the endpoints to the same stack of D-Branes. In particular a stack of N coincident D-Branes is classically described by an effective action known as the Dirac-Born-Infeld action that, in the low-energy limit, generates a non-abelian $U(N)$ Yang-Mills action. In order to determine the gauge structure of the Standard Model it is natural to think that one would need three stacks of such D-Branes giving rise to a $U(3) \times U(2) \times U(1)$ gauge symmetry. However this turns out to be not enough to also encode in the description the appropriate fundamental matter fields interacting with the gauge bosons. Of course the gauge sector of the model constructed in this way would live in higher dimensions, precisely in $p+1$ space-time dimensions corresponding to the world-volume of the brane. In order to achieve a four-dimensional physics it is necessary to introduce into these models a compactified target space-time in a form of a tensor product between a flat four-dimensional Minkowski space-time and a compact six-dimensional internal manifold. In particular this compactification is necessary to obtain also a four-dimensional theory of gravity. It is worth mentioning here that another possibility has been explored in the literature [23] to achieve the confinement of gravity to four dimensions, namely by considering curved but non-compact extra-dimensions in a warped geometrical background, which effectively generate a particular potential for the graviton that confines it to propagate in a four-dimensional Minkowski space-time. Restricting anyway to a factorised and compactified background geometry, again the choice of toroidal compactifications provides a simple and fully solvable conformal field theory (CFT) on the world-sheet of the strings and it has proven in the literature to be enough as a starting point to encode many phenomenologically desirable features into the explicitly constructed models. Moreover in general for the sake of simplicity the toroidal compactifications have often been chosen to be factorised in the form of $T^6 = T^2 \times T^2 \times T^2$.

In this context chiral fermions describing the fundamental interacting matter arise from strings stretched between two different stacks of D-Branes which are intersecting at angles in the internal compactified directions of the target space and filling the flat Minkowski four dimensions of space-time. They are inserted in bi-fundamental representations of the gauge symmetry group. Indeed strings stretched between the stacks a and b containing respectively N_a and N_b superposed D-Branes will give rise to states in the representation $(\bar{N}_a, N_b)$ consistently with the fact that the two endpoints of such strings have opposite charges. Of course in order to obtain in the low-energy spectrum of the model right-handed leptons it is necessary to have strings stretched between two different branes (to accommodate the necessary chirality) which should both be endowed with a $U(1)$ gauge symmetry. This is the main reason why at least four stacks of D-Branes must be present in the so-called Intersecting Brane Worlds. The explicit most successful models constructed in the literature involve D6-Branes intersecting at angles in the type IIA superstring theory [24, 25, 26, 27, 28, 29] (details can also be found in the PhD-theses [30, 31, 32, 33]). In a factorised geometry where the compactified internal manifold is a direct product of three two-dimensional tori, D6-Branes simply become 1-cycles in each of the tori and it is here that they actually intersect one another. Furthermore the periodicity of the internal compactified directions makes it possible to replicate the intersections between two given D-Branes and, since in these intersections live the strings that describe the interacting fundamental matter of the Standard Model, it is manifest that a description of the family replication of fundamental particles with the same quantum numbers (apart from their masses) is very naturally contained in the models under study. Much effort has been devoted to reproduce the exact chiral spectrum of the Standard Model [34] in the stringy models constructed in the literature where supersymmetry is in general not preserved and it can be restored only with particular choices of the intersection angles between different branes. Reproducing the exact spectrum of particles in the Standard Model requires the elimination of extra gauge symmetries, in particular of extra abelian $U(1)$ symmetries that are unavoidable in consistent brane models, and an appropriate choice of the wrapping numbers of the 1-cycles in the internal compactified directions to have the right number of families. The first problem can be solved by observing that actually some of the extra $U(1)$'s in these models are anomalous and the cancellation of such anomalies is possible via a so-called generalised Green-Schwarz mechanism (see [25] for a recent review). Basically stated, there is a coupling between anomalous gauge bosons and two-forms arising from the closed string spectrum of the theory, namely the Ramond-Ramond forms, which contributes at tree-level to cancel the one-loop triangle anomaly involving these $U(1)$ gauge bosons as external legs. The same coupling is also responsible for a Stueckelberg mechanism which effectively makes the gauge bosons associated to the anomalous extra symmetries massive (see for instance [35]). From the low-energy description of the model, this implies that the symmetry still persists in the effective lagrangian, but there is no gauge boson related to it, hence the symmetry becomes global. In fully realistic models it has been possible to show that such global symmetries can be directly related for instance to the conservation of the lepton and baryon numbers which in turn guarantee some phenomenologically important properties like the stability of the proton. In general the only $U(1)$ symmetry which persists as a gauge symmetry in the Brane World Models must be the correct hypercharge of the chiral fermions describing the fundamental matter.

Finally closed strings in this setups are not confined to live in the D-Branes world-volumes and can propagate in the bulk of the ten-dimensional target space-time. However upon the compactification that makes for instance the physics of the gauge sector four-dimensional, also gravity contained in the massless excitations of closed strings becomes four-dimensional and consistent with the real world. Apart from the graviton in this sector some other important massless fields are also present, namely the Ramond-Ramond forms. D-Branes are known to be charged under such fields and the coupling is described classically by the Wess-Zumino contribution to the brane action. The presence of charged objects in compact spaces of course can lead to some inconsistencies related to the Gauss law, hence it must be possible, in order to construct coherent Brane World Models in string theory, to cancel the tadpoles arising from the RR-fields. In the particular setups involving intersecting D6-Branes as discussed in the literature it is shown that there is an interdependence between the cancellation of such tadpoles and the one of the non-abelian gauge anomalies. In non-supersymmetric models full consistency is ensured quite easily by a suitable choice of the wrapping numbers of the 1-cycles in the internal compactified directions. If supersymmetry is restored however this is no longer sufficient as recalling that D-Branes are BPS solitons it is clear that the cancellation of their total charge would imply the cancellation of their total tension as well or, in other words, the fact of having no D-Branes at all in the constructed model. In this case the difficulty is circumvented by adding orientifold planes to the models, which, being objects with negative tension, can compensate for the tension of the D-Branes and still allow for the cancellation of the RR-tadpoles. Adding orientifold planes requires considering orientifolded models in which new D-Branes, namely mirror cycles with respect to the orientifold planes, arise as well as new sectors for the open strings. For instance strings stretched between a stack of D-Branes a and the mirror image of the stack of branes b will be characterised by new bi-fundamental representations (N_a, N_b) while strings whose endpoints are attached to a stack of D-Branes and its own mirror image generate some chiral fermions in the symmetric or antisymmetric two-index representation. Observe that historically the first explicit model which actually provided at low energy only the spectrum of the Standard Model was built in this context using orientifolded theories involving D6-Branes and their mirror images [34]. As a final remark regarding the spectrum of such models one should underline that in non-supersymmetric constructions NS-NS tadpoles are instead in general not canceled (an example where they are is [36]) as of course they do not represent an inconsistency of the theory. Nevertheless they imply that the backgrounds chosen do not really solve the equations of motion of the theory and should as a consequence be subject to backreactions that lead them to a stability point. Clearly the complete equations of string theory are not known yet, so it could still be that these backgrounds represent a good approximation of the true solutions.

It is worth mentioning that even if the greater amount of literature for the study of the Brane World Models has been devoted to the construction of explicit configurations of intersecting D6-Branes in type IIA theory, the T-dual version of such models involving D9-Branes in type IIB with non trivial background magnetic fields turned on in their world-volume has also been explored [37, 38, 39, 40, 41, 42, 43, 44]. In particular just after the discovery of D-Branes in string theory the use of background magnetic fields in the world-volumes of D-Branes was recognised as an effective means to break supersymmetry [45]. Hence independently of the analysis of the spectra associated to strings stretched between branes intersecting at angles, the study of the open strings charged with respect to these magnetic fields led to the idea that the mentioned breaking of supersymmetry could also yield the presence of four-dimensional chiral fermions upon suitable compactifications. Recall in fact that the main difficulty in accommodating for chiral multiplets in four dimensions in the low-lying states of oscillations of the strings is the big amount of supersymmetry in the mother ten-dimensional theory. In particular for type II theories in presence of (parallel) D-Branes the supersymmetry preserved is $\mathcal{N} = 1$ SUSY which becomes $\mathcal{N} = 4$ SUSY in four dimensions after compactifying six of the directions of the target space-time. This spoils the possibility to have chiral multiplets in the Minkowski space-time unless some other mechanism to break SUSY further down to $\mathcal{N} = 1$ is implemented. The presence of background magnetic fields in the brane world-volumes is exactly a means to reduce SUSY and accommodate for the presence of chirality in 4D. T-Duality relates the models constructed in the Intersecting Brane Worlds with the ones in the context of the Magnetised Brane Worlds, which as a consequence have the same features analysed so far. In particular here the family replication of the fundamental particles is given by the presence of degenerate vacua for the strings stretched between branes with different magnetic fields turned on (charged strings), which play the same role as the well-known degeneration of the Landau Levels for charged particles propagating in a magnetic field background in a compact space. The compactness of the space indeed guarantees the finiteness of such degeneration which can then be interpreted naturally as the family replication of the Standard Model. Furthermore it can be argued that models constructed employing magnetised D9-Branes in type IIB theories can be more versatile in the sense that they do not rely on a visible geometric picture of intersecting cycles on simple two-dimensional tori. The mathematical techniques used in the Magnetised Brane Worlds result to be applicable to any kind of toroidal flat compactification and allow for the exploration of a broader spectrum of phenomenologically interesting vacua of string theory. Of course many of them can still be related to geometric constructions involving intersecting D6-Branes by means of T-Duality, but in this context it is possible to find more general setups that do not appear to be easily geometrisable. We will return to this point in the present work which is indeed devoted to a detailed analysis of the mathematical techniques necessary to describe Magnetised Brane Worlds in full generality and, where possible, Intersecting Brane Worlds obtained via duality.

As already mentioned earlier one of the appealing features of all of these models resides in the fact that the string world-sheet theory is a free CFT and it is exactly calculable. Differently to the generic Calabi-Yau compactifications in which it is only possible to determine explicitly some information on the spectrum of the strings that reproduce the physics of the Standard Model, in the discussed toroidal compactifications of type II theories in presence of D-Branes it is possible to go beyond the level of the spectrum and construct the low-energy effective action to encode also the interactions of the strings in the realistic constructed models. The effective lagrangian should reproduce the low-energy contribution to the scattering amplitudes involving open and closed strings in the Brane World Models. It is important to make the distinction between the states arising in these models from the oscillations of the strings in the four flat Minkowski directions and the ones in the compactified directions of the space-time. The former should in fact reproduce the action of the (spontaneously broken) $\mathcal{N} = 1$ super Yang-Mills coupled with matter [46], while the latter, which from the four dimensional point of view give rise to scalar particles, should be considered as moduli. Of course there is no evidence of the existence of scalar particles (apart from the experimentally sought Higgs boson), thus it must be possible to find mechanisms to freeze the dynamics of such bosons and to ultimately give them vacuum expectation values which are simply remnant of the characteristics of the compactified extra-dimensions. This is the so-called moduli-stabilisation thoroughly investigated in the literature [47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]. In the BWM explored in the literature the dynamics of the closed-string moduli is in general ignored and such fields are given a non vanishing v.e.v. in the compact directions corresponding to the metric of the compactification torus, or the Kalb-Ramond field on it. The information carried by these fields enters in the parameters that characterise the low-energy action, giving its closed string-moduli dependence. The open string-moduli dynamics is instead in general not frozen and it is still described in terms of a $\mathcal{N} = 1$ super Yang-Mills action. These fields indeed arise from the oscillations of the strings attached with both endpoints to a stack of D-Branes in the internal dimensions and are associated to the positions of the D-Branes in the compact space and the background gauge fields turned on in their world-volume. They must ultimately be stabilised as well. Many of the properties of the effective actions can then be determined already by suitably compactifying the Dirac-Born-Infeld lagrangian that describes D-Branes at low energies. Upon compactification it is in fact possible to obtain the Yang-Mills Lagrangian for the four-dimensional gauge bosons and the action for the so-called untwisted moduli related to open strings with both endpoints attached to the same stack of D-Branes. The analysis of the twisted fields arising from open strings stretched between different stacks of D-Branes is instead more involved and cannot be directly derived by means of compactification. In this case it is necessary to be able to compute string scattering amplitudes that contribute to the relevant coupling of the low-energy Lagrangian which is to be reproduced [44]. The vertex operators which describe twisted fields have been explicitly constructed [59, 60, 44] and they involve in the bosonic sector of the theory particular conformal operators known as twist fields for which a simple representation in terms of free fields is not yet available. Most of the difficulties in computing scattering amplitudes with these fields arise from the fact that it is not possible to exploit the usual Wick theorem and the simple Green function of the free fields to calculate them. It is unavoidable in this case to employ other CFT techniques to compute the amplitudes. Nevertheless it has been possible to derive the Kähler metric for the twisted matter fields and its (frozen) moduli dependence by computing mixed scattering amplitudes involving open and closed string vertex operators on a disk [60, 44] and to address the problem of determining the superpotential by computing the Yukawa couplings [61, 62, 63, 60]. These couplings are in particular the main field of application of the new string techniques to determine the low-energy action of the Brane World Models in the present work [64, 65]. They are of course related to the computation of the correlation function involving three twist fields, which has been addressed in several different contexts in the existing literature. These amplitudes appear in fact in the computation of the Yukawa couplings not only in realistic models employing D-Branes but also in orbifold models of the Heterotic String theory. In the latter context the quantum contribution of these correlation functions was soon determined via CFT arguments while the classical part of the amplitudes involving three twist fields, which is the low-energy contribution that enters directly in the effective lagrangian, led to developing techniques to compute the so-called world-sheet instantons [59, 66, 12, 13, 14, 15]. The same techniques were then used in the context of Intersecting D-Branes models [63, 61] where the correlation function of three twisted fields indeed receives a classical correction given by instantons associated to the area of the triangles formed by three intersecting D-Branes. On the side of the Magnetised Brane Worlds the result of the classical contribution to the couplings has been calculated in a low-energy field theory context, lacking a full string theoretic derivation [67]. The novelty of the techniques presented here resides exactly in the derivation of the most general moduli dependence of the Yukawa couplings in generic T^6-compactifications involving magnetised D-Branes and in a complete string-theoretic analysis.

1.1 Outline

- The second chapter is a brief review of the main properties of the tree-level Lagrangian of the Standard Model. Particular emphasis is given to the symmetries of the action and the discussion of the Yukawa couplings.

- The third chapter contains some techniques which are of fundamental importance throughout the whole presented work. In particular the first part is devoted to a simple study of the characteristics of gauge bundles associated to magnetised compact toroidal manifolds from the field-theoretic point of view. There follows an introduction to closed strings in toroidal compactifications and the detailed analysis of T-Duality transformations.

- The fourth chapter presents the general techniques to describe twisted strings in the Brane World Models via the so-called "doubling-trick" and the definition of reflection and monodromy matrices for the world-sheet fields. A careful analysis is performed of the twisted spectrum of these strings in the Magnetised Brane Worlds in generic toroidal compactifications. The spectrum is then related to the known results obtained in the context of the Intersecting Brane Worlds via T-Duality. It is shown how chiral fermions arise in these models and how they are replicated to naturally describe the families of particles of the Standard Model. Also a discussion of the supersymmetry preserved in these models can be found.

- The fifth chapter is devoted to the untwisted open strings. Particular care is paid to describe how the extra abelian symmetries acquire mass through a Stueckelberg mechanism and yield a natural description of the global symmetries of the Standard Model. Finally some general consistency conditions for the Brane World Models such as tadpole and anomaly cancellation are introduced.

- The sixth chapter is the description of magnetised D-Branes from the closed strings point of view via boundary states. The boundary state for a generic wrapped magnetised D-Brane is computed in two different alternative approaches and, by means of T-Duality, also the boundary state for intersecting D-Branes is discussed. We are also able to determine all of the phase factors of the magnetised boundary state which are of fundamental importance for our final results.

- The seventh chapter introduces the Reggeon vertex for the emission of off-shell closed strings. Its zero-modes dependence in toroidal compactifications is discussed with particular attention as it leads to some cocycle factors necessary for its consistency. Moreover it is shown that a fully coherent picture for the interactions is only possible if the well-known T-Duality rules are also generalised including some phase factors.

- The eighth chapter is the computation of the amplitude obtained by sewing $g+1$ wrapped magnetised D-Branes boundary states with the Reggeon vertex on a generic $2d$-dimensional torus. The focus is on the detailed computation of the zero-modes dependence including all of the phase factors previously determined.

- The ninth chapter contains the modular transformation which leads to the interpretation of the computed amplitude in the open string channel as the g-loop twisted partition function on a generic toroidal compactification. The focus is again on the zero-modes contribution. Specifying to the $g=2$ case and considering the degeneration limit in which the Schwinger time of the string propagators is taken to infinity, the amplitude is then factorised by unitarity to obtain the correlation function between three twist fields. The classical contribution of the computation yields the general moduli dependence of the Yukawa couplings in toroidally compactified Magnetised Brane Worlds.

- The tenth chapter contains some remarks and conclusions on the results found and the comparison with the previously determined results in the literature.

2. THE STANDARD MODEL TREE-LEVEL LAGRANGIAN

We will introduce the main characteristics of the Lagrangian of the Standard Model at tree-level [68], highlighting the structure of its symmetries whilst considering each of the four contributions it contains. The first part of the Lagrangian will be the Yang-Mills contribution, which describes the dynamics and the interactions of the gauge bosons that are the mediators of the forces between fundamental particles; the second contribution will encode the kinetic terms for the massless matter fields and their couplings with the gauge bosons, while the remaining two terms are related to the mass generation for the fundamental particles themselves. In particular they will involve the introduction of a scalar particle, known as the Higgs boson, and its couplings with the matter.
We will analyse each of these contributions separately in the following subsections where the conventions used will be such that $\hbar = c = 1$ and the four dimensional flat metric is $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

2.1 The Yang-Mills Lagrangian

The pure gauge content of the Standard Model is encoded in a four dimensional Yang-Mills theory with gauge group $SU(3)_c \times SU(2)_w \times U(1)_y$. The subscripts indicate the colour, weak isospin and hypercharge quantum numbers of the fields interacting in the sector of the theory with the corresponding gauge symmetry.

$$\mathcal{L}_{YM} = -\frac{1}{4g_c^2}\sum_{A=1}^{8} G^A_{\mu\nu}G^{\mu\nu A} - \frac{1}{4g_w^2}\sum_{a=1}^{3} F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{4g_y^2}B_{\mu\nu}B^{\mu\nu} \tag{2.1.1}$$

The Lagrangian above is defined in terms of the field strengths of the gauge fields in the following fashion

$$\begin{aligned} G^A_{\mu\nu} &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f^{ABC}A^B_\mu A^C_\nu \\ F^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon^{abc}W^b_\mu W^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned} \tag{2.1.2}$$

thus it contains eight gluons A^A_μ, mediators of the colour interaction, three intermediate bosons W^a_μ which are responsible for the weak interaction and one

abelian gauge field B_μ associated to the hypercharge. While the latter defines a genuinely free theory, as the respective Lagrangian term is just the kinetic term for the field, the two former gauge bosons are self interacting, as a consequence of the fact that they are related to non abelian symmetries. Indeed in the definition of their field strengths, there appear contributions with no derivatives involving the structure constants of the $SU(3)$ gauge group, f^{ABC}, and of $SU(2)$, ϵ^{abc}.
Finally g_c, g_w and g_y are the dimensionless coupling constants for the three sectors of the gauge theory.

2.2 Matter fields Lagrangian

The matter fields are in the form of (left-handed) chiral four dimensional Weyl fermions

$$\gamma^5\psi_L = \psi_L \tag{2.2.3}$$

where $\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ is the chirality operator and the γ's satisfy the four dimensional Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{2.2.4}$$

in the Dirac representation, i.e.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.2.5}$$

σ^i being the three Pauli matrices. The constraint (2.2.3) halves the degrees of freedom of a generic four dimensional Dirac spinor and leads to a two components chiral Weyl fermion. The other chirality (right-handedness) is obviously obtained by considering the same kind of constraint with the opposite eigenvalue for the γ^5 operator.
The matter content of the Standard Model involves three families of massless particles which have identical quantum numbers and, hence, are indistinguishable unless a mass term is turned on through a spontaneous breaking of symmetry mechanism which we will not consider in what follows. They are inserted in either the trivial or the fundamental representation of the non-abelian gauge groups of the model, depending on whether they feel the corresponding gauge interaction.
The leptons do not strongly interact, i.e. they are colour singlets,

$$L_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \sim (1_c, 2_w)_{y_1} \quad , \quad \bar{e}_{iL} \sim (1_c, 1_w)_{y_2} \tag{2.2.6}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$. Here the notation reveals the representation of $(SU(3), SU(2))_{U(1)}$ in which each of the fields is inserted and the index $i = 1, 2, 3$ labels the three families of identical particles of the model.
Quarks are instead in the (anti)-fundamental representation also of the colour group

$$\begin{gathered} \mathbf{Q}_i = \begin{pmatrix} \mathbf{u}_i \\ \mathbf{d}_i \end{pmatrix}_L \sim (3_c, 2_w)_{y_3} \\ \bar{\mathbf{u}}_{iL} \sim (\bar{3}_c, 1_w)_{y_4} \quad , \quad \bar{\mathbf{d}}_{iL} \sim (\bar{3}_c, 1_w)_{y_5} \end{gathered} \tag{2.2.7}$$

where bold symbols indicate that the field is also a three components column vector in the (anti)-fundamental representation of $SU(3)$.
The values of the hypercharges have not been fixed yet since they can be derived, as it will be shown soon, by the consistency requirement of anomaly cancellation for the gauge theory.
The Lagrangian for the matter fields reads

$$\begin{aligned} \mathcal{L}_M = \sum_{i=1}^{3} & \Big(L_i^\dagger \sigma^\mu D_\mu L_i + \bar{e}_i^\dagger \sigma^\mu D_\mu \bar{e}_i + \mathbf{Q}_i^\dagger \sigma^\mu D_\mu \mathbf{Q}_i \\ & + \bar{\mathbf{u}}_i^\dagger \sigma^\mu D_\mu \bar{\mathbf{u}}_i + \bar{\mathbf{d}}_i^\dagger \sigma^\mu D_\mu \bar{\mathbf{d}}_i \Big) \end{aligned} \tag{2.2.8}$$

The derivatives in the Dirac Lagrangian for the Weyl fermions are covariant with respect to the full gauge group of the Standard Model and their definitions depend on the representations of the matter fields they act on, namely

$$\begin{aligned} D_\mu L_i &= \left(\partial_\mu + \frac{\mathrm{i}}{2} W_\mu^a \tau^a + \frac{\mathrm{i}}{2} y_1 B_\mu \right) L_i \\ D_\mu \bar{e}_i &= \left(\partial_\mu + \frac{\mathrm{i}}{2} y_2 B_\mu \right) \bar{e}_i \\ D_\mu \mathbf{Q}_i &= \left(\partial_\mu + \frac{\mathrm{i}}{2} A_\mu^A \lambda^A + \frac{\mathrm{i}}{2} W_\mu^a \tau^a + \frac{\mathrm{i}}{2} y_3 B_\mu \right) \mathbf{Q}_i \\ D_\mu \bar{\mathbf{u}}_i &= \left(\partial_\mu - \frac{\mathrm{i}}{2} A_\mu^A \lambda^{*A} + \frac{\mathrm{i}}{2} y_4 B_\mu \right) \bar{\mathbf{u}}_i \\ D_\mu \bar{\mathbf{d}}_i &= \left(\partial_\mu - \frac{\mathrm{i}}{2} A_\mu^A \lambda^{*A} + \frac{\mathrm{i}}{2} y_5 B_\mu \right) \bar{\mathbf{d}}_i \end{aligned} \tag{2.2.9}$$

where τ^a and λ^A are respectively the generators of $SU(2)$ and $SU(3)$ and their indices are summed with the ones of the corresponding gauge bosons. As it is well known not only do the covariant derivatives defined above contain the purely kinetic terms for the matter fields, but they also introduce the couplings between the Weyl fermions and the gauge bosons.

The Lagrangian (2.2.8) has a very large global symmetry, as all of the fields involved come in three, so far undistinguishable, copies given by the three families of the Standard Model. In fact a relabeling of the family indices through a generic unitary transformation does not change the form of the aforementioned Lagrangian (2.2.8). This leads to a $U(3) \times U(3) \times U(3) \times U(3) \times U(3)$ global symmetry of the model. Of course this huge symmetry group will be broken once one takes into account the contributions to the Standard Model Lagrangian related to the mass generation terms as we will see when introducing the Yukawa couplings for the matter chiral fermions with the Higgs boson.

2.3 Cancellation of Gauge and Mixed Anomalies and Hypercharge

The values of the hypercharge quantum numbers for the matter fields introduced in the previous subsection can be almost uniquely fixed by considering a necessary internal consistency requirement for the quantum theory defined by the Lagrangian terms analysed so far. Indeed a gauge theory like the Standard Model, in order to be well defined at the quantum level, must be free from gauge symmetry anomalies. An anomalous gauge symmetry would in fact spoil fundamentally crucial characteristics of the quantum theory such as Ward identities and renormalisability, making the theory itself effectively inconsistent beyond the classical tree-level approximation.

Gauge anomalies in the Standard Model can only occur at the one loop level in interactions involving triangle Feynman diagrams with three external gauge boson legs and chiral fermions circulating in the loops as in figure 2.1. The diagram related to the one loop interaction of three $SU(2)$ bosons does not contribute to any anomalies since, by group theory arguments

$$\mathrm{Tr}\left[\tau^a \tau^b \tau^c\right] = 0 \tag{2.3.10}$$

Similarly there is no pure $SU(3)$ anomaly, although the trace above would be non vanishing in this case, as in the loop there circulates the same number of quarks and antiquarks. Hence the only diagrams that can contribute to gauge anomalies are the ones involving three $U(1)$ gauge bosons or mixed diagrams with one $U(1)$ gauge boson interacting with either two intermediate $SU(2)$ bosons or two $SU(3)$ gluons.

These three diagrams separately give vanishing contributions as expected only if

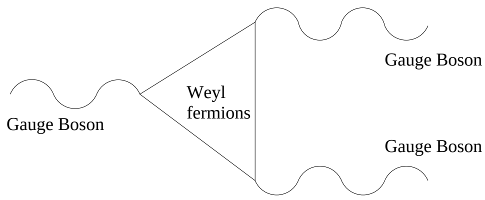


Fig. 2.1: One loop triangle diagram with three external gauge boson legs and Weyl fermions circulating in the loop

the following relations between the hypercharges of the matter fields are satisfied

$$\begin{cases} 2y_3 + y_4 + y_5 = 0 \\ y_1 + 3y_3 = 0 \\ 2y_1^3 + y_2^3 + 3(2y_3^3 + y_4^3 + y_5^3) = 0 \end{cases} \tag{2.3.11}$$

This is not enough to fix all of the quantum numbers, but, taking into account that one of their values can be normalised by suitably redefining the hypercharge coupling constant g_y, only one of the hypercharges still remains undetermined. It is very interesting to notice that this left out value can be actually derived by considering also the cancellation of mixed gauge and gravitational anomalies. As gravity is still decoupled from the Standard Model Lagrangian and can not be consistently quantised as a gauge theory like all the other interactions in nature, in principle it would not be allowed to consider such mixed diagrams involving interactions of gauge bosons with gravitons. Nevertheless it turns out that the actual values of the hypercharges for the fundamental particles as determined from experimental data do cancel these anomalies as well as it should happen if we were able to incorporate quantum gravity in the theory. Thus the cancellation of the mixed anomaly leads to an additional relation between the hypercharges that read

$$2y_1 + y_2 + 6y_3 + 3y_4 + 3y_5 = 0 \tag{2.3.12}$$

There are two possible solutions to these equations:

$$y_1 = y_2 = y_3 = 0 \quad \text{and} \quad y_4 = -y_5 \tag{2.3.13}$$

or

$$y_1 = -3y_3 \;\; , \;\; y_2 = 6y_3 \;\; , \;\; y_4 = -4y_3 \;\; , \;\; y_5 = 2y_3 \tag{2.3.14}$$

At this stage both the possibilities would be acceptable, but only the second one is actually realised in nature with $y_3 = 1/3$.

2.4 The Higgs sector

All of the matter fields in the Standard Model Lagrangian must be massless in order not to spoil the renormalisability of the theory[1]. The mass generation is obtained by coupling these fields with an additional scalar field known as the Higgs boson whose vacuum expectation value after a spontaneous symmetry breaking mechanism is related to the masses of the three families of Weyl fermions.
Potential mass terms for quarks and leptons must have the form

$$\hat{\mathbf{Q}}\bar{\mathbf{u}} \quad , \quad \hat{\mathbf{Q}}\bar{\mathbf{d}} \quad , \quad \hat{L}\bar{e} \tag{2.4.15}$$

where we have introduced the notation $\hat{\psi} = {}^t\psi\sigma_2$. It is manifest that these terms, despite being colour singlets and Lorentz invariant, do violate the weak isospin symmetry as they contain combinations of fields in the fundamental representation of $SU(2)$, with isospin $I_w = 1/2$, with weak singlets with isospin zero. Hence, in order to compensate for the breaking of the weak symmetry by half a unit introduced by these combinations, they have to be coupled to another weak doublet in the trivial $SU(3)$ representation, which is given by the Higgs field

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim (1_c, 2_w)_{y_h} \tag{2.4.16}$$

All the hypercharges of the matter fields will be, when introducing the actual Yukawa couplings with the Higgs field, determined in units of y_h, which will be reabsorbed into the definition of the hypercharge coupling constant g_y.
The Higgs Lagrangian reads

$$\mathcal{L}_H = (D_\mu H)^\dagger (D^\mu H) - V(H) \tag{2.4.17}$$

The covariant derivative in the expression above takes into account the coupling of the Higgs scalar particle with the weak intermediate and hypercharge bosons in the following fashion

$$D_\mu H = \left(\partial_\mu + \frac{\mathrm{i}}{2} W^a_\mu \tau^a + \frac{\mathrm{i}}{2} y_h B_\mu\right) H \tag{2.4.18}$$

while the potential $V(H)$ is simply the most generic renormalisable potential, invariant under the weak and hypercharge symmetry, that one can introduce for a scalar field, namely

$$V = \lambda \left(H^\dagger H\right)^2 - \mu^2 H^\dagger H \tag{2.4.19}$$

[1] To be precise this applies to non-abelian theories, while a renormalisable $U(1)$ gauge theory could also allow for massive fields.

Here λ is a dimensionless coupling which has to be taken positive for the potential to be bounded from below, and μ^2 is the only parameter of the theory with mass units.
This term of the Standard Model Lagrangian has a large $SO(4)$ symmetry related to rotations of the four components of the Higgs complex doublet. This large symmetry group, as well as the one of the matter term of the Lagrangian, is partly broken by the Yukawa couplings sector yet to be analysed. The broken symmetry becomes accidental of this part of the full theory.

2.5 Yukawa Couplings and CKM Matrix

The mass generation for the three families of fundamental particles in the Standard Model follows from a spontaneous breaking of symmetry mechanism in which the Higgs field acquires a non vanishing vacuum expectation value, transforming its couplings with the Weyl fermions into mass terms of the Lagrangian. These are known as Yukawa couplings written in the form of 3×3 matrices $Y^{[e,u,d]}$ in the Yukawa sector of the Lagrangian

$$\mathcal{L}_{Yu} = \mathrm{i}\hat{L}_i \bar{e}_j H^* Y^{[e]}_{ij} + \mathrm{i}\hat{\mathbf{Q}}_i \bar{\mathbf{d}}_j H^* Y^{[d]}_{ij} + \mathrm{i}\hat{\mathbf{Q}}_i \bar{\mathbf{u}}_j \tau_2 H Y^{[u]}_{ij} + \text{c.c.} \tag{2.5.20}$$

The fact that they are non diagonal matrices is just again the most generic choice of couplings one can do in accordance with the request that they contain potential mass generation terms upon spontaneous breaking of symmetry, and it implies that there can be a mixing between particles belonging to different families as it is actually the case realised in nature.
Of course this added term in the Lagrangian must be invariant under all of the gauge symmetries of the theory. We have already highlighted in the discussion of the Higgs sector that these couplings are weak isospin singlets as a consequence of the fact that the Higgs field is in the fundamental representation of $SU(2)$. The term (2.5.20) of the Lagrangian is also manifestly Lorentz and colour invariant, and it conserves the hypercharge provided that

$$y_h = y_1 + y_2 = -y_3 - y_4 = y_3 + y_5 \tag{2.5.21}$$

This new relation combined with the ones in eq.(2.3.14) yields the unique determination of the hypercharges of the fields in the Standard Model in units of y_h, which can be reabsorbed in the definition of the hypercharge coupling constant g_y. Indeed

$$y_1 = -1 \quad , \quad y_2 = 2 \quad , \quad y_3 = \frac{1}{3} \quad , \quad y_4 = -\frac{4}{3} \quad , \quad y_5 = \frac{2}{3} \tag{2.5.22}$$

The choice in eq. (2.3.13) would not have worked because the combinations

$$\hat{\mathbf{Q}}\bar{\mathbf{d}} \quad \text{and} \quad \hat{L}\bar{e} \tag{2.5.23}$$

would have had different hypercharges hence it would not have been possible to couple them with the same field.
The large global symmetry under the group $U(3) \times U(3) \times U(3) \times U(3) \times U(3)$ of the Yang-Mills and Dirac-Weyl Lagrangians can now be exploited to simplify the form of the Yukawa couplings $Y^{[e,d,u]}$. In fact any matrix can be decomposed into the product of a unitary matrix times a diagonal real matrix times another unitary matrix, namely

$$Y = {}^{t}UMV \tag{2.5.24}$$

where U and V are two 3×3 unitary matrices and M is a diagonal real matrix. Considering for instance the case of the lepton Yukawa couplings $Y^{[e]}$, the corresponding unitary matrices of the above decomposition, U_e and V_e, can be used to redefine the lepton fields in the following fashion

$$L \to U_e L \quad \text{and} \quad \bar{e} \to V_e \bar{e} \tag{2.5.25}$$

which is nothing else than a unitary rotation of the family indices of the leptons and it does not affect the rest of the Lagrangian of the model. In this way the leptonic Yukawa couplings are reduced to real diagonal matrices directly related to the masses of the new three families of leptons after breaking of symmetry.
The situation is slightly more complicated when focusing on the quark sector of the couplings, as here the field $\hat{\mathbf{Q}}$ is involved in two different couplings. Indeed if one tries to reabsorb the unitary matrix U_d from the decomposition of $Y^{[d]}$ in the definition of $\hat{\mathbf{Q}}$ then the adjoint of the same matrix reappears in the contribution related to the up quark. Nevertheless it is still possible to exploit the matrices V_u and V_d to relabel

$$\bar{\mathbf{u}} \to V_u \bar{\mathbf{u}} \quad \text{and} \quad \bar{\mathbf{d}} \to V_d \bar{\mathbf{d}} \tag{2.5.26}$$

This yields the following simplified version of the Yukawa sector of the theory

$$\mathcal{L}_{Yu} = \mathrm{i}\hat{L}_i \bar{e}_i H^* y^{[e]}_{ii} + \mathrm{i}\hat{\mathbf{Q}}_i \bar{\mathbf{d}}_i H^* y^{[d]}_{ii} + \mathrm{i}\hat{\mathbf{Q}}_i \left(\mathcal{V}\right)_{ji} \bar{\mathbf{u}}_j \tau_2 H y^{[u]}_{jj} + \text{c.c.} \tag{2.5.27}$$

where $y^{[e,u,d]}_{ii}$ are three diagonal real matrices and

$$\mathcal{V} = U_u U_d^{\dagger} \tag{2.5.28}$$

which is a unitary matrix.
Some further simplifications can still be introduced by using the so called Iwasawa

decomposition of a unitary matrix. In fact any unitary matrix $\mathcal{V}$ can be written as

$$\mathcal{V} = {}^t\mathcal{P}\mathcal{U}\mathcal{P}' \tag{2.5.29}$$

where $\mathcal{P}$ and $\mathcal{P}'$ are diagonal phase matrices that belong to the Cartan subalgebra of the unitary group, and $\mathcal{U}$ is a unitary matrix that contains the remaining parameters. Once again the matrix $\mathcal{P}$ can be used to relabel the definition of $\bar{\mathbf{u}}$ while $\mathcal{P}'$ can be reabsorbed first in $\hat{\mathbf{Q}}$ to reappear in the coupling with the down quark and be finally incorporated in the definition of $\bar{\mathbf{d}}$.
This yields the final version of the Yukawa term in the Lagrangian of the Standard Model

$$\mathcal{L}_{Yu} = \mathrm{i}\hat{L}_i y^{[e]}_{ii} \bar{e}_i H^* + \mathrm{i}\hat{\mathbf{Q}}_i y^{[d]}_{ii} \bar{\mathbf{d}}_i H^* + \mathrm{i}\hat{\mathbf{Q}}_i \mathcal{U}_{ji} y^{[u]}_{jj} \bar{\mathbf{u}}_j \tau_2 H + \text{c.c.} \tag{2.5.30}$$

It is manifest from the form of the Lagrangian term above that the global symmetry group of the Dirac-Weyl term for the chiral fermions, $U(3)\times U(3)\times U(3)\times U(3)\times U(3)$, has been broken down to the following $U(1)$ transformations

$$L_i \to e^{\mathrm{i}\alpha_i} L_i \quad , \quad \bar{e}_i \to e^{-\mathrm{i}\alpha_i} \bar{e}_i \tag{2.5.31}$$

for the leptons and

$$\mathbf{Q}_i \to e^{\mathrm{i}\alpha} \mathbf{Q}_i \quad , \quad \bar{\mathbf{u}}_i \to e^{-\mathrm{i}\alpha} \bar{\mathbf{u}}_i \quad , \quad \bar{\mathbf{d}}_i \to e^{-\mathrm{i}\alpha} \bar{\mathbf{d}}_i \tag{2.5.32}$$

for the quarks. The three phase transformations for the leptons give rise to the three leptonic numbers which are separately conserved, while in the case of the quarks, essentially due to the mixing of the families given by the matrix $\mathcal{U}$, only one quantum number is actually conserved, which is the baryon number independent of the family. At the one loop level, however, this is no longer true as, due to a global symmetry anomaly for the $U(1)$'s related to leptonic and baryonic numbers, only three quantum numbers are really conserved, corresponding to the differences of the three leptonic and the baryonic numbers.
In order to count the number of parameters contained in the mixing matrix $\mathcal{U}$ one can start from the matrix $\mathcal{V}$. In the case of two families of chiral fermions, for instance, the matrix $\mathcal{V}$ would contain four real parameters. One of them is simply an over-all phase, two enter the diagonal phase matrices of the Iwasawa decomposition and only one parameter would be left in $\mathcal{U}$ to mix the families of quarks. This parameter would indeed be the Cabibbo angle. In the case of three families instead the mixing matrix $\mathcal{U}$, known as the Cabibbo-Kobayashi-Maskawa or, for short, CKM matrix, contains four real parameters in the form of the three real angles of the $SO(3)$ subgroup of $SU(3)$ and a phase that must be left out of

the real subgroup.
The fact that the CKM mixing matrix is not fully real, as it would have been with only two families of particles in nature, is very interesting from the phenomenological point of view, since the non-reality of the Yukawa couplings is directly related to the phenomenon of the CP symmetry violation in the Standard Model, actually observed in experimental data.
A CP transformation acts on the chiral Weyl fermions in the following fashion

$$\psi_L \to \sigma_2 \psi_L^* \tag{2.5.33}$$

A generic Yukawa term is of the form

$$\mathcal{L} = \mathrm{i}y\hat{\psi}_L \chi_L H - \mathrm{i}y^* \psi_L^\dagger \sigma_2 \chi_L^* H^* \tag{2.5.34}$$

Hence under a CP transformation

$$\begin{aligned} \mathcal{L} \to \quad & \mathrm{i}y\psi_L^\dagger \sigma_2{}^t \sigma_2 \sigma_2 \chi_L^* H^{CP} - \mathrm{i}y^{*t} \psi_L \sigma_2 \sigma_2 \sigma_2^* \chi_L H^{*CP} \\ & = -\mathrm{i}y\psi_L^\dagger \sigma_2 \chi_L^* H^{CP} + \mathrm{i}y^{*t} \psi_L \sigma_2 \chi_L H^{*CP} \end{aligned} \tag{2.5.35}$$

Since CP simply acts as a complex conjugation on the scalar Higgs field, provided that the other terms of the Lagrangian of the Standard Model are already invariant under its action, it is evident that CP is a symmetry of the full model only if the Yukawa couplings are real, namely $y = y^*$. In the case of three particle families, the mixing CKM matrix contains a phase, which is, as a consequence, responsible for the observed CP violation of the Standard Model.
As a concluding remark we can count the number of parameters we have had to introduce in the full Lagrangian of the model. There are three dimensionless gauge coupling constants, g_c, g_w and g_y in the Yang-Mills Lagrangian, then 12 real parameters and one phase in the Yukawa couplings and two couplings also in the Higgs sector of the model, one dimensionless and one with mass units. Hence the classical Lagrangian of the Standard Model contains 18 parameters[2].

[2] The number of parameters is actually 19 once the ϑ-term is also introduced as in Eq.(5.1.2).

3. TOROIDAL COMPACTIFICATIONS, WRAPPED D-BRANES AND T-DUALITY

As already mentioned in the introduction one of the realisations of the Standard Model spectrum in String Theory is achieved by considering the so called Intersecting or Magnetised Brane worlds.
Before entering the details of the spectrum of the open and closed strings in these setups, we will introduce, assuming the fundamental basics of String Theory and D-Branes, some fundamental ingredients involved in the construction of these Brane World Models. We will in particular introduce the necessary geometrical background to embed such models, analysing toroidal compactifications involving wrapped D-Branes. We will define the main background fields that characterise a generic toroidal compactification and then we will see how to describe (especially magnetised) D-Branes wrapping the cycles of these compact manifolds (for the material covered in these sections see for instance [69, 70, 71, 72, 19]). Notice that with respect to the general introduction to the Standard Model tree-level Lagrangian in the previous chapter, in the remainder of the work the background gauge fields will have dimensions of length, while the gauge field strengths will be chosen dimensionless. We will start from a simple example on a two-dimensional torus, then we will generalise the results found in the case of a T^{2d}. Finally we will introduce T-Duality [73] as the main means to investigate the relations between Magnetised and Intersecting Brane Worlds, which will be exploited mostly in the subsequent chapters.

3.1 Wrapped D-Branes and gauge bundles on a two-dimensional torus

We will start from a simple configuration of a D-Brane doubly wrapped along the b-cycle of a two-dimensional torus, without introducing any background gauge field yet, in order to show that a brane wrapped twice along a compactified direction is equivalent to a collection of two coincident branes in the same direction, glued together by suitably chosen holonomy matrices acting on the fields living in

their world-volumes [72, 19]. We aim to describe such D-Brane by using only the zero-modes of the open strings living on its world-volume. Hence we will restrict ourselves to the point particle fields that describe the dynamics of these modes, without needing the details of the stringy nature of the objects involved. These details will be introduced in subsequent chapters.
A doubly wrapped D-Brane along the b-cycle along x^2 of a two-torus in the covering space of the compact target manifold has, from the world-volume point of view, two inequivalent copies which are superposed due to the periodicity of the cycle. Hence there are two types of open strings to be considered: strings whose both extrema end on the same copy of the wrapped brane in the covering space of the torus, and strings stretched between the two copies that are identified on the compact direction. Imagining the two copies of the brane as two coincident D-Branes, then one would distinguish between these two types of strings through their Chan-Paton factors. The diagonal components of the Chan-Paton matrices represent the two copies of the brane, while the off-diagonal ones are related to strings stretched between them. Hence the fields related to the zero-modes of the two types of strings mentioned above can be described as the following adjoint scalar fields

$$\Psi^{(1)}(x^1,x^2) = \begin{pmatrix} e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2}{2R_2}x^2} & 0 \\ 0 & e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2}{2R_2}(x^2+2\pi R_2)} \end{pmatrix} \quad \text{and}$$
$$\Psi^{(2)}(x^1,x^2) = \begin{pmatrix} 0 & e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2}{2R_2}x^2} \\ e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2}{2R_2}(x^2+2\pi R_2)} & 0 \end{pmatrix} \tag{3.1.1}$$

where we have made use of the quantisation of the momentum ($n_M \in \mathbb{Z}$) along a compactified direction x^M with radius R_M, taken into account that on the wrapped brane the cycle x^2 has a double effective length as $x^2 = x^2 + 4\pi R_2$, and shifted x^2 by $2\pi R_2$ when considering the second copy of the brane in the torus covering space. The expression above can be further simplified by distinguishing the cases in which n_2 is even or odd. By doing so one finds four different states

$$\Psi^{(1)}_{\text{even}}(x^1,x^2) = e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2'}{R_2}x^2} \times 1 \tag{3.1.2}$$
$$\Psi^{(1)}_{\text{odd}}(x^1,x^2) = e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{2n_2'+1}{2R_2}x^2} \times \sigma_3 \tag{3.1.3}$$
$$\Psi^{(2)}_{\text{even}}(x^1,x^2) = e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2'}{R_2}x^2} \times \sigma_1 \tag{3.1.4}$$
$$\Psi^{(2)}_{\text{odd}}(x^1,x^2) = e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{2n_2'+1}{2R_2}x^2} \times \mathrm{i}\sigma_2 \tag{3.1.5}$$

Notice that the Chan-Paton factors arisen resemble the ones that describe strings attached to two coincident D-Branes where all the states must transform under a

$U(2)$ gauge symmetry. In fact a wrapped D-Brane can be seen at least classically as a particular multi D-Brane state [72, 19], whose world-volume fields have a non trivial holonomy. Indeed the case studied above is equivalent to a system of two D-Branes with holonomies

$$U_1 = 1 \quad \text{and} \quad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1.6}$$

This means that an open string field $\Psi(x^1, x^2)$ which is $U(2)$-valued should satisfy the holonomy conditions

$$\Psi(x^1 + 2\pi R_1, x^2) = \Psi(x^1, x^2) \quad \text{and} \quad \Psi(x^1, x^2 + 2\pi R_2) = U_2 \Psi(x^1, x^2) U_2^\dagger \tag{3.1.7}$$

Indeed, from this request, one can easily recover the aforementioned zero-mode dependence of the string fields living on the wrapped D-Brane world-volume. This situation is actually continuously connected to the standard case of two D-Branes with trivial holonomy, where the $U(1) \times U(1)$ gauge group, that here arises from the massless open strings with Chan-Paton factors given by the identity matrix and σ_1, is enhanced to $U(2)$. Recall, in fact, that, in a toroidal compactification in which the space-time is a direct product of a four-dimensional Minkowski and a compact space, the masses for the Kaluza-Klein modes of the scalar fields Ψ transforming in the adjoint representation are given by the eigenvalues of the Klein-Gordon operator in the internal, compactified directions. Here it is manifest that the KK-modes of the fields which are proportional to σ_2 and σ_3 can never have zero mass, for any choice of the Kaluza-Klein momenta n_1 and n_2. In string theory compactifications, furthermore, these scalar fields arise as internal components of the gauge fields in the spectrum of the open strings attached with both the extrema to the same (stack of) D-Brane [20], thus, only the massless fields can be related to an unbroken gauge symmetry.

The breaking of the gauge group $U(2)$ down to $U(1) \times U(1)$ is achieved by introducing a non abelian Wilson line on the wrapped D-Brane, which commutes only with the fields that remain massless after the two copies of the D-Brane are glued into a single wrapped object. Moreover the setups of two coincident D-Branes and of a unique wrapped D-Brane can be interpolated from one to the other by defining the Wilson line

$$A_2 = \alpha(-1 + \sigma_1) R_1 \tag{3.1.8}$$

that depends on the real parameter $\alpha \in [0, \frac{1}{4}]$. If the holonomy generated by such constant background gauge field on the brane is

$$U_{2,\alpha} = \exp\left\{ \frac{\mathrm{i}}{R_1 R_2} \int_0^{2\pi R_2} A_2 dx^2 \right\} \tag{3.1.9}$$

then the two extremal values for the parameter α reproduce $U_{2,\alpha=0} = 1$, which describes the configuration with a $U(2)$ gauge symmetry, and $U_{2,\alpha=1/4} = \sigma_1$ which breaks the symmetry down to $U(1)\times U(1)$. All of the intermediate values that the parameter α can assume will be related to smooth classical deformations which interpolate between the two extremal situations.

Let us now introduce a non zero background magnetic field on the world volume of the wrapped D-Brane. A magnetic field defined on a compact manifold is actually a monopole field and, according to the Dirac quantization, in the case of a doubly wrapped D-Brane, it should be half-integer

$$F = \begin{pmatrix} 0 & \frac{p}{2} \\ -\frac{p}{2} & 0 \end{pmatrix} \tag{3.1.10}$$

where $p \in \mathbb{Z}$ is known as the Chern class associated to the magnetic field. Of course any gauge potential (gauge bundle) for the monopole field involves a linear, hence non-periodic, function on the torus. This non-periodicity has to be compensated for by requiring that the gauge potentials on two adjacent fundamental domains of the two-torus in its covering space are related by a gauge transformation, namely

$$\begin{array}{l} A_M(x^1 + 2\pi R_1, x^2) = U_1 \left(2\pi \mathrm{i} R_1 R_2 \partial_M + A_M(x^1, x^2)\right) U_1^\dagger \\ A_M(x^1, x^2 + 2\pi R_2) = U_2 \left(2\pi \mathrm{i} R_1 R_2 \partial_M + A_M(x^1, x^2)\right) U_2^\dagger \end{array} \tag{3.1.11}$$

with $M = 1, 2$. In our case we can choose the following asymmetric gauge potential

$$\begin{array}{l} A_1 = 0 \\ A_2 = \frac{p}{2} x^1 \end{array} \tag{3.1.12}$$

for which the suitable gauge transformation that encodes the transition between two different copies of the fundamental region of the torus along x^1 is $V_1 = \exp\left(\mathrm{i}\frac{p}{2}\frac{x^2}{R_2}\right)$ for $0 \leqslant x^2 < 4\pi R_2$, as the D-Brane is wrapped twice along the b-cycle.

In the language of the bundles described through holonomy matrices this configuration corresponds to the holonomies

$$\begin{array}{l} U_1 = \sigma_3 e^{\mathrm{i}\frac{p}{2}\frac{x^2}{R_2}} \\ U_2 = \sigma_1 \end{array} \tag{3.1.13}$$

where now $0 \leqslant x^2 < 2\pi R_2$. Similarly to the case of an unmagnetised D-Brane as seen earlier, the holonomy matrix U_2 glues the two D-Branes along the x^2 axis of the torus to create a unique doubly wrapped D-Brane. Notice that U_2 is

also consistent with the gauge transformations (3.1.11) on the background gauge potential as the latter, being proportional to the identity matrix, commutes with σ_1. The holonomy matrix U_1 is constructed in such a way that the $(1,1)$-entry corresponds to the gauge transformation for the gauge potential with the choice $0 \leqslant x^2 < 2\pi R_2$, while in the other diagonal element the x^2 coordinate has been shifted by $2\pi R_2$.
Of course any field charged under the gauge potential (3.1.12) should obey similar conditions to Eq.(3.1.11). For instance, fields transforming in the fundamental (Φ) or in the adjoint (Ψ) representation must satisfy

$$\Phi(x+2\pi R_M) = U_M(x)\Phi(x) \quad \text{and} \quad \Psi(x+2\pi R_M) = U_M(x)\Psi(x)U_M^\dagger(x) \tag{3.1.14}$$

The first of these relations also implies that, since

$$\Phi\left((x+2\pi R_1)+2\pi R_2\right) = \Phi\left((x+2\pi R_2)+2\pi R_1\right) \tag{3.1.15}$$

then, in order to have a consistent bundle, the holonomy matrices must satisfy the overlap condition

$$U_2^\dagger(x)U_1^\dagger(x+2\pi R_2)U_2(x+2\pi R_1)U_1(x) = 1 \tag{3.1.16}$$

which is indeed the case for the choice made in Eq.(3.1.13).
We can now return to the analysis of the scalar fields in the adjoint representation living on the world-volume of the doubly wrapped D-Brane. As a consequence of the relations in Eq.(3.1.14), it is possible to show that one can distinguish, as in the unmagnetised case, between four different fields

$$\Psi_1(x^1,x^2) = e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{n_2}{R_2}x^2} \times 1 \tag{3.1.17}$$

$$\Psi_2(x^1,x^2) = e^{\mathrm{i}\frac{n_1}{R_1}x^1+\mathrm{i}\frac{2n_2+1}{2R_2}x^2} \times \sigma_3 \tag{3.1.18}$$

$$\Psi_3(x^1,x^2) = e^{\mathrm{i}\frac{2n_1+1}{2R_1}x^1+\mathrm{i}\frac{n_2}{R_2}x^2} \times \sigma_1 \tag{3.1.19}$$

$$\Psi_4(x^1,x^2) = e^{\mathrm{i}\frac{2n_1+1}{2R_1}x^1+\mathrm{i}\frac{2n_2+1}{2R_2}x^2} \times \sigma_2 \tag{3.1.20}$$

with $n_M \in \mathbb{Z}$. This is due to the fact that here the holonomy matrices involve σ_1 and σ_3 at the same time, thus the scalar fields in the adjoint representation can be massless only if they commute with both. This only applies to fields proportional to the identity matrix as it is manifest from the result above. Hence in a string theoretical compactification, where these states are the internal components of the gauge fields living on the world-volume of the wrapped D-Brane, we can argue that a doubly wrapped D-Brane with a non trivial background magnetic field turned on is related to a $U(1)$ low energy gauge theory, as it would be expected

for a single unwrapped D-Brane. Similarly stacks of N coincident D-Branes will be associated, as usual, to $U(N)$ gauge symmetries.
Another fundamental result which can be found from the transformation of the charged fields (3.1.14) is the behaviour of the holonomy matrices introduced under a generic gauge transformation. Focusing in particular on the fields in the fundamental representation one has that

$$\Phi'(x) = \gamma(x)\Phi(x) \quad \Rightarrow \quad U'_M(x) = \gamma(x + 2\pi R_M)U_M(x)\gamma^\dagger(x) \tag{3.1.21}$$

Notice that $\gamma(x)$ is a $U(2)$ transformation even if the actual gauge symmetry associated to the wrapped D-Brane is broken down to a single $U(1)$. This follows from the fact that the fundamental and adjoint fields in Eq.(3.1.14) are $U(2)$ valued, but, as already stressed, only one of them can be massless. Furthermore no periodicity constraints have to be imposed on the transformation $\gamma(x)$.

Let us now reverse the roles of the two wrappings of the D-Brane and consider the same kind of configuration where the D-Brane wraps the a-cycle twice and has the same magnetic field as before turned on in its world-volume. A gauge bundle appropriate to describe this system is given by the following gauge choice

$$\begin{aligned} A_1 &= -\tfrac{p}{2}x^2 \\ A_2 &= 0 \end{aligned} \tag{3.1.22}$$

together with the holonomy matrices

$$\begin{aligned} U_1 &= \sigma_1 \\ U_2 &= \sigma_3 e^{-\mathrm{i}\frac{p}{2}\frac{x^1}{R_1}} \end{aligned} \tag{3.1.23}$$

The analysis of the scalar fields transforming in the adjoint representation reveals that the only modification with respect to the case in which the D-Brane is wrapped along the b-cycle is, as expected, the exchange of the roles of the two coordinates of the torus. The mass spectrum and the unbroken gauge symmetry of the configurations perfectly match. Hence there is a one-to-one relation that maps the states associated to the open strings attached to a doubly wrapped D-Brane with wrapping numbers $(1, 2)$ along the two cycles of the torus to the states of the a similar D-Brane with wrapping numbers $(2, 1)$. This can be easily understood by considering this second system and performing the gauge transformation given by $\gamma(x) = \exp\left(\mathrm{i}\frac{p}{2}\frac{1}{2\pi R_1 R_2}x^1x^2\right)$. Then the gauge potential is mapped into the one chosen for the first setup with wrappings $(1, 2)$ and the holonomy matrices are transformed to

$$\begin{aligned} U_1 &= e^{\mathrm{i}\frac{p}{2}\frac{x^2}{R_2}}\sigma_1 \\ U_2 &= \sigma_3 \end{aligned} \tag{3.1.24}$$

By finally introducing a change of basis for the non abelian group indices one can transform $\sigma_1 \leftrightarrow \sigma_3$ and obtain the same gauge bundle as the one describing the D-Brane with wrapping numbers $(1,2)$. Thus already from this simple example one could argue that on a two-dimensional torus configurations of wrapped magnetised D-Branes are classically equivalent if they are characterised by the same Chern class and by the same product of wrapping numbers along the two cycles of the torus. Indeed such configurations are identified by gauge bundles which are related to each other by suitable gauge transformations. It is not difficult, in fact, to generalise the arguments above for wrapping numbers (w_1, w_2) and convince oneself that this statement is always true for a two-dimensional compactification. From the string theoretical point of view this will also be manifest when considering the duality relating configurations of magnetised and intersecting D-Branes, as in the latter picture the only information that identifies the corresponding D-Brane at angle will be given by the Chern class and the product of the wrappings of the magnetised brane on the T^2.

3.2 Magnetised wrapped D-Branes on higher dimensional tori

Since we will be considering compactifications of string theory on generic $2d$-dimensional tori, involving magnetised D-Branes and D-Branes intersecting at angles, it is convenient to generalise the discussion in the previous section to such higher dimensional manifolds, introducing the properties of the gauge bundles defined on them. Before entering these details, we first give a brief overview on the general characteristics of a T^{2d}.

A $2d$-dimensional real torus T^{2d} is defined by a collection of $2d$ vectors a_M in the Euclidean space $\mathbb{R}^{2d}$. In particular T^{2d} is simply $\mathbb{R}^{2d}$ modulo the identification with integer shifts along the a_M's

$$x \equiv x + 2\pi\sqrt{\alpha'} \sum_{M=1}^{2d} c^M a_M \quad , \quad \forall x \in \mathbb{R}^{2d} \text{ and } c^M \in \mathbb{Z} \tag{3.2.25}$$

In what follows the conventions chosen will be such that the real coordinates on the torus x^M, with $M = 1, \ldots, 2d$, will all have periods $2\pi\sqrt{\alpha'}$ and will be referred to as the integral basis. One can choose a Cartesian reference frame and write the components of each vector a_M as a^a_M. Then the metric on T^{2d} is $G_{MN} = \sum_{a=1}^{2d} a^a_M a^a_N$. The components of each a_M can be used to fill in the column vectors of a square matrix $E^a_M \equiv a^a_M$; then, by construction, E is a vielbein matrix satisfying

$$G = {}^t E E \quad , \quad 1 = {}^t E^{-1} G E^{-1} \tag{3.2.26}$$

The inverse matrix E^{-1} instead has the dual vectors $\hat{a}^M$ as rows, which are defined by the relations $\sum_{a=1}^{2d} \hat{a}_a^M a_N^a = \delta_N^M$. Of course any other matrix obtained by means of an orthogonal $SO(2d)$ rotation of E is a good vielbein matrix, which generates the same metric G_{MN} on the torus.

In this framework, similarly to the discussion in the previous section devoted to a single T^2, a D-Brane with multiple wrappings is better described in terms of a non-trivial gauge bundle on the torus T^{2d}. In absence of background magnetic fields and in the simpler case of a D-Brane whose non trivial wrapping w is only along one particular cycle of the compactification torus, the matrix that glues the w copies of the brane in a single wrapped object is given by the straightforward generalisation of σ_1 introduced in the previous section, namely

$$U = P_{w\times w} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2.27}$$

Of course when the D-Brane has non trivial wrappings w_M, $M = 1, \ldots, 2d$, along each of the directions of the T^{2d}, the gluing matrices assume the form of tensor products

$$U_M = 1_{w_1\times w_1} \otimes \cdots \otimes P_{w_M\times w_M} \otimes \cdots \otimes 1_{w_{2d}\times w_{2d}} \tag{3.2.28}$$

Observe that the dimensionality of the holonomy matrices is always given by the product of all the wrappings $w = \prod_{M=1}^{2d} w_M$.

A background magnetic field in the integral basis is quantised as a consequence of the compactness of the torus as

$$F_{MN} = \frac{p_{MN}}{w_M w_N} \tag{3.2.29}$$

where p_{MN} is an integer matrix [1]. It is possible to show [74] (see Appendix A.3 for a more pedestrian proof) that any $2d \times 2d$ antisymmetric integer matrix M can

[1] Defining the parallel transport $\Phi(x_f) = \exp\left[\frac{\mathrm{i}}{2\pi\alpha'}\int_{x_f}^{x_i} A\right]\Phi(x)$ of a fundamental field Φ living in the world-volume of the wrapped brane with background gauge field A and considering the particular closed path $x^M \to x^M + 2\pi\sqrt{\alpha'}w_M \to x^M + 2\pi\sqrt{\alpha'}w_M + 2\pi\sqrt{\alpha'}w_N \to x^M + 2\pi\sqrt{\alpha'}w_N \to x^M$ one obtains $\Phi(x_f \equiv x_i) = \exp\left[\frac{\mathrm{i}}{2\pi\alpha'}\oint A\right]\Phi(x_i)$. Applying the Gauss theorem yields $\Phi(x_f \equiv x_i) = \exp\left[\frac{\mathrm{i}}{2\pi\alpha'}\int F\right]\Phi(x_i)$ where now the integral is over the surface whose boundary is the closed path defined above. As the brane world-volume is a compact manifold it is possible to choose two different surfaces with the same boundary, one enclosing the full magnetic field flux, the other being trivial. Consistency then requires the full magnetic flux to be quantised as shown.

be put in a block-diagonal form by means of a unimodular integer transformation O, namely

$$ {}^t\!OMO = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_2 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_d \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.2.30} $$

We can apply this theorem to rewrite the magnetic field (3.2.29) on a generic magnetised D-Brane in a rather simple form. In order to do so it is first convenient to define the integer matrix $\hat{p}$ associated to the matrix in Eq.(3.2.29) which instead has rational entries

$$ \hat{p} = F \times \text{l.c.m.}\,\{w_M w_N,\ \forall M, N = 1, \ldots, 2d\} = \omega F \tag{3.2.31} $$

Here l.c.m. is the least common multiple of all the pairs of wrappings that appear in the denominators of Eq.(3.2.29). Since the transformation O contains integer entries, it can be seen as a redefinition of the lattice (3.2.25) of the torus, where the vectors a_M are mapped into integer linear combinations of themselves. Of course the map is consistent only if there is a one-to-one correspondence between the points of the lattices before and after the transformation induced by O. This is ensured by the fact that O is a unimodular matrix, hence its inverse has exactly the same characteristics. Notice that we have not requested O to be also an orthogonal rotation, thus we expect the metric G_{MN} of the torus to be non-trivially modified by this transformation. In geometrical terms the basis for the lattice in which $\hat{p}$ is block-diagonal simply corresponds to the most convenient choice of the fundamental domain of the T^{2d} to describe the fields defined on it. Thus

$$ \hat{p} \to {}^t\!O\hat{p}O = \omega\, {}^t\!OFO = \omega F_{\text{block}} \tag{3.2.32} $$

where F_{block} has the same form as Eq.(3.2.30) with rational entries. Observe that the numerators and denominators of the elements of F_{block} could still have common factors, expurgating which we can write the magnetic field on the wrapped D-Brane world volume in the form [2]

$$ F = \begin{pmatrix} 0 & \frac{p_1}{W_1} & 0 & 0 & \cdots \\ -\frac{p_1}{W_1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{p_2}{W_2} & \cdots \\ 0 & 0 & -\frac{p_2}{W_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.2.33} $$

[2] With a small abuse of notation we will indicate again with F the magnetic field in this final form, which will be the only one used in the subsequent chapters.

where $p_\alpha \in \mathbb{Z}$ and $W_\alpha \in \mathbb{Z} - \{0\}$, $\forall \alpha = 1, \ldots, d$. Notice that even if the field F now describes a direct product of d T^2's inside the T^{2d}, the compactification is in general non factorisable as a consequence of the form of the metric. The p_α's can be interpreted as the Chern classes of the magnetic fields while the W_α's are the products of the couples of wrapping numbers on each of the T^2's inside T^{2d}. Two comments are in order now. First, expurgating the common factors between numerators and denominators in F_{block} corresponds to smoothly deforming the brane configuration, at the classical level, to a new configuration with co-prime p_α and W_α with the same ratio. Indeed, similarly to the situation discussed in the context of a single T^2 compactification, one can find a whole family of consistent bundles that interpolate between these different configurations. Second, in the final form of the magnetic field in Eq.(3.2.33), it is not necessary to specify all the wrappings of the brane along each cycle of the torus. In fact as already shown in the previous section, configurations of magnetised D-Branes with the same product of wrappings along the pairs of cycles of the T^2's inside the T^{2d} are classically equivalent. The holonomy matrices defining the gauge bundle of a brane wrapped along a T^2 are related by a gauge transformation if they describe branes with the same Chern class p and the same product of the wrappings W. Thus we can choose, for instance, to wrap the branes W_α times along the even directions $x_M \equiv x_{2\alpha}$ only. We will also make the following gauge choice for the gauge potential associated to the field strength (3.2.33)

$$A_{M\equiv 2\alpha}(x) = \frac{p_\alpha}{W_\alpha} x_{2\alpha-1} \quad \text{and} \quad A_{M\equiv 2\alpha-1}(x) = 0 \quad \forall \alpha = 1, \ldots, d \tag{3.2.34}$$

The gauge bundle that describes the magnetised D-Brane on the $2d$-dimensional torus is identified by the gauge choice above together with the set of holonomy matrices that encode both the gauge transformation of the non periodic potential (3.2.34) defined on two adjacent fundamental regions of the torus in its covering space, namely

$$A_M\left(x + 2\pi\sqrt{\alpha'}a_N\right) = U_N(x)\left(2\pi\alpha' \mathrm{i}\partial_M + A_M(x)\right)U_N^\dagger(x) \tag{3.2.35}$$

and the gluing of the copies of the wrapped D-Brane along the even directions of the compact manifold. The result is given by

$$\begin{aligned} U_{2\alpha}(x) &= 1_{W_1\times W_1} \otimes \ldots \otimes P_{W_\alpha\times W_\alpha} \otimes 1_{W_{\alpha+1}\times W_{\alpha+1}} \otimes \ldots \otimes 1_{W_d\times W_d} && (3.2.36) \\ U_{2\alpha-1}(x) &= 1_{W_1\times W_1} \otimes \ldots \otimes (Q_{W_\alpha\times W_\alpha})^{p_\alpha} \otimes 1_{W_{\alpha+1}\times W_{\alpha+1}} \otimes \ldots \otimes 1_{W_d\times W_d}\ e^{\frac{\mathrm{i}}{\sqrt{\alpha'}}\frac{p_\alpha}{W_\alpha}x_{2\alpha}} \end{aligned}$$

where $Q_{W_\alpha\times W_\alpha} = \mathrm{diag}\left\{1, \mathrm{e}^{2\pi\mathrm{i}/W_\alpha}, \ldots, \mathrm{e}^{2\pi\mathrm{i}(W_\alpha-1)/W_\alpha}\right\}$. Let us stress that the bundle is consistent as it satisfies the relation

$$U_N^\dagger(x)U_M^\dagger\left(x + 2\pi\sqrt{\alpha'}a_N\right)U_N\left(x + 2\pi\sqrt{\alpha'}a_M\right)U_M(x) = 1 \tag{3.2.37}$$

similar to the one already written for a single T^2 compactification. Again charged fields transforming in the fundamental (Φ) or in the adjoint (Ψ) representation must satisfy

$$\Phi\left(x+2\pi\sqrt{\alpha'}a_N\right)=U_N(x)\Phi(x)\quad,\qquad\Psi\left(x+2\pi\sqrt{\alpha'}a_N\right)=U_N(x)\Psi(x)U_N^\dagger(x) \tag{3.2.38}$$

and, as a consequence, under a generic (not necessarily periodic) gauge transformation $\gamma(x)$, the gluing matrices U_N transform in the following fashion

$$U_N(x)\to\gamma\left(x+2\pi\sqrt{\alpha'}a_N\right)U_N(x)\gamma^\dagger(x) \tag{3.2.39}$$

3.3 Closed strings on toroidal manifolds and T-Duality

T-Duality is a symmetry of string theory that holds order by order in string perturbation theory. We will consider in this section compactifications of the bosonic closed string sector of the theory with toroidal flat internal manifolds, and we will discuss the duality group focusing in particular on the invariance of the spectrum of bosonic closed strings in such compactifications [73]. We will leave instead the discussion of the duality acting on the open strings, and the D-Branes to which they are attached, to the sections devoted to the analysis of their spectrum.

Let us first review the basics of closed bosonic string theory compactifications on toroidal flat manifolds. The Lorentzian action

$$S_{\text{bos}}=-\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left[G_{MN}\partial^\alpha x^M\partial_\alpha x^N+\epsilon^{\alpha\beta}\partial_\alpha x^M\partial_\beta x^N B_{MN}\right] \tag{3.3.40}$$

with $\epsilon^{\sigma\tau}=1$, describes the dynamics of the string coordinates, $x^M(\tau,\sigma)$ with $M=1,\ldots,2d$, propagating on a toroidal background T^{2d} characterised by a constant metric G_{MN} and a constant Kalb-Ramond antisymmetric field B_{MN}. Of course, in order for the theory to be at the critical dimension, in which the Weyl anomaly is cancelled and one can choose, as we have done in Eq.(3.3.40), the metric of the world-sheet to simply be the orthonormal $\eta_{\alpha\beta}$, it is necessary to think of the toroidal background, T^{2d}, as tensored for example with a $(26-d)$-dimensional flat Minkowski manifold [20], whose presence will be neglected in what follows.

Since the geometrical background encoded in the metric and B-field is described by constant matrices, the sigma-model defined by the action (3.3.40) is free and exactly solvable. In particular one can easily show that the equations of motion reduce to the free propagation of left and right handed fields, while the boundary

conditions which set the surface term in the variation of the action to zero can be chosen to be periodic, $x^M(\sigma + 2\pi, \tau) = x^M(\sigma, \tau) + 2\pi m^M\sqrt{\alpha'}$, to describe a closed string. Notice that here $m^M \in \mathbb{Z}^{2d}$ is a collection of integer numbers that indicates that the string can be wound along the M-th direction of the torus, with radius $\sqrt{\alpha'}$, m^M times. This is an important novelty of String Theory on a compact space, with respect for instance to compactifications of quantum field theories, and it will be a crucial starting point to investigate T-Duality as a new symmetry of purely stringy models.
The general solution to the equations of motion reads

$$x^M(\sigma, \tau) = \frac{1}{2}\left[X^M(\tau + \sigma) + \tilde{X}^M(\tau - \sigma)\right] \tag{3.3.41}$$

where

$$X^M(\tau + \sigma) = x_0^M + \sqrt{2\alpha'}(\tau + \sigma)\alpha_0^M + \sqrt{2\alpha'}\sum_{m\neq 0}\frac{\alpha_m^M}{m}e^{-im(\tau+\sigma)} \tag{3.3.42}$$

and $\tilde{X}(\sigma, \tau)$ is obtained by replacing $\alpha_n^M \to \tilde{\alpha}_n^M\ \forall n \in \mathbb{Z}$.
By defining the conjugate momentum of the closed string coordinate x^M as

$$2\pi\alpha' P_M = 2\pi\alpha'\frac{\delta\mathcal{L}}{\delta\partial_\tau x^M} = G_{MN}\partial_\tau x^M + B_{MN}\partial_\sigma x^N \tag{3.3.43}$$

one can construct the Hamiltonian associated to the action (3.3.40), which contains the information of the energy spectrum of the theory,

$$H = \int_0^{2\pi} d\sigma\left(P_M\partial_\tau x^M - \mathcal{L}\right) \tag{3.3.44}$$

and express it in terms of the string coordinates and their conjugate momenta, eliminating the dependence on their worldsheet time derivatives, namely

$$\begin{aligned} H = \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma\ [\ &(2\pi\alpha')^2 P_M(G^{-1})^{MN}P_N + \partial_\sigma x^M(G - BG^{-1}B)_{MN}\partial_\sigma x^N + \\ &+ 4\pi\alpha'\partial_\sigma x^M(BG^{-1})_M{}^N P_N] \end{aligned} \tag{3.3.45}$$

If we introduce now the adimensional left and right momenta

$$\begin{aligned} P_{La} &= \tfrac{1}{\sqrt{2\alpha'}}\left[2\pi\alpha' P_M + (G - B)_{MN}\partial_\sigma x^N\right](E^{-1})^M{}_a \\ P_{Ra} &= \tfrac{1}{\sqrt{2\alpha'}}\left[2\pi\alpha' P_M - (G + B)_{MN}\partial_\sigma x^N\right](E^{-1})^M{}_a \end{aligned} \tag{3.3.46}$$

where E_M^a is the vielbein that defines the metric as in Eq.(3.2.26), the Hamiltonian simply becomes

$$H = \frac{1}{4\pi}\int_0^{2\pi} d\sigma\left(P_L^2 + P_R^2\right) \tag{3.3.47}$$

Let us focus for a moment on the zero-modes contributions to the Hamiltonian above. In our conventions

$$\begin{aligned}(P_{0L})_M &= (G\alpha_0)_M = \tfrac{1}{\sqrt{2}}\left[n_M + (G-B)_{MN}m^N\right] \\ (P_{0R})_M &= (G\tilde{\alpha}_0)_M = \tfrac{1}{\sqrt{2}}\left[n_M - (G+B)_{MN}m^N\right]\end{aligned} \tag{3.3.48}$$

where the relation between the adimensional momenta and the string zero modes follows from Eq.(3.3.41) and Eq.(3.3.43), while the last equality descends from the fact that the zero mode of the conjugate momentum is quantised, $2\pi P_{0M} = \frac{n_M}{\sqrt{\alpha'}}$, and so is $\partial_\sigma x^M$ in terms of the winding numbers m^M. Hence the zero modes contribution to the Hamiltonian is

$$H_0 = \frac{1}{2}\left[n_M(G^{-1})^{MN}n_N + m^M(G - BG^{-1}B)_{MN}m^N + 2m^M(BG^{-1})_M{}^N n_N\right] \tag{3.3.49}$$

Notice also that since the Lorentzian length $P_L^2 - P_R^2 = 2m^M n_M$ is even, (P_{La}, P_{Ra}) define a self-dual even Lorentzian lattice $\Gamma^{(d,d)}$, also known as Narain lattice [20]. We can now consider the well known particular case of the compactification of bosonic closed strings on a circle, in order to start determining the T-Duality transformations in this simple example.

In the notation we have been using the periodicities of the compact directions are all equal to $2\pi\sqrt{\alpha'}$. In order to introduce a generic radius of compactification for the circle we can simply take

$$G = \left(\frac{R}{\sqrt{\alpha'}}\right)^2 \tag{3.3.50}$$

where R is the radius of the circle of compactification. Substituting this form of the metric back into the formula Eq.(3.3.49) restricted to one dimension only, and hence also with $B = 0$ one finds

$$H_0 = \frac{\alpha'}{4}\left[\left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2\right] \tag{3.3.51}$$

As expected also in compactifications of quantum field theories on a circle, in the limit $R \to \infty$, the Kaluza-Klein states corresponding to the quantised momenta n form a continuous spectrum, while the stringy winding states simply become infinitely massive. The great novelty of String Theory is found in the limit in which $R \to 0$. In this case, in fact, there is a continuum of states as well given by the winding states, which would not appear in a quantum field theory. Thus the two limits look identical from the physical point of view as though even in the case of $R \to 0$ there was an uncompactified direction.

The well known T-Duality transformation on the closed string states, which leaves the spectrum of the theory invariant in this simple example, is

$$R \leftrightarrow \frac{\alpha'}{R} \quad \text{together with} \quad n \leftrightarrow m \tag{3.3.52}$$

which also means $P_L \leftrightarrow P_L$ and $P_R \leftrightarrow -P_R$. In terms of the field in the definition (3.3.41) the duality transformation acts as follows

$$x(\tau,\sigma) \rightarrow x'(\tau,\sigma) = \frac{1}{2}\left[X(\tau+\sigma) - \tilde{X}(\tau-\sigma)\right] \tag{3.3.53}$$

leading exactly to the change of sign of the right-handed part of the momentum zero-mode. This last representation of the action of T-Duality on the string fields is also very useful for its generalisation to the open string sector [20].
This is just a very small portion of the full symmetry group that acts on the closed string states in a generic toroidal compactification. Indeed going back to the most general case, in order to find the symmetry group of the model defined by the action (3.3.40) it is useful first to determine the transformations that generate all the possible Lagrangians, i.e. all the possible backgrounds in the action (3.3.40), that are consistent with the properties discussed above. In particular one can show that all the even self-dual Lorentzian lattices are related to one another via $O(2d, 2d, \mathbb{R})$ transformations, which can be conveniently represented as follows

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2d, 2d, \mathbb{R}) \Leftrightarrow {}^t g J g = J \tag{3.3.54}$$

where a, b, c, d are $2d \times 2d$ matrices and

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.3.55}$$

This request constrains the elements of g in the following fashion

$${}^t a c + {}^t c a = 0 \quad , \quad {}^t b d + {}^t d b = 0 \quad , \quad {}^t a d + {}^t c b = 1 \tag{3.3.56}$$

Notice that as $J^{-1} = J$ also ${}^t g \in O(2d, 2d, \mathbb{R})$ and, as a consequence, we find more constraints on the matrices above, namely

$$a\,{}^t b + b\,{}^t a = 0 \quad , \quad c\,{}^t d + d\,{}^t c = 0 \quad , \quad a\,{}^t d + b\,{}^t c = 1 \tag{3.3.57}$$

The Hamiltonian (3.3.49) can be rewritten as

$$H = \frac{1}{2}\,{}^t Z M Z \quad \text{, with} \quad Z = (m, n) \quad \text{and} \quad M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{3.3.58}$$

and the matrix M transforms under the $O(2d,2d,\mathbb{R})$ transformation g as

$$M_g = gM\,{}^t g \tag{3.3.59}$$

However it is invariant, as it is manifest from Eq.(3.3.47), under the action of the maximal compact subgroup of $O(2d,2d,\mathbb{R})$, which is $O(2d,\mathbb{R})\times O(2d,\mathbb{R})$, hence the moduli space of the models compactified on a $2d$-dimensional torus is given by the coset

$$\mathcal{M} = \frac{O(2d,2d,\mathbb{R})}{O(2d,\mathbb{R})\times O(2d,\mathbb{R})} \tag{3.3.60}$$

Let us now consider explicitly the transformation of the geometrical background in the action (3.3.40) in terms of the elements of $g\in O(2d,2d,\mathbb{R})$. It is useful to define the action of the group element on a generic $2d\times 2d$ matrix, H, as a fractional linear transformation

$$g(H) = (aH+b)(cH+d)^{-1} \tag{3.3.61}$$

and it is easy to check that this representation is consistent with the group property $g(g'(H)) = (gg')(H)$. By introducing

$$g_{(G+B)} = \begin{pmatrix} {}^tE & BE^{-1} \\ 0 & E^{-1} \end{pmatrix} \tag{3.3.62}$$

E being again the vielbein of Eq.(3.2.26), one indeed embeds the geometrical moduli space $O(2d,2d,\mathbb{R})/O(2d,\mathbb{R})\times O(2d,\mathbb{R})$ into $O(2d,2d,\mathbb{R})$ because $g_{(G+B)}(1) = G+B$. Since from Eq.(3.3.58) $M = g_{(G+B)}\,{}^t g_{(G+B)}$, then from Eq.(3.3.59) $M_g = gg_{(G+B)}\,{}^t g_{(G+B)}\,{}^t g = g_{(G+B)'}\,{}^t g_{(G+B)'}$, hence $g_{(G+B)'} = gg_{(G+B)}$, from which

$$(G+B)' = g_{(G+B)'}(1) = gg_{(G+B)}(1) = g(G+B) = [a(G+B)+b]\,[c(G+B)+d]^{-1} \tag{3.3.63}$$

Furthermore, taking the transpose of the expression above and using the relations in (3.3.56), it is possible to show that

$$(G-B)' = [a(G-B)-b]\,[d-c(G-B)]^{-1} \tag{3.3.64}$$

These transformations, together with the constraints in Eq.(3.3.56) and Eq.(3.3.57), define all the possible backgrounds for a toroidal compactification model as in Eq.(3.3.40).

We want now to discuss the symmetries of these models focusing on the transformations that preserve the energy spectrum as given by the Hamiltonian (3.3.47). Of course this is not enough to guarantee that the transformations involved are

symmetries that persist order by order in string perturbation theory, but this actually turns out to be the case as shown for example in [73].
We can identify three generators of the subgroup of $O(2d, 2d, \mathbb{R})$ that preserve the spectrum of the compactified theory. The first one is a representation of the integer shifts of the Kalb-Ramond field B_{MN}. The contribution of the B-field to the action is in fact a total derivative, thus its integer shifts change the action only by an integer times 2π and do not contribute to the path integral. An integer shift of the B-field can be represented by the $O(2d, 2d, \mathbb{R})$ matrix

$$g_\Theta = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix} \tag{3.3.65}$$

where Θ is an antisymmetric integer $2d \times 2d$ matrix. In fact under this transformation $(G + B)' = g_\Theta(G + B) = G + B + \Theta$. The invariance of the spectrum can be also checked through the definition of the Hamiltonian in Eq.(3.3.58), as the transformation above boils down to $m \to m$ and $n \to n - \Theta m$ which is a symmetry of the partition function of the model.
The second generator is just a change of basis in the lattice that defines the compactification torus of the model and can be represented as

$$g_A = \begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix} \tag{3.3.66}$$

Indeed here $(G + B)' = g_A(G + B) = A(G + B)\,{}^t\!A$ which is a good change of basis in the lattice if A is a unimodular integer $2d \times 2d$ matrix. The spectrum is manifestly invariant because, starting from Eq.(3.3.58), the transformation is nothing else than $m \to {}^t\!Am$ and $n \to A^{-1}n$.
The third type of symmetry, which is called factorised duality D_i, is a generalisation of the usual $R \leftrightarrow \frac{\alpha'}{R}$ symmetry on a circle [20], and reduces to this case if the compactified target manifold of the model can be seen as a direct product of a circle in the i-th direction and a lower dimensional manifold that is left invariant. The $O(2d, 2d, \mathbb{R})$ transformation that encodes this duality is

$$g_{D_i} = \begin{pmatrix} 1 - e_i & e_i \\ e_i & 1 - e_1 \end{pmatrix} \tag{3.3.67}$$

Here e_i is $2d \times 2d$ matrix whose entries are all zero except from $e_{ii} = 1$, and the spectrum is left invariant as in Eq.(3.3.58) this becomes the exchange $m_i \leftrightarrow n_i$.
The three symmetries described above generate the $O(2d, 2d, \mathbb{Z})$ group which is the duality group of the bosonic closed string sector of string theory compactified on a torus. From now on the elements of this group will be indicated by $T \in$

$O(2d, 2d, \mathbb{Z})$. In addition there is a parity symmetry, given by the inversion $\sigma \rightarrow -\sigma$ together with $B \rightarrow -B$, that exchanges the roles of P_{0L} and P_{0R} in Eq.(3.3.48). As already mentioned it is possible to show that these symmetries persist beyond the level of the partition function of the theory and hold order by order in string perturbation theory [73].
We will briefly now include the non-zero modes contributions to the spectrum in Eq.(3.3.58) in order to determine the duality action on the oscillators in the string coordinates expansion (3.3.41). It is convenient to first write down the $O(2d, 2d, \mathbb{Z})$ transformation for the metric of the torus as

$$ {}^tT_+ G T_+ = {}^tT_- G T_- = G' \tag{3.3.68} $$

where

$$ T_\pm = [c(\pm G + B) + d]^{-1} \tag{3.3.69} $$

These relations can be shown by using that $G' = \frac{1}{2}[(G+B)' + {}^t(G+B)'] = \frac{1}{2}[(G-B)' + {}^t(G-B)']$ combined with Eq.(3.3.63) and Eq.(3.3.64) and with the constraints in Eq.(3.3.56) to evaluate the l.h.s. of the two equations above.
The full Hamiltonian can be rewritten as the sum of the zero-modes contribution as in Eq.(3.3.49) and of the number operators $N = \sum_{n>0} \alpha^M_{-n} G_{MN} \alpha^N_n$, $\tilde{N} = \sum_{n>0} \tilde{\alpha}^M_{-n} G_{MN} \tilde{\alpha}^N_n$

$$ H = \frac{1}{2}\, {}^tZ M Z + N + \tilde{N} \tag{3.3.70} $$

From the definition of the number operators and the transformations in Eq.(3.3.68) it is easy to see that the full spectrum is invariant under an $O(2d, 2d, \mathbb{Z})$ transformation of the type in Eq.(3.3.54) if

$$ \alpha_n = T_+ \alpha'_n \quad \text{and} \quad \tilde{\alpha}_n = T_- \tilde{\alpha}'_n \tag{3.3.71} $$

These transformations are also canonical on the oscillators as they leave their commutation relations invariant

$$ \left[\alpha^M_n, \alpha^N_m\right] = \left[\tilde{\alpha}^M_n \tilde{\alpha}^N_m\right] = n G^{MN} \delta_{m+n,0} \quad , \quad \left[\alpha^M_m, \tilde{\alpha}^N_m\right] = 0 \tag{3.3.72} $$

Finally under worldsheet parity, together with the exchange $B \rightarrow -B$, left and right-handed oscillators are interchanged with each other and the full spectrum is manifestly invariant under this transformation.

4. THE TWISTED SPECTRUM IN THE BRANE WORLD MODELS

We will describe the so-called twisted sector of the Brane World models involving open strings whose endpoints are attached to different stacks of D-Branes, either with different background magnetic fields or intersecting at angles. The relation between these two descriptions will be investigated using the results on T-Duality introduced in the previous chapter. We will mostly focus on the analysis of the bosonic sector of the open string action, reinstating subsequently the fermionic contribution to discuss the GSO projection and supersymmetry in the models under consideration. We will then see how from the twisted sector of the model four-dimensional chiral Weyl fermions arise as fundamental ingredients to embed the Standard Model Lagrangian in String Theory. Finally we will show how the spectrum determined has a finite multiplicity both in the magnetised and in the intersecting D-Branes pictures, allowing for a nice interpretation in the context of String Theory of the family replication of the fundamental particles of the Standard Model.

4.1 The general framework

Let us reconsider the action (3.3.40) for open strings in its Euclidean version, which is obtained by defining the analytic continuation of the world-sheet time as

$$\tau_e = \mathrm{i}\tau \tag{4.1.1}$$

and by allowing for the presence in the world-sheet Lagrangian of a boundary term which describes the coupling of the strings with external gauge fields living on D-Branes. Hence the action for the bosonic open strings reads

$$\begin{aligned} S_{\text{bos}} = &- \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left[\partial^\alpha x^M \partial_\alpha x^N G_{MN} + \mathrm{i}\epsilon^{\alpha\beta}\partial_\alpha x^M \partial_\beta x^N B_{MN}\right] \\ &- \frac{\mathrm{i}}{2\pi\alpha'} \sum_\sigma q_\sigma \int_{C_\sigma} dx^M A^\sigma_M \end{aligned} \tag{4.1.2}$$

where $\sigma = 0, \pi$ in the boundary term labels the two endpoints of the string, with charges $q_\pi = -q_0 = 1$, ending on the boundaries C_σ that coincide with two D-Branes. In fact for the sake of simplicity, as already done in the T-Duality discussion of the previous section, we will neglect for now the existence of a four dimensional flat Minkowski space-time in the target space of the open strings, focusing only in the internal compactified directions. We will actually need to reintroduce the flat space when dealing with the GSO projection in the spectrum of the open supersymmetric strings to discuss how chiral Weyl fermions arise in the model.

In the general framework under analysis we will choose the same geometrical background introduced in the previous chapter for closed strings in toroidal compactifications, described by a constant metric G_{MN} and a constant Kalb-Ramond field B_{MN}, $M = 1 \dots 2d$, while the background gauge field will be at most linear in the string coordinates. Having made this choice, the action (4.1.2) describes a free conformal field theory (CFT), whose equations of motion simply are

$$\partial_\alpha \partial^\alpha x^M(\tau, \sigma) = 0 \tag{4.1.3}$$

The general solution of this kind of model is given in terms of a superposition of left and right-handed fields, namely $X^M(\tau + \sigma)$ and $\tilde{X}^M(\tau - \sigma)$. The variation of the action necessary to obtain the equations of motion also yields a boundary term, which has to be suitably canceled through the choice of appropriate boundary conditions. In the discussion of the closed strings on toroidal compactifications as in section 3.3 the boundary term was simply canceled by requiring that the fields assumed the same value on two extrema of the string, up to a non-trivial winding of the string on the cycles of the torus. Here, instead of choosing periodic conditions on the string field, we can eliminate the boundary term by relating the left and right-handed fields to one another

$$\left[(1 + R_\sigma)_{MN} \partial_\sigma x^N - \mathrm{i}(1 - R_\sigma)_{MN} \partial_\tau x^N\right] \Big|_{\sigma=0,\pi} = 0 \tag{4.1.4}$$

where R_σ are the so-called reflection matrices. We can introduce complex coordinates on the world sheet

$$z = e^{\tau + \mathrm{i}\sigma} \quad , \quad \bar{z} = e^{\tau - \mathrm{i}\sigma} \tag{4.1.5}$$

to rewrite the boundary conditions as

$$\begin{aligned} \bar{\partial} x^M(z, \bar{z})\big|_{\sigma=0} &= (R_0)^M{}_N \, \partial x^N(z, \bar{z})\big|_{\sigma=0} \\ \bar{\partial} x^M(z, \bar{z})\big|_{\sigma=\pi} &= (R_\pi)^M{}_N \, \partial x^N(z, \bar{z})\big|_{\sigma=\pi} \end{aligned} \tag{4.1.6}$$

Let us start focusing on one of the two boundaries the open string ends on, identified with the D-Brane at $\sigma = 0$. Then by integrating the first line in Eq.(4.1.6) we find a relation between the left and right moving fields, solutions of the equations of motion, and write the string coordinates in terms of a meromorphic field as follows

$$x^M(z, \bar{z}) = x_0^M + \frac{1}{2}\left[X^M(z) + (R_0)^M{}_N X^N(\bar{z})\right] \tag{4.1.7}$$

where the complex variable z is defined on the upper half plane. This is often referred to as the "doubling trick". From this equation, compared also to the similar result for closed strings in Eq.(3.3.41), it is manifest that the reflection matrices act as identifications between the left and right-handed fields as $\tilde{X}^M(\bar{z}) \equiv (R_0)^M{}_N X^N(\bar{z})$. Using also the mode expansion in Eq.(3.3.42) at the level of the string oscillators this can be reinterpreted as $\tilde{\alpha}_m^M \equiv (R_0)^M{}_N \alpha_m^N$. Notice that the zero-modes of such expansion become non-commutative. Referring to the literature (see for instance [75]) for a derivation we only state the final result which reads

$$[x^M, x^N] = \mathrm{i}2\pi\alpha' \left(\frac{1}{G+B}\right)^{[MN]} \tag{4.1.8}$$

denoting with $[MN]$ the antisymmetric part of the matrix.
In absence of the identifications we are considering here, the free CFT encoded in the action (4.1.2) would be characterised by a holomorphic and an anti-holomorphic component of the stress-energy tensor

$$T(z) = -\frac{1}{2\alpha'}\partial X^M(z) G_{MN} \partial X^N(z) \ , \ \ \bar{T}(\bar{z}) = -\frac{1}{2\alpha'}\bar{\partial}\tilde{X}^M(\bar{z}) G_{MN} \bar{\partial}\tilde{X}^N(\bar{z}) \tag{4.1.9}$$

whose Laurent expansion would give two independent copies of the Virasoro generators

$$L_m = \oint \frac{dz}{2\pi\mathrm{i}} z^{m+1} T(z) \quad \text{and} \quad \tilde{L}_m = \oint \frac{d\bar{z}}{2\pi\mathrm{i}} \bar{z}^{m+1} \bar{T}(\bar{z}) \tag{4.1.10}$$

Once the identification is imposed to describe open strings, the two components of the stress energy tensor are also related as $\bar{T}(\bar{z}) = -\frac{1}{2\alpha'}\bar{\partial}X\, {}^tR_0 G R_0 \bar{\partial}X$. Of course now there should exist only one copy of the Virasoro algebra written in terms of the sole oscillator α_m^M. This is consistent with the picture above only if

$${}^tR_\sigma G R_\sigma = G \tag{4.1.11}$$

where we have generalised the constraint to any σ since the boundary at $\sigma = 0$ does not play any special role with respect to the other. Notice that from the identification between left and right-handed fields written in terms of the strings non-zero modes, recalling the relations in Eq.(3.3.71), one can determine the transformation of the reflection matrices under T-Duality, namely

$$R'_\sigma = T_-^{-1} R_\sigma T_+ \tag{4.1.12}$$

If we now reinstate the presence also of the second boundary at $\sigma = \pi$ and substitute the ansatz (4.1.7) back into the second line of Eq.(4.1.6), we find a contradiction unless it is required that the meromorphic field be multivalued, introducing a branch cut just below the negative real axis on the complex plane. If the monodromy of the field $X^M(z)$ is

$$X^M(e^{2\pi \mathrm{i}} z) = \left(R_\pi^{-1} R_0\right)^M{}_N X^N(z) = (R)^M{}_N X^N(z) \tag{4.1.13}$$

then the second line of Eq.(4.1.6) with the ansatz (4.1.7) is automatically satisfied. The matrix we have introduced in the last equality of the previous equation is the monodromy matrix for the meromorphic chiral fields in Eq.(4.1.7). Notice that this matrix preserves the background metric as well as the single R_σ's and it transforms with a similarity transformation under T-Duality.

In summary the presence of a boundary in the open string action introduces reflection matrices R_σ that identify the left and right-handed fields in terms of a single meromorphic field defined on the full complex plane. They preserve the metric and transform as in Eq.(4.1.12) under duality. The simultaneous presence of two boundaries at $\sigma = 0$ and $\sigma = \pi$ imposes a non trivial monodromy on the holomorphic field through the monodromy matrix $R = R_\pi^{-1} R_0$, which also preserves the metric and is subject to a similarity transformation under duality. This general framework is applicable to the description of twisted open strings both in magnetised and in intersecting brane world models, depending on the explicit forms of the matrices introduced above. As we will explore in more details in the following section, magnetised D-Branes models are characterised by $R_\sigma = (G - \mathcal{F}_\sigma)^{-1}(G + \mathcal{F}_\sigma)$ where we have introduced the gauge invariant combination $\mathcal{F}_\sigma = B + F_\sigma$ involving the magnetic field on the brane and the background Kalb-Ramond field. In the intersecting brane models instead, D-Branes will in general have d Neumann directions and d Dirichlet directions on the compactification torus. Clearly from Eq.(4.1.4) the reflection matrices in this picture should have d eigenvectors with eigenvalue 1 and d eigenvectors with eigenvalue -1. This implies that $1 \pm R_\sigma$ are projector operators and that $R_\sigma^2 = 1$, which is a general property of all the fully geometrical (i.e. without background gauge fields turned on) configurations of D-Branes. Let us finally stress that, in toroidal compactifications, in order to have a consistent configuration it must be possible to write all of the eigenvectors mentioned above as linear integer combinations of the vectors specifying the torus lattice as in Eq.(3.2.25).

4.2 Twisted strings in Magnetised Brane Worlds

We will use the results found in the previous section to describe the particular case of the bosonic sector of the twisted strings stretched between two magnetised space-filling D-Branes on a T^{2d}. The sigma-model is encoded in the action (4.1.2) where the gauge field A^σ_M is linear in the strings coordinates, with constant field-strength $(F_\sigma)_{MN}$. The stacks of D-Branes, the open strings end to, are chosen to wrap the cycles of the internal torus of compactification non trivially, which means that they will be characterised not only by the magnetic fields on their world-volumes, but also by consistent gauge bundles that encode the information of the gluing of different copies of the branes on the cycles of the compactification torus. We will not need in the following discussion the explicit form of the magnetic field on the D-Branes world-volume, but the bundle already constructed in the previous chapter, associated to F written in the form of Eq.(3.2.33), will be useful to relate the results we will find with the picture of intersecting D-Branes. The appropriate boundary conditions for the twisted open strings in the magnetised D-Brane picture are [44]

$$\left(G_{MN}\partial_\sigma x^N + \mathrm{i}(\mathcal{F}_\sigma)_{MN}\partial_\tau x^N\right)\Big|_{\sigma=0,\pi} = 0 \tag{4.2.14}$$

where we have again defined the gauge invariant combination $(\mathcal{F}_\sigma)_{MN} = B_{MN} + (F_\sigma)_{MN}$. The expression in Eq.(4.2.14) can be rewritten in the following form using complex coordinates on the string world-sheet as in (4.1.5)

$$\left((G+\mathcal{F}_\sigma)_{MN}\partial x^N(z,\bar{z}) - (G-\mathcal{F}_\sigma)_{MN}\bar{\partial} x^N(z,\bar{z})\right)\Big|_{\sigma=0,\pi} = 0 \tag{4.2.15}$$

Comparing these constraints on the fields to the result found in Eq.(4.1.6) it is manifest that the reflection matrices in this context are given by $R_\sigma = (G - \mathcal{F}_\sigma)^{-1}(G+\mathcal{F}_\sigma)$ [44, 48], as anticipated in the previous section. Let us explicitly observe that these reflection matrices indeed do leave the metric invariant as

$$\begin{aligned} {}^tR_\sigma G R_\sigma &= (G-\mathcal{F}_\sigma)(G+\mathcal{F}_\sigma)^{-1}G(G-\mathcal{F}_\sigma)^{-1}(G+\mathcal{F}_\sigma) = \\ &= (G-\mathcal{F}_\sigma)(1+G^{-1}\mathcal{F}_\sigma)^{-1}(1-G^{-1}\mathcal{F}_\sigma)^{-1}(1+G^{-1}\mathcal{F}_\sigma) = \\ &= (G-\mathcal{F}_\sigma)(1+G^{-1}\mathcal{F}_\sigma)^{-1}(1+G^{-1}\mathcal{F}_\sigma)(1-G^{-1}\mathcal{F}_\sigma)^{-1} = \\ &= (G-\mathcal{F}_\sigma)(1-G^{-1}\mathcal{F}_\sigma)^{-1} = G \end{aligned} \tag{4.2.16}$$

and so does as a consequence the monodromy matrix $R = R_0^{-1}R_\pi$. Moreover by introducing the vielbein E^a_M such that $G = {}^tEE$ the previous equation can be recast in terms of the reflection matrices in the Cartesian basis $ER_\sigma E^{-1}$, which turn out to be elements of the $SO(2d)$ group, as a consequence of the fact that in the Cartesian basis the metric is simply given by the identity matrix. In order to rewrite the monodromy matrix in a rather simple form and solve the relations (4.1.13) explicitly for the open string fields, it is convenient to introduce complex coordinates on the torus. Recall that T^{2d} is in fact a complex (Kähler) manifold, which means that it admits a globally defined complex structure encoded in the mixed tensor

$$\mathcal{I} = \mathrm{i}dz \otimes \frac{\partial}{\partial z} - \mathrm{i}d\bar{z} \otimes \frac{\partial}{\partial \bar{z}} \tag{4.2.17}$$

The choice of a complex structure fully determines the complexification of the real coordinates on the manifold. We can start from the Cartesian coordinates defined by E as in Eq.(3.2.26) and introduce the complex vielbein $\mathcal{E}$, such that

$$\mathcal{E} = SE \quad , \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \tag{4.2.18}$$

where all four blocks of S are proportional to the $d \times d$ identity matrix. Notice that in the complex coordinates the flat metric $\mathcal{G}$ is off-diagonal

$$\mathcal{G} = {}^t\mathcal{E}^{-1} G \mathcal{E}^{-1} = \bar{S} S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.2.19}$$

Two sets of complex coordinates are inequivalent if they cannot be connected by a unitary transformation. Thus the $SO(2d)$ ambiguity in the definition of the vielbein E implies that on the same real torus there is a set of inequivalent complex structures which is parametrized by $SO(2d)/U(d)$. We can fix a particular complex structure on the torus such that the corresponding complex vielbein defines a basis in which the orthogonal matrix R is diagonal

$$\mathcal{E} R \mathcal{E}^{-1} = \mathrm{diag}\left(e^{2\pi \mathrm{i}\theta_1}, \ldots, e^{2\pi \mathrm{i}\theta_d}, e^{-2\pi \mathrm{i}\theta_1}, \ldots, e^{-2\pi \mathrm{i}\theta_d}\right) \tag{4.2.20}$$

for $0 \leqslant \theta_i < 1$. Switching to this complex basis, the monodromy properties (4.1.13) for the eigenvectors of R are

$$\mathcal{Z}^i(e^{2\pi \mathrm{i}} z) = e^{2\pi \mathrm{i}\theta_i} \mathcal{Z}^i(z) \quad \text{and} \quad \bar{\mathcal{Z}}^i(e^{2\pi \mathrm{i}} z) = e^{-2\pi \mathrm{i}\theta_i} \bar{\mathcal{Z}}^i(z) \tag{4.2.21}$$

where $\mathcal{Z}^i = (\mathcal{E})^i_M X^M$ for $i = 1, \ldots, d$, and thus it is easier to explicitly write the mode expansion for the meromorphic fields that, upon canonical quantization, reads

$$\partial \mathcal{Z}^i(z) = -\mathrm{i}\sqrt{2\alpha'} \left(\sum_{m=1}^{\infty} \bar{a}^i_{m-\theta_i} z^{-m+\theta_i-1} + \sum_{m=0}^{\infty} a^{\dagger i}_{m+\theta_i} z^{m+\theta_i-1} \right) \tag{4.2.22}$$

$$\partial \bar{\mathcal{Z}}^i(z) = -\mathrm{i}\sqrt{2\alpha'} \left(\sum_{m=0}^{\infty} a^i_{m+\theta_i} z^{-m-\theta_i-1} + \sum_{m=1}^{\infty} \bar{a}^{\dagger i}_{m-\theta_i} z^{m-\theta_i-1} \right) \tag{4.2.23}$$

Notice that the modes of the open strings attached to the magnetised D-Brane are shifted by the eigenvalues of the monodromy matrix and so are their commutation relations due to the canonical quantisation

$$\left[\bar{a}^i_{m-\theta_i}, \bar{a}^{\dagger j}_{r-\theta_j}\right] = (m-\theta_i)\delta^{ij}\delta_{m,r} \qquad \forall r,m \geqslant 1 \tag{4.2.24}$$

$$\left[a^i_{m+\theta_i}, a^{\dagger i}_{r+\theta_j}\right] = (m+\theta_i)\delta^{ij}\delta_{m,r} \qquad \forall r,m \geqslant 0 \tag{4.2.25}$$

The Virasoro generators are given by

$$L^{(\mathcal{Z})}_m = \oint \frac{dz}{2\pi \mathrm{i}} z^{m+1} T(z) = -\frac{1}{2\alpha'} \oint \frac{dz}{2\pi \mathrm{i}} z^{m+1} \sum_{i=1}^{d} \partial \mathcal{Z}^i \partial \bar{\mathcal{Z}}^i \tag{4.2.26}$$

$T(z)$ being the holomorphic stress energy tensor of the CFT defined by the twisted strings. In order to determine the bosonic contribution to the spectrum of such strings it is useful to focus on the form of the normal-ordered $L^{(\mathcal{Z})}_0$ which reads

$$L^{(\mathcal{Z})}_0 = \sum_{i,m} \left[\bar{a}^{\dagger i}_{m-\theta_i}\bar{a}^i_{m-\theta_i} + a^{\dagger i}_{m+\theta_i} a^i_{m+\theta_i}\right] + \sum_{i=1}^{d} \frac{1}{2}\theta_i \left(1-\theta_i\right) \tag{4.2.27}$$

Notice the appearance of the zero-point energy which can be derived by means of a Riemann zeta-function regularisation of the infinity related to the sum over m of the commutator of creation and annihilation operators needed to normal order $L^{(\mathcal{Z})}_0$[1]. Hence for instance the ground state $|\Theta\rangle \equiv |\{\theta_i\}\rangle$, which is the usual open string tachyon in absence of fluxes has a mass squared given by

$$\alpha' M^2 |\Theta\rangle = \left[-1 + \sum_{i=1}^{d} \frac{1}{2}\theta_i(1-\theta_i)\right] |\Theta\rangle \tag{4.2.28}$$

From this expression, one can see that $|\Theta\rangle$ must be related to the usual $SL(2,\mathbb{R})$ invariant vacuum for the untwisted open strings, $|0\rangle$, through the action of d twist fields [59] $\sigma_{\theta_i}(z)$ of conformal dimensions

$$h_{\sigma_{\theta_i}} = \frac{1}{2}\theta_i \left(1-\theta_i\right) \tag{4.2.29}$$

[1] Actually the described procedure would lead to a zero-point energy given by $\sum_{i=1}^{d} \frac{1}{2}\theta_i(1-\theta_i) - \frac{d}{12}$, but the final shift will always be reabsorbed in the total zero-point energy of the ground state which receives contributions from the normal ordering of L_0 in the four dimensional flat space-time which has to be finally incorporated into the model and from the (b,c) ghost system with conformal weights $(2,-1)$ respectively.

as follows[2]

$$|\Theta\rangle = \lim_{z\to 0} \prod_{i=1}^{d} \sigma_{\theta_i}(z)|0\rangle \tag{4.2.30}$$

A twist field σ_{θ_i} is then a CFT operator that implements a change in the boundary conditions for the (complexified) bosonic coordinates by a phase $e^{2\pi i\theta_i}$. Observe that it is not possible to describe such operators in terms of free fields and this is the source of all of the difficulties one encounters in computing amplitudes involving them. The main characteristics of these fields, which we will not explicitly need in what follows, are discussed for instance in [44].

4.3 T-Duality and Intersecting Branes

The same kind of mode-expansion for the open string fields leading to the same quantisation and spectrum as in the previous section can be obtained in a configuration of D-Branes intersecting at angles [22, 27, 25], where the parameters θ_i in this case have the nice geometrical interpretation of being the intersection angles between different branes. We consider here some particular examples explicitly to show the duality between the two models in full detail. We start in particular with the analysis of the case of a single T^2 compactification. The closed string background moduli that encode the geometrical structure of a two-dimensional torus, which is a Kähler complex manifold, can be redefined in terms of its complex structure modulus $U = U_1 + iU_2$ and Kähler modulus $T = T_1 + iT_2$ as follows

$$G = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -T_1 \\ T_1 & 0 \end{pmatrix} \tag{4.3.31}$$

In the compactifications under consideration where the periods of the coordinates x^M, $M = 1, 2$, are given by $2\pi\sqrt{\alpha'}$, the complex structure modulus is simply $U = e^{i\gamma}$, where γ is the angle between the two cycles (see Fig. 4.1) of the generic tilted torus, while the Kähler modulus contains the area of the fundamental domain $T_2 = \alpha' \sin\gamma$ complexified with the off-diagonal component of the Kalb-Ramond field $T_1 = B_{12}$. The magnetic flux in the D-Brane world-volume has the form

$$F_\sigma = \begin{pmatrix} 0 & f_\sigma \\ -f_\sigma & 0 \end{pmatrix} \tag{4.3.32}$$

where f_σ is real and quantised as in eq.(3.2.29).
Like in the previous section the reflection and monodromy matrices that charac-

[2] Observe that different twist fields, $\sigma_{\theta_i}(z)$ and $\sigma_{\theta_j}(z)$, with $i \neq j$, have a trivial OPE, $\sigma_{\theta_i}(z)\sigma_{\theta_j}(w) \sim 0$

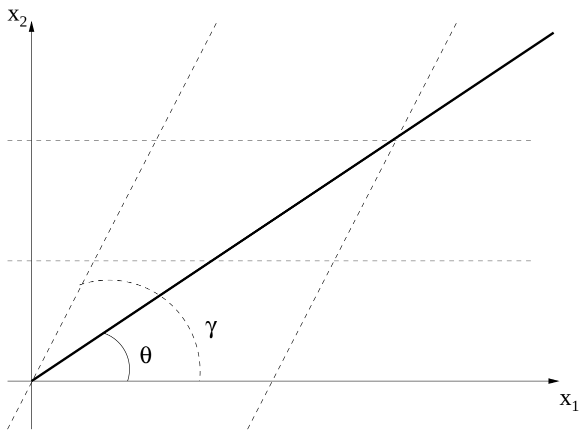


Fig. 4.1: D-Brane at angle on a two-dimensional tilted torus

terise the boundary conditions for twisted strings in this setup have the simplest form in the complex basis which is identified by the following choice of the complex vielbein

$$\mathcal{E} = \sqrt{\frac{T_2}{2U_2}} \begin{pmatrix} 1 & U \\ 1 & \bar{U} \end{pmatrix} \quad \text{and} \quad \mathcal{E}^{-1} = \mathrm{i}\sqrt{\frac{1}{2T_2U_2}} \begin{pmatrix} \bar{U} & -U \\ -1 & 1 \end{pmatrix} \tag{4.3.33}$$

Starting from the definition of the reflection matrices as

$$R_\sigma = (G - B - \mathcal{F}_\sigma)^{-1} (G + B + \mathcal{F}_\sigma) \tag{4.3.34}$$

upon change of basis, we can compute

$$\mathcal{R}_\sigma = \mathcal{E} R_\sigma \mathcal{E}^{-1} = - \begin{pmatrix} \frac{\bar{T}-f_\sigma}{T-f_\sigma} & 0 \\ 0 & \frac{T-f_\sigma}{\bar{T}-f_\sigma} \end{pmatrix} \tag{4.3.35}$$

Notice that in the simple compactification under analysis it is also possible to write the explicit form of the monodromy matrix, in the complex basis where it is diagonal, to find an explicit expression for the shift θ entering the mode expansion of the twisted string field and the commutation relations for the shifted oscillators, in terms of the background moduli of the T^2

$$e^{2\pi \mathrm{i}\theta} = \frac{T - f_\pi}{\bar{T} - f_\pi} \frac{\bar{T} - f_0}{T - f_0} \tag{4.3.36}$$

Notice that in type IIB magnetised configurations these shifts depend only on the Kähler moduli.
In order to make contact between this configuration and the corresponding picture

of D-Branes intersecting at angles we can perform a T-Duality along the b-cycle of the two-dimensional torus T^2, obtained by considering a $O(1,1,\mathbb{Z})$ transformation of the type in Eq.(3.3.67) where

$$e_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3.37}$$

In the definition of the duality matrix in Eq.(3.3.54) this choice corresponds to

$$a = d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3.38}$$

Recalling the transformation for the reflection matrices under T-Duality as expressed in Eq.(4.1.12) it is not difficult to find the result

$$\mathcal{R}'_\sigma = \begin{pmatrix} 0 & -\frac{1+\bar{U}f_\sigma}{1+Uf_\sigma} \\ -\frac{1+Uf_\sigma}{1+\bar{U}f_\sigma} & 0 \end{pmatrix} \tag{4.3.39}$$

written already in the complex basis. Observe that in order to determine the explicit expression above it is also necessary to relate the background moduli of the two dual T^2's to one another. This can be achieved by reconsidering the T-Duality identified by the $O(1,1,\mathbb{Z})$ with entries as in Eq.(4.3.38) together with the transformation of the background fields given in Eq.(3.3.63)-(3.3.64). In terms of the Kähler and complex structure moduli it is not difficult to see that the duality transformation boils down to the well-known [20] exchange

$$T \leftrightarrow -\frac{1}{U} \tag{4.3.40}$$

This implies that the shifts of the modes in the twisted open strings in the type IIA intersecting D-Branes configurations depend only on the complex structure of the torus as follows

$$e^{2\pi i\theta'} = \frac{1+Uf_\pi}{1+\bar{U}f_\pi}\frac{1+\bar{U}f_0}{1+Uf_0} \tag{4.3.41}$$

as seen both from the definition of the monodromy matrix in the intersecting branes picture and from the transformation of the twist θ in Eq.(4.3.36) under the exchange $T \leftrightarrow -1/U$.

If we simplify the expression (4.3.41) by considering for instance $f_0 = 0$, which, in the magnetised picture, simply means that one of the two D-Branes the twisted string ends to is unmagnetised, we find that

$$\tan(\pi\theta') = \frac{f_\pi U_2}{1+f_\pi U_1} = \frac{pU_2}{w+pU_1} \tag{4.3.42}$$

having used the quantisation of the magnetic field as in eq.(3.2.29) introducing $p = p_{12} \in \mathbb{Z}$ and $w = w_1 \times w_2 \in \mathbb{Z}$. This last formula describes exactly the configuration of a D-Brane at angle on the dual two-torus, wrapping p and w times respectively the b and a-cycle as it is manifest from Fig.(4.1). Obviously there is no loss of generality with the choice $f_0 = 0$ and, indeed, if $f_0 \neq 0$ the expression above is still valid, θ' being in this case the angle between the two D-Branes the twisted open strings are attached to. Notice finally from the T-Dual picture of intersecting branes, that, in the case of a two-dimensional torus compactification, the product of the two wrappings, w, together with the Chern number p_{12} is sufficient to characterise the magnetised D-Brane configuration, while the information of the single wrappings w_1 and w_2 is not necessary. This is in perfect agreement with result found in the picture of magnetised D-Branes seen as gauge bundles in section 3.1, in which we showed that on a T^2 the bundles associated to configurations of magnetised D-Branes characterised by the same Chern class and product of wrappings are all related to each other via gauge transformations.

The analysis of the T-Duality relating magnetised and intersecting D-Branes can be easily generalised to a T^{2d} compactification. As it has already been stressed in the description of a gauge bundle related to a wrapped magnetised D-Branes in section 3.2, the field F_σ present in the reflection matrices $R_\sigma = (G-\mathcal{F}_\sigma)^{-1}(G+\mathcal{F}_\sigma)$ of the magnetised picture can be put in the form of Eq.(3.2.33) by means of a unimodular integer transformation O that preserves the lattice of the torus. Since the T-Duality rules are not affected by the particular expression of the metric of the torus, the form of the magnetic field implies that the setup behaves effectively as a direct product of d T^2's. The generalisation of the duality Eq.(4.3.38) that relates the magnetised configuration to the intersecting branes one is given by an $O(2d, 2d, \mathbb{Z})$ matrix with

$$a = d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1_{d\times d} \quad \text{and} \quad b = c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes 1_{d\times d} \tag{4.3.43}$$

which is nothing else than a duality along all of the even cycles of the T^{2d}. In the corresponding intersecting D-Branes description, on each of the d T^2's defined by the form of F_σ one has the same picture as Fig.4.1 where the magnetised D-Brane at $\sigma = \pi$ has become a lower dimensional brane with wrapping numbers p_α and W_α on the a and b cycle of the α-th T^2, following the conventions in section 3.2. Notice, however, that, when considering more than one magnetised boundary at the same time to construct, for instance, the monodromy matrix associated to the twisted strings, in order to use the same picture described so far, it is necessary to show that the magnetic fields on all of the D-Branes considered can be simultaneously put in the form (3.2.33) by means of a unique unimodular integer transformation O. This is certainly the case if the fluxes on the boundaries commute with each other. In the nomenclature used in the literature (see for example [48, 76]) this setup corresponds to having *parallel* fluxes, as opposite to the so called *oblique* fluxes. In the first scenario in which the fluxes do commute the system of magnetised D-Branes can indeed be fully geometrised by means of the T-Duality encoded in the matrices in Eq.(4.3.43). Observe that, due to the generic form of the metric in the basis in which the magnetic fields are written as in Eq.(3.2.33), the explicit expression for the twists (angles) in terms of the background geometry is more involved than in the case of a single T^2 and it can be found, in particular examples of higher dimensional toroidal compactifications, for instance in [48]. Let us underline that we have not managed to fully geometrise via T-Duality a setup with oblique fluxes. We will actually come back to this fundamental point when we consider the specific configurations involving three boundaries that coincide with D-Branes with different magnetic fields in their world-volumes or intersecting at angles, for the realisation of the Yukawa couplings in the brane world models. There we will indeed compute the Yukawa couplings for configurations of magnetised D-Branes which, even if they are always described in terms of the general framework analysed in this chapter, do not seem to be directly related to setups involving only intersecting D-Branes by means of T-Duality.

4.4 Fermionic Sector

Focusing on the construction of the full spectrum of the twisted strings stretched between magnetised or intersecting D-Branes, we reinstate now the fermionic sector of the open strings action, whose Euclidean expression reads [77]

$$S = -\frac{\mathrm{i}}{4\pi\alpha'}\int d\sigma d\tau \bar{\chi}^M \rho^\alpha \partial_\alpha \chi^N (G_{MN} + B_{MN}) - \frac{\mathrm{i}}{4\pi\alpha'}\sum_\sigma q_\sigma \int_{C_\sigma} d\tau \bar{\chi}^M \rho^\tau \chi^N F^\sigma_{MN} \tag{4.4.44}$$

where χ^M is a world-sheet spinor and ρ^α are two-dimensional Dirac matrices

$$\rho^\tau = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \rho^\sigma = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{4.4.45}$$

Since the background fields G_{MN}, B_{MN} and F^σ_{MN}, with $M, N = 1 \ldots 2d$, are taken constant, the action is free like the bosonic contribution. It is convenient to write the spinor in terms of its chiral components as follows

$$\chi^M = \begin{pmatrix} \chi_-^M \\ \chi_+^M \end{pmatrix} \tag{4.4.46}$$

Hence the equations of motion are

$$\partial_\pm \chi_\mp^M = 0 \tag{4.4.47}$$

The boundary conditions, which cancel the boundary term arising from the variation of the action above to obtain the equations of motion, are

$$\chi_-^M\Big|_{\sigma=0} = (R_0)^M{}_N\, \chi_+^N\Big|_{\sigma=0} \quad \text{and} \quad \chi_-^M\Big|_{\sigma=\pi} = -\eta\, (R_\pi)^M{}_N\, \chi_+^N\Big|_{\sigma=\pi} \tag{4.4.48}$$

Here $\eta = \pm 1$ is a parameter that introduces two different sectors of the fermionic twisted strings. This also happens when one analyses untwisted open fermionic strings and is due to the fact that, unlike the bosonic action, the Dirac Lagrangian is linear in the world-sheet derivatives. The value $\eta = 1$ corresponds to what is known as the Neveu-Schwarz, or NS, sector, while $\eta = -1$ leads to the Ramond, or R, sector. We will see that, similarly to the traditional untwisted case, the spectrum of the open superstrings in the NS and R sectors contains space-time bosons and fermions respectively.

Since world-sheet fermions have conformal dimension $1/2$, as it is manifest considering that the fermionic action above must be invariant under conformal transformations, they are non-local fields, i.e. they introduce branch points in the complex plane defined by the world-sheet variables $z = e^{\tau + \mathrm{i}\sigma}$ and $\bar{z} = e^{\tau - \mathrm{i}\sigma}$ when contracted with any local operator. In order to use the same kind of derivation of the mode expansions already exploited in the bosonic sector analysis, it is then useful to introduce the following local fields

$$\psi_+^M(z) \equiv z^{-1/2}\chi_+^M(z) \quad \text{and} \quad \psi_-^M(\bar{z}) \equiv \bar{z}^{-1/2}\chi_-^M(\bar{z}) \tag{4.4.49}$$

for any z with $\operatorname{Im} z \geqslant 0$, to solve the boundary conditions (4.4.48). Following the same steps as in eq.(4.1.6)-eq.(4.1.13) we can introduce a meromorphic chiral field defined on the whole complex plane such that

$$\psi_+^M(z) = \psi^M(z) \quad \text{and} \quad \psi_-^M(\bar{z}) = (R_0)^M{}_N\, \psi^N(\bar{z}) \tag{4.4.50}$$

and assign to it a non-trivial monodromy

$$\psi^M\left(e^{2\pi \mathrm{i}} z\right) = \eta \left(R_\pi^{-1} R_0\right)^M{}_N\, \psi^N(z) \tag{4.4.51}$$

Introducing the vielbein $\mathcal{E}$ that defines the same set of complex coordinates which diagonalise the monodromy matrix in the bosonic sector of the theory, the monodromy conditions on the chiral meromorphic fields read

$$\Psi^i\left(e^{2\pi \mathrm{i}}z\right) = \eta e^{2\pi \mathrm{i}\theta_i}\Psi^i(z) \quad \text{and} \quad \bar{\Psi}^i\left(e^{2\pi \mathrm{i}}z\right) = \eta e^{-2\pi \mathrm{i}\theta_i}\bar{\Psi}^i(z) \tag{4.4.52}$$

with $i = 1 \dots d$. Upon canonical quantisation the mode expansion for the holomorphic fields written in the complex basis reads

$$\Psi^i(z) = \sqrt{2\alpha'}\sum_{m=0+\nu}^{\infty}\left(\bar{\Psi}^i_{m-\theta_i}z^{-m+\theta_i-\frac{1}{2}} + \Psi^{\dagger i}_{m+\theta_i}z^{m+\theta_i-\frac{1}{2}}\right) \tag{4.4.53}$$

$$\bar{\Psi}^i(z) = \sqrt{2\alpha'}\sum_{m=0+\nu}^{\infty}\left(\Psi^i_{m+\theta_i}z^{-m-\theta_i-\frac{1}{2}} + \bar{\Psi}^{\dagger i}_{m-\theta_i}z^{m-\theta_i-\frac{1}{2}}\right) \tag{4.4.54}$$

where $\nu = 0$ in the R sector, $\nu = 1/2$ in the NS sector and the modes obey the following anticommutation relations

$$\left\{\Psi^i_{m+\theta_i}, \Psi^{\dagger j}_{r+\theta_j}\right\} = \left\{\bar{\Psi}^i_{m-\theta_i}, \bar{\Psi}^{\dagger j}_{r-\theta_j}\right\} = \delta^{ij}\delta_{m,r} \quad \forall m, r \geqslant 0+\nu \tag{4.4.55}$$

Let us concentrate on the NS sector ($\nu = 1/2$). The oscillators $\Psi^i_{m+\theta_i}$ and $\bar{\Psi}^i_{m-\theta_i}$ are annihilation operators, while $\Psi^{\dagger i}_{m+\theta_i}$ and $\bar{\Psi}^{\dagger i}_{m-\theta_i}$ are creation operators with respect to the fermionic twisted NS vacuum $|\Theta\rangle_{\mathrm{NS}}$, i.e.

$$\Psi^i_{m+\theta_i}|\Theta\rangle_{\mathrm{NS}} = \bar{\Psi}^i_{m-\theta_i}|\Theta\rangle_{\mathrm{NS}} = 0 \quad \forall m \geqslant \frac{1}{2} \tag{4.4.56}$$

Notice that this definition of creation/annihilation operators is natural only for $0 \leqslant \theta_i < \frac{1}{2}$. In this range, in fact, the oscillator $\bar{\Psi}^{\dagger i}_{\frac{1}{2}-\theta_i}$ is a true creation operator since it increases the energy by the positive amount $(\frac{1}{2}-\theta_i)$. If, instead, $\frac{1}{2} < \theta_i < 1$, the oscillator $\bar{\Psi}^{\dagger i}_{\frac{1}{2}-\theta_i}$ decreases the energy of the state it acts on. Thus, in this case the roles of the NS vacuum $|\Theta\rangle_{\mathrm{NS}}$ and of $\bar{\Psi}^{\dagger i}_{\frac{1}{2}-\theta_i}|\Theta\rangle_{\mathrm{NS}}$ are exchanged and the latter becomes the true vacuum of the theory, since it has lower energy.
Like in the bosonic sector analysis we will finally introduce the Virasoro generators also for the fermionic strings

$$L^{\Psi}_m = \oint \frac{dz}{2\pi \mathrm{i}}z^{m+1}T(z) = -\frac{1}{4\alpha'}\oint \frac{dz}{2\pi \mathrm{i}}z^{m+1}\sum_{i=1}^{d}\left(\Psi^i\partial\bar{\Psi}^i + \bar{\Psi}^i\partial\Psi^i\right) \tag{4.4.57}$$

where $T(z)$ is the fermionic contribution to the holomorphic stress energy tensor of the twisted open superstrings. The normal ordered version of L^{Ψ}_0 then can be written as

$$L^{\Psi}_0 = \sum_{i=1}^{d}\sum_{m=0+\nu}^{\infty}\left[(m+\theta_i)\Psi^{\dagger i}_{m+\theta_i}\Psi^i_{m+\theta_i} + (m-\theta_i)\bar{\Psi}^{\dagger i}_{m-\theta_i}\bar{\Psi}^i_{m-\theta_i}\right] + c_{\nu} \tag{4.4.58}$$

The zero-point energy here depends on the value of $\nu = 0, 1/2$, i.e. on the choice of the R or NS sector and it is determined by using a Riemann Zeta-function regularisation of the infinity arising from the anticommutation of the oscillators in L_0^Ψ. The final result is

$$c_{\mathrm{R}} = -\sum_{i=1}^{d} \frac{1}{2}\theta_i \left(1-\theta_i\right) \quad \text{and} \quad c_{\mathrm{NS}} = \sum_{i=1}^{d} \frac{1}{2}\theta_i^2 \tag{4.4.59}$$

Similarly to the bosonic case, we deduce that the twisted vacua in the R and NS sectors can be related to the $SL(2,\mathbb{R})$ invariant vacuum through suitable twist fields as follows

$$|\Theta\rangle_{\mathrm{NS}} = \lim_{z\to 0} \prod_{i=1}^{d} s_{\theta_i}|0\rangle \quad \text{and} \quad |\Theta\rangle_{\mathrm{R}} = \lim_{z\to 0} \prod_{i=1}^{d} s_{\theta_i - \frac{1}{2}}|0\rangle \tag{4.4.60}$$

having conformal weights

$$h_{s_{\theta_i - \frac{1}{2}+\nu}} = \frac{1}{2}\left(\theta_i - \frac{1}{2} + \nu\right)^2 \tag{4.4.61}$$

for $\nu = 0, 1/2$ respectively in the R and NS sectors. Differently from the bosonic case (in which there is no simple way to define the twist fields in terms of free fields), thanks to the bosonisation procedure [78], it is possible to compute correlation functions involving fermionic twist fields by exploiting the Wick theorem as these fields can be rewritten in terms of free bosons H_i

$$s_{\theta_i - \frac{1}{2}+\nu} \equiv e^{\mathrm{i}\left(\theta_i - \frac{1}{2}+\nu\right)H_i} \quad , \quad \Psi^i \equiv \sqrt{2\alpha'} e^{\mathrm{i}H_i} \quad \text{with} \quad H_i(z)H_j(w) \sim -\delta_{ij}\ln(z-w) \tag{4.4.62}$$

4.5 GSO Projection, low energy twisted spectrum and supersymmetry

In this section we will make use of the discussion of the quantisation of the twisted open strings both in the bosonic and in the fermionic sector to determine the low energy spectrum of configurations of magnetised D9-Branes, which are related to configurations of D6-Branes intersecting at angles. The general set-up will be a generic toroidal compactification of superstring theory from ten dimensions down to four, hence the target space will be chosen to be of the type $M^4 \times T^6$, where M^4 is the four-dimensional flat Minkowski space-time and T^6 is a generic internal six-dimensional torus. The twisted open strings whose spectrum will be here considered are the ones starting and ending on different D9-Branes with different magnetic fields turned on in their world volumes, or, in the T-Dual picture the ones stretched between two D6-Branes intersecting at angles. Recalling that a stack of N D-Branes is associated to a $U(N)$ gauge group, it is manifest that twisted strings will transform in the bi-fundamental representation $(\bar{N}_1, N_2)$ of the two stack of Branes they are attached to. This is the first similarity between these objects and the matter fields of the Standard Model we want to reproduce. Let us start from the analysis of the NS sector. The physical spectrum of the superstrings is given by all the excitations of the vacuum $|\Theta\rangle_{\rm NS}$ which satisfy the constraint[3]

$$\left(L_0^{\rm NS} - \frac{1}{2}\right)|\phi\rangle_{\rm NS} = 0 \tag{4.5.63}$$

and the GSO projection that we will introduce later. Here the Virasoro generator has to contain all of the contributions, both bosonic and fermionic in the NS sector, coming from the four-dimensional flat directions and from the ones compactified in the T^6, hence

$$L_0^{NS} = L_0^{(x)} + L_0^{(\psi_{\rm NS})} + L_0^{(\mathcal{Z})} + L_0^{(\Psi)} \tag{4.5.64}$$

The two contributions related to the compact directions have already been calculated in eq.(4.2.27) and eq.(4.4.58), while the remainders are easily shown to give

$$L_0^{(x)} + L_0^{(\psi_{\rm NS})} = \alpha' p_\mu p^\mu + \sum_{m=1}^{\infty} a_m^{\dagger\mu} a_{m\,\mu} + \sum_{r=\frac{1}{2}}^{\infty} r\psi_r^{\dagger\mu}\psi_{r\,\mu} \tag{4.5.65}$$

as they can be obtained from the internal compactified ones, simply by taking $\theta_i = 0$ and by adding the momentum contribution from the bosonic zero-modes of untwisted strings. Thus in the NS sector the physical states will have to satisfy

$$\left(\alpha' p_\mu p^\mu + N^{(x)} + N^{(\psi)} + N^{(\mathcal{Z})} + N^{(\Psi)} - \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{3}\theta_i\right)|\phi\rangle_{\rm NS} = 0 \tag{4.5.66}$$

where the N's introduced are the number operators that count the excitations of each of the contributions to the sector. From the constraint above one can read

[3] Here the $-\frac{1}{2}$ is the collective θ-independent contribution to the zero-point energy given by the normal ordering of the Virasoro generators in the flat and compact directions of the space-time together with the ground state energy coming from the ghost systems (b,c) and (β,γ). Recall that the latter are described by first order Lagrangians of conjugate fields whose weights are generically given by λ and $1-\lambda$. The (b,c)-system has $\lambda = 2$ and the (β,γ)-system has $\lambda = \frac{3}{2}$.

off the masses of the physical spectrum of the theory.
Physical states are also subject to the GSO projection traditionally introduced into Superstring Theory to eliminate the presence of the tachyonic vacuum from the physical states and to achieve a supersymmetric spectrum from the space-time point of view. This is obtained by projecting out half of the states of the spectrum keeping only the ones which have an odd number of fermionic excitations on the vacuum in the NS sector. The opposite choice would correspond instead to the description of strings stretched between a D-Brane and an anti-D-Brane. It follows from the observation made just before the definition of the Virasoro generators in the fermionic sector that we can now interpolate continuously between these two situations. In fact when one of the angles θ_i is bigger than 1/2 the usual GSO projection selects the vacuum $\bar{\Psi}^{\dagger i}_{\frac{1}{2}-\theta_i}|\Theta\rangle_{\rm NS}$ as well as the states with an even number of fermionic oscillators acting on it. Thus we have two possibilities: we can limit the range of the angles to $[0, 1/2]$ and specify in each case whether we take the brane/brane or the brane/anti-brane GSO; otherwise we keep the interval $0 \leqslant \theta_i < 1$ but we always stick to the same GSO. Here we will use this second option. Notice also that the case $\theta_i = 1/2$ is special and requires a separate treatment due to the appearance of further zero-modes. Then the first low-lying states in the spectrum are [44]:

$$\text{1 vector} \quad \psi^{\dagger\mu}_{\frac{1}{2}}|k,\Theta\rangle_{\rm NS} \quad 2\alpha' M^2 = \sum_{j=1}^{3}\theta_j \tag{4.5.67}$$

$$\text{3 scalars} \quad \Psi^{\dagger i}_{\frac{1}{2}+\theta_i}|k,\Theta\rangle_{\rm NS} \quad 2\alpha' M_i^2 = \sum_{j\neq i}^{3}\theta_j + 3\theta_i \tag{4.5.68}$$

$$\text{3 scalars} \quad \bar{\Psi}^{\dagger i}_{\frac{1}{2}-\theta_i}|k,\Theta\rangle_{\rm NS} \quad 2\alpha' M_i^2 = \sum_{j\neq i}^{3}\theta_j - \theta_i \tag{4.5.69}$$

where $|k,\Theta\rangle_{\rm NS}$ is the twisted vacuum with four-dimensional momentum k^μ. In our convention it is manifest that, in generic configurations, the vectors and the scalars in the second line of the equation above cannot survive in the low-energy limit, $\alpha' \to 0$. Indeed they can be present in the low-energy effective theory only if the angles involved in their masses are chosen to be zero in the stringy model. The situation is slightly different when dealing with the scalars in the last line of the equation above, as here finite values for the masses can also occur in the aforementioned low energy limit. In fact one can rewrite

$$\theta_i = \theta_i^{(0)} + 2\alpha'\epsilon_i \tag{4.5.70}$$

having introduced the "field theory value" for the i-th twist, $\theta_i^{(0)}$, and its stringy corrections ϵ_i with the dimensions of a mass squared. Both $\theta_i^{(0)}$ and ϵ_i are kept fixed in the limit $\alpha' \to 0$. Then the mass for the scalars in the last line of eq.(4.5.67) reads

$$M_i^2 = \frac{1}{2\alpha'}\left(\sum_{j\neq i}^{3}\theta_j^{(0)} - \theta_i^{(0)}\right) + \sum_{j\neq i}^{3}\epsilon_j - \epsilon_i \tag{4.5.71}$$

By suitably choosing the field theory values $\theta_i^{(0)}$ for the twists it is possible to cancel the term in brackets in the previous expression and find a finite mass in the $\alpha' \to 0$ limit. Notice that in the low-energy effective theory in general these scalars will enter with finite, non zero, masses hence the theory itself will not be supersymmetric. We will soon discuss how particular choices of the twists actually restore supersymmetry in the effective action, after introducing the space-time fermionic spectrum arising in the R sector of the model.
Similarly to the discussion of the NS sector, in order to determine the contribution to the spectrum from the R sector, one has to consider the constraint on the physical states given by[4]

$$L_0^{(R)}|\psi\rangle_R = 0 \tag{4.5.72}$$

where $L_0^{(R)}$ contains again all the contributions from both the fermionic and bosonic strings either in four dimensions or in the compactified torus, namely

$$L_0^{\rm R} = L_0^{(x)} + L_0^{(\psi_R)} + L_0^{(\mathcal{Z})} + L_0^{(\Psi)} \tag{4.5.73}$$

where $L_0^{(\mathcal{Z},\Psi)}$ have been defined in eq.(4.2.27) and eq.(4.4.58) and

$$L_0^{(x)} + L_0^{(\psi_R)} = \alpha' p_\mu p^\mu + \sum_{m=1}^{\infty} a_m^{\dagger\mu} a_{m\,\mu} + \sum_{r=0}^{\infty} r\psi_r^{\dagger\mu}\psi_{r\,\mu} \tag{4.5.74}$$

Let us highlight that in this case all the zero-point energies (including the θ-dependent contributions) cancel out in the sum above and hence in the R sector the low energy spectrum contains massless fields. These fields, which actually correspond to the vacuum of the sector, are not projected out like the NS vacuum, as the GSO projection acts differently here. The R vacuum is degenerate due to the presence of zero-modes in the fermionic mode expansion of the untwisted four dimensional string fields. Upon canonical quantisation these modes have the following anticommutation relations

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \tag{4.5.75}$$

[4] Here the net contribution to the θ-independent zero-point energies given by the normal ordering of the Virasoro generators and the ghost systems cancels out.

By redefining

$$\psi_0^\mu = \frac{1}{\sqrt{2}}\gamma^\mu \tag{4.5.76}$$

one can see that the resulting γ's satisfy a Clifford algebra. Hence the degenerate vacuum, whose components are reshuffled by the action of the ψ_0's as four dimensional Dirac gamma matrices, must be a four dimensional Dirac spinor, which we will denote as $|\alpha, \dot{\alpha}\rangle$.
The GSO projection in this sector selects one of the four-dimensional chiralities for this spinor, halving the number of degrees of freedom and reducing it for example to the left-handed $|\alpha\rangle$.
Thus we have found that in the spectrum of twisted strings stretched between D9-Branes with different magnetic fields on their world volumes, or, via T-Duality, D6-Branes intersecting at angles, there appear four dimensional chiral left-handed Weyl spinors $|\alpha\rangle$ in the bi-fundamental representation of the two gauge groups associated to the D-Branes to which their endpoints are attached, $(\bar{N}_1, N_2)$. These spinors are, then, good candidates to represent the matter fields in the Standard Model Lagrangian.
When one of the twist parameters is zero, one of the internal complex fermions Ψ^i ceases to be twisted and two extra fermionic real zero-modes appear. In this case the vacuum becomes doubly degenerate and one finds two massless fermions in four dimensions. When all the twists are vanishing, all the internal fermions have zero-modes and, upon compactification, one finds four massless fermions in the resulting four-dimensional theory.

In summary, the low energy spectrum of open strings stretched between two stacks of D9-Branes consists of one chiral massless fermion and a number of scalars that are generically massive or tachyonic. For specific values of the fluxes and hence of the twists, one or more scalars may become massless and supersymmetric configurations may be realised. This situation can be conveniently represented in terms of a tetrahedron in the twist parameters space [39], as shown in Fig.4.2 This represents supersymmetric configurations and separate an inner region, where the scalars are all massive, from an outer region, where the scalars become tachyonic. Faces, edges and vertices of the tetrahedron correspond to $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ configurations, respectively. Considering for example the face identified by the equation $\theta_1 - \theta_2 - \theta_3 = 0$, the only complex scalar that becomes massless is $\bar{\Psi}^{\dagger 1}_{\frac{1}{2}-\theta_1}|k;\Theta\rangle_{\rm NS}$. This is the partner of the unique massless Weyl fermion that is already present for generic θ_i's. On the edges, instead, one of the angles always becomes zero, hence in the low-energy spectrum there appear two four-dimensional fermions with opposite chirality coming from a chiral six dimensional spinor after

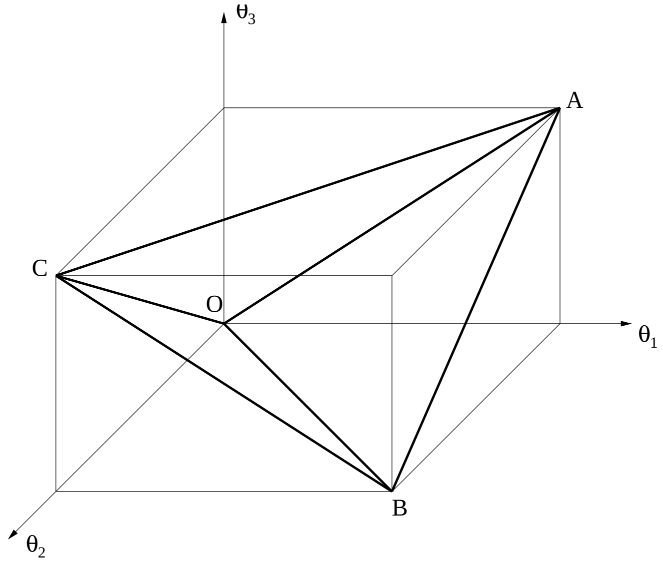


Fig. 4.2: The tetrahedron in θ-space

compactification. Thus in the $\mathcal{N} = 2$ case chirality in four dimensions is lost. An explicit example could be the edge $\theta_1 = \theta_2$ and $\theta_3 = 0$. Here two complex scalars become massless, $\bar{\Psi}^{\dagger 1}_{\frac{1}{2}-\theta_1}|k;\Theta\rangle_{\rm NS}$ and $\bar{\Psi}^{\dagger 2}_{\frac{1}{2}-\theta_2}|k;\Theta\rangle_{\rm NS}$ and, together with the aforementioned two four dimensional spinors, fill in a $\mathcal{N} = 2$ hypermultiplet. When all of the twists are vanishing, then the vector $\psi^{\dagger\mu}_{\frac{1}{2}}|k;\Theta\rangle_{\rm NS}$ and all of the complex scalars are massless. At the same time the ten dimensional chiral spinor is decomposed into four four-dimensional fermions with opposite chirality. These fields complete the $\mathcal{N} = 4$ supersymmetric gauge multiplet and four dimensional chirality is again lost.

Notice that in our conventions, where the twists θ_i's are taken in the range $[0, 1)$, the vertices A,B and C, and the face (ABC) are not part of the moduli space. Of course one could change conventions and choose a different parametrization without changing the physical conclusions.

4.6 Zero-modes and degeneration of the states

In the discussion of the previous sections we have focused on the spectrum of the twisted open strings, attached either to magnetised or to intersecting D-Branes, as a result of the action of the non-zero modes on the twisted vacuum. The action and the physical meaning of the bosonic zero-modes x_0^M introduced in Eq.(4.1.7) has been so far overlooked. In this section we will fill in this gap explaining how their presence in the mode expansion of bosonic twisted strings reveals that the twisted vacuum has a finite dimensional degeneration. This comes as no surprise if one takes into account that the zero-modes contribution to a string field should encode the information of the dynamics of the string seen as a point particle concentrated in its centre of mass. Recalling that the field in Eq.(4.1.7) in the magnetised brane worlds describes a charged string on a magnetised torus, from field theory arguments we expect that for a fixed energy the corresponding charged particle is described by finitely many inequivalent wave functions known as Landau levels [67].

From the string theoretical point of view, following the discussion in [79], it is not difficult to show that the complexified bosonic twisted string zero-modes, $x_0^i = \mathcal{E}^i_M x_0^M$, constructed by means of the complex vielbein that diagonalises the monodromy matrices R, should obey non trivial commutation relations, namely

$$\left[x_0^i, x_0^j\right] = \alpha' \frac{2\pi}{F_\pi^{ij} - F_0^{ij}} \tag{4.6.77}$$

where we have considered the magnetic fields on the two D-Branes, the twisted string is stretched between, invertible as it is indeed the case if $\theta_i \neq \theta_j$ and they are also not equal to zero and one at the same time. This is the situation in which the string is actually charged. In what follows we will concentrate on the simple example of a string attached to two D-Branes with magnetic fields $F_0 = 0$ and $F_\pi \equiv F$ wrapped only once along a two-dimensional square torus. For the non vanishing magnetic field we will use the conventions introduced in Eq.(4.3.32).

From the commutation relation above it is clear that the real zero-mode x_2 behaves like the conjugate momentum with respect to x_1, as

$$[x_1, x_2] = \alpha' \frac{2\mathrm{i}\pi}{f} \tag{4.6.78}$$

thus the wave function describing the charged string on the magnetised torus has the property

$$x_2\, e^{\frac{\mathrm{i}}{\sqrt{\alpha'}} k x_1} = \sqrt{\alpha'} \frac{2\pi k}{f}\, e^{\frac{\mathrm{i}}{\sqrt{\alpha'}} k x_1} \tag{4.6.79}$$

k being an integer number to ensure the single-valuedness of the wave function along x_1. A constant magnetic field on a torus is actually a monopole field and no smooth single-valued gauge potential A_M exists in this case (see also section 3.1 on gauge bundles on a T^2). We may separate the torus along the x_2 axis of its covering space into two regions characterised by $0 \leqslant x_2 < a$ and $a \leqslant x_2 < 2\pi\sqrt{\alpha'}$ and place here two smooth potentials that are related by a gauge transformation

$$A_1 = \begin{cases} -f x_2 & 0 \leqslant x_2 < a \\ -f(x_2 - 2\pi\sqrt{\alpha'}) & a \leqslant x_2 < 2\pi\sqrt{\alpha'} \end{cases} \qquad A_2 = 0 \tag{4.6.80}$$

Thus the two regions are connected by the transformation $x_2 \to x_2 - 2\pi\sqrt{\alpha'}$ which acting on the wave function of the charged particle simply means that $k \to k - f$. Notice that the transition function $e^{-\frac{\mathrm{i}}{\sqrt{\alpha'}} f x_1}$ is single valued along x_1 only if the magnetic field $f \in \mathbb{Z}$, which is consistent with the form we have already given in Eq.(3.2.29) for singly wrapped D-Branes on the torus. Now since $x_2 \equiv x_2 + 2\pi\sqrt{\alpha'}$ then

$$e^{\frac{\mathrm{i}}{\sqrt{\alpha'}} k x_1} \equiv e^{\frac{\mathrm{i}}{\sqrt{\alpha'}} (k+f) x_1} \tag{4.6.81}$$

and hence one only has $f \in \mathbb{Z}$ inequivalent choices for k.

In the T-Dual language of intersecting D-Branes the analogues of the zero-modes x_0^M in Eq.(4.1.7) would simply represent the positions of the intersections between the two D-Branes where the twisted strings are confined. Here of course it is not possible to speak of charged particles and Landau levels, but, nevertheless, the finite degeneracy of the twisted vacuum discussed above should persist also in this picture. Sticking with the simple example analysed so far, in fact, the dual picture would be of strings stretched between two D-Branes with wrappings $(1, f)$ and $(1, 0)$ along the cycles of the dual two-torus. It is then manifest that the system of these two D-Branes contains exactly f inequivalent intersections, in a fundamental domain of the torus, where all the vacua are described by the same quantum numbers and result to be degenerate.

The generalisation of this simple picture follows from the observation that the degenerate vacua correspond to separate, but equivalent, Hilbert spaces which also contain chiral fermions from the discussion of the spectrum of the twisted strings in the previous sections. Hence the counting of the number of such Hilbert spaces becomes the evaluation of the kernel of the internal Dirac operator in generic toroidal compactifications. In particular the difference between the number of chiral fermions and the number of anti-chiral fermions in the kernel is given by the Witten index

$$I_{ij} = \int_{T^{2d}} \mathrm{ch}(F_i - F_j) = \mathrm{Tr} \int_{T^{2d}} (F_i - F_j)^d \tag{4.6.82}$$

Here the indices i and j label the two magnetised D-Branes the twisted string under consideration ends to, and in the last equation the trace is related to non abelian gauge fields and the power of the magnetic fields is interpreted as an external product. Recall in fact from section 3.2 that non abelian magnetic fields arise in the description of non-trivially wrapped magnetised D-Branes.

In the picture of D-Branes intersecting at angles the topological numbers in Eq.(4.6.82) are related to the so called intersection numbers. For two cycles i and j identified by the wrappings numbers $(m_i^{(\alpha)}, n_i^{(\alpha)})$ and $(m_j^{(\alpha)}, n_j^{(\alpha)})$ in each bidimensional plane inside the T^{2d} indicated with the index α

$$I_{ij} = \prod_{\alpha=1}^{d} \left(m_i^{(\alpha)} n_j^{(\alpha)} - n_i^{(\alpha)} m_j^{(\alpha)} \right) \tag{4.6.83}$$

In a setup with parallel fluxes, as seen in the discussion on the duality between magnetised and intersecting brane worlds, all of the magnetic fields on the D-Branes can be put in the form (3.2.33) and so the two descriptions of the degeneracy of the twisted vacua in terms of the Witten index or the intersection numbers coincide if one identifies by means of T-Duality $(m^\alpha, n^\alpha) \equiv (W_\alpha, p_\alpha)$ for each of the boundaries. Recall also that for non-trivially wrapped D-Branes the dimensionality of the matrices which specify the gauge bundle associated to F_i is given by the product of all the wrappings of the branes along the cycles of the compactification torus. Of course the topological number (4.6.82) is also valid in configurations with oblique fluxes which are not fully geometrisable and in this sense it turns out to be more general.

5. OPEN STRING UNTWISTED SECTOR, MODEL BUILDING AND CONSISTENCY CONDITIONS

In this chapter we will take into consideration the untwisted spectrum of the open strings attached with both ends on one stack of D-Branes. As already seen in the discussion of the gauge bundles related to wrapped magnetised D-Branes (section 3.1) a single, even if non-trivially wrapped, magnetised D-Brane carries a $U(1)$ gauge symmetry. This is associated to the strings whose Chan-Paton factors are proportional to the identity matrix, which give rise to massless states in their spectrum. Stacks of N coincident D-Branes will describe non-abelian $U(N)$-gauge symmetries. In what follows we will discuss the minimal setup to embed the gauge group of the Standard Model in the Brane worlds and we will stress that any possible choice actually leads to a bigger group than $SU(3) \times SU(2) \times U(1)$. In particular extra $U(1)$'s are always present. These abelian groups can however be shown to acquire a mass under the so called Stueckelberg mechanism, due to the couplings of the involved D-Branes with the RR fields in the Wess-Zumino term of their low-energy description. These couplings are also responsible for the presence of RR tadpoles, whose cancellation also ensures that the considered models are free from cubic gauge anomalies. Moreover the remaining mixed anomalies involving external legs with both non-abelian and $U(1)$ gauge symmetries turn out to cancel as a result of the Green-Schwarz mechanism. We will not consider in detail the explicit constructions that yield the spectrum of the Standard Model, for which a large literature is already available, but we will focus on the general characteristics of consistent brane world models.

5.1 The gauge group, extra symmetries and Stueckelberg mechanism

Since a stack of N coincident D-Branes is associated to a non-abelian $U(N)$ gauge symmetry, in order to reproduce the gauge sector of the Standard Model, it would be expected that the minimal possible setup involves only three stacks of D-Branes yielding a $U(3)\times U(2)\times U(1)$ symmetry. Already in this very simple case it is manifest that the presence of extra $U(1)$'s in the brane world models is unavoidable. Furthermore this cannot be sufficient to consistently describe the matter content of the Standard Model. We have seen in the previous chapter how chiral Weyl fermions arise as twisted strings stretched between D-Branes with different magnetic fields on their world-volumes or intersecting at angles. Their spectrum contains fields in the bi-fundamental representation $(\bar{N}_i, N_j)$ where with i,j we indicate the two D-Branes the twisted string ends to. Hence [34] as the left-handed antileptons have the following quantum numbers

$$\bar{e}_{iL} \sim (1_c, 1_w)_2 \tag{5.1.1}$$

they have to be described by twisted strings stretched between two different single $U(1)$ D-Branes. As a consequence the minimal model which accommodates the full spectrum of the Standard Model should contain four D-Branes yielding a $U(3)\times U(2)\times U(1)\times U(1)$ gauge symmetry. In the explicit models analysed in the literature [80, 61, 34] linear combinations of the charges of the twisted strings under these four $U(1)$'s can always be associated to physical quantum numbers such as the baryon and lepton numbers, the hypercharge and a Peccei-Quinn symmetry. Without entering the details, the latter is an additional global symmetry of a generalisation of the Standard Model tree-level Lagrangian due to Peccei and Quinn to solve the so-called strong CP-problem [68]. It is in fact possible to add to the QCD Lagrangian describing the colour interaction a CP violating term with the form

$$\mathcal{L} = \frac{\mathrm{i}\vartheta}{32\pi^2}\mathrm{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu} \tag{5.1.2}$$

where the tilde stands for the Hodge dual of the field and ϑ is an additional parameter which weighs the violation of the CP symmetry in this sector. Experimental data lead to a very small value for ϑ close to zero. This is a fine tuning problem in the theory and a solution was proposed by Peccei and Quinn who promoted ϑ to be a scalar field arising as a Goldstone boson from the breaking of an additional $U(1)$ axial symmetry to the theory, known as Peccei-Quinn symmetry. The term in the Lagrangian above indeed comes as a result of the fact that the symmetry is anomalous at one loop and hence the Goldstone boson has a linear coupling with the gluons, while all of the other couplings have to involve its derivatives.

From the physical point of view, however, we expect only one of these abelian symmetries to survive as a gauge symmetry to be associated to the hypercharge

of the chiral fermions, while the remaining ones should only be global symmetries in the context of the Standard Model Lagrangian. We will now see that indeed in the brane worlds constructions the extra abelian symmetries can acquire a mass through the so called Stueckelberg mechanism.
The low-energy Dp-Brane action contains [21], apart from the Dirac-Born-Infeld term, also a Wess-Zumino contribution in the form

$$\int_{\Sigma_{p+1}} \sum_q C_q \mathrm{Tr}\left(e^{B+F}\right) \tag{5.1.3}$$

where C_q are the RR-forms, arising in the closed string spectrum when both the left and right handed world-sheet fields are taken in the Ramond sector, and $B + F$ is the usual gauge invariant combination of the Kalb-Ramond and gauge field strength on the brane. The sum is over all the RR-forms that, together with the expansion of the exponential above, give a top $p + 1$-form. Dp-Branes are in fact coupled in this way not only with the C_{p+1} RR-forms, but also with the lower rank ones, provided that the integrand is always in total a top form. In order for the extra $U(1)$'s to have a mass term it is necessary that the corresponding gauge fields acquire an additional degree of freedom to be interpreted as a longitudinal polarisation. In the brane world models under consideration such d.o.f. is given by a scalar Stueckelberg field which will be indicated as a_S. The corresponding form is then Hodge dual in the flat four dimensions to a two-form $B_S = B_{S\mu\nu} dx^\mu \wedge dx^\nu$. In the models involving magnetised D9-Branes these two-forms are derived from the dimensional reduction of the Wess-Zumino term with $q = 2, 6$ respectively, while in the intersecting D6-Branes models $q = 3, 5$. Let us focus on the magnetised picture. The gauge fields should here have a 10 dimensional dynamics. Upon toroidal compactification we will indicate with A_i and $F_i = dA_i$ the four-dimensional components of the $U(1)$ fields on the i-th brane along the flat directions on M^4. The internal compactified components of the $U(1)$ field strength in the magnetised brane picture are frozen and given a non-vanishing vacuum expectation value corresponding to the magnetic field on the brane world volume as in Eq.(3.2.33). The coupling between a magnetised D-Brane with the Stueckelberg two-form results in the following two possibilities

$$N_i \prod_{\alpha=1}^{3} p_{i,\alpha} \int_{M^4} B_{S,0} \wedge F_i \tag{5.1.4}$$

$$N_i \prod_{\alpha\neq\beta=1}^{3} W_{i,\alpha} p_{i,\beta} \int_{M^4} B_{S,\beta} \wedge F_i \tag{5.1.5}$$

where we have used the conventions introduced in Eq.(3.2.33) for the magnetic fields on the i-th brane wrapped on a generic T^6. In the first possibility $C_2 \equiv B_{S,0}$ and the product of three internal magnetic fields has been integrated on the T^6 to give the Chern numbers on the three T^2's inside the six-torus defined by Eq.(3.2.33). In the second possibility instead four indices of C_6 are integrated on the internal directions identified by α, giving a result proportional to the wrappings $W_{i,\alpha}$ while the integral of the magnetic field is calculated on the β-th T^2 inside T^6. The prefactor of N_i takes into account that the integration is performed for N_i coincident D-Branes. Of course the actual presence of the couplings above depends on the particular features of the model considered. Notice that they can be rewritten as

$$\int_{M^4} B_S \wedge F_i = -\int_{M^4} dB_S \wedge A_i = \int_{M^4} A_i \wedge *da_S = \int_{M^4} d^4x\, A_{i\mu}\partial^\mu a_S \qquad (5.1.6)$$

There is still one fundamental ingredient to be taken into account in order to determine the full Stueckelberg Lagrangian. From the string theoretical point of view the remaining coupling is found from an open string diagram with two borders and four punctures involving two external $U(1)$ gauge bosons and two magnetic fields on the D-Brane world volume, characterised by a string-coupling constant power of $g_s^{-2+2g+b+c} = g_s^0$. Here g is the genus (number of handles) of the Riemann surface of the string world sheet involved in the interaction considered, and b and c are respectively its number of borders and crosscaps. Observe that since the diagram contains more than one border, its contribution to the effective action cannot be determined by the simple dimensional reduction of the combination of the DBI action and the Wess-Zumino term that describes the low energy theory of a D-Brane. This is due to the fact that these terms arise as effective descriptions of the string theoretical disk interactions only. The result of this contribution has the form

$$\frac{k_{10}^2}{2g_s^4}\mathrm{Tr}\left(A_i \wedge \langle F\rangle_{\mathrm{m}}\right) \wedge *\mathrm{Tr}\left(A_i \wedge \langle F\rangle_{\mathrm{m}}\right) \qquad (5.1.7)$$

where $k_{10} \sim g_s^2$ is the gravitational coupling in 10 dimensions and, to avoid confusion, we have introduced the notation $\langle F\rangle_{\mathrm{m}}$ to indicate the internal magnetic field on the brane world-volume (the angular brackets remind that this is a background field that acquires a non vanishing vacuum expectation value). Notice that the combination of Eqs.(5.1.6) and (5.1.7) is responsible for the familiar shift, in the 10D supergravity action associated to the Type I superstring theory [21], of the kinetic term for the two form C_2 which, there, plays the same role as B_S

$$H = dC_2 \to dC_2 - \frac{k_{10}^2}{g_s^2}\omega_3 \quad \text{with} \quad \omega_3 = \mathrm{Tr}\left(A \wedge F - \frac{2\mathrm{i}}{3}A \wedge A \wedge A\right) \qquad (5.1.8)$$

In our case in fact $A \equiv A_i$ is an abelian field, hence the second contribution in the Chern-Simons term ω_3 disappears, while F is taken in the internal compactified directions and given a v.e.v. to become $\langle F \rangle_{\rm m}$. It is not difficult to see that $\frac{1}{2k_{10}^2} H \wedge *H$ yields the kinetic term for the two-form together with the couplings considered in Eqs.(5.1.6) and (5.1.7). The full Stueckelberg Lagrangian in the flat four dimensions reads

$$\begin{aligned} \mathcal{L}_S &= \frac{1}{2}\partial_\mu a_S \partial^\mu a_S - \frac{1}{4}\left(\partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}\right)^2 + M A_{i\mu}\partial^\mu a_S + \frac{1}{2}M^2 A_{i\mu}A_i^\mu \\ &= -\frac{1}{4}\left(\partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}\right)^2 + \frac{1}{2}\left(\partial_\mu a_S + M A_{i\mu}\right)^2 \end{aligned} \quad (5.1.9)$$

where M is a constant that reproduces the overall factors in front of the integrals in Eq.(5.1.4). The Lagrangian above has the gauge symmetry

$$\begin{cases} A_{i\mu} \to A_{i\mu} + \partial_\mu \Lambda \\ a_S \to a_S - M\Lambda \end{cases} \quad (5.1.10)$$

exploiting which, the field a_S can be reabsorbed in the definition of A_i that, as a consequence, acquires a non vanishing mass term.
In the explicit models constructed in the literature the combination of the $U(1)$ fields associated to the Baryon number B becomes massive and hence this is reinterpreted as a global symmetry of the string model that also ensures the stability of the proton. The same happens to the Lepton number L, which is conserved only as the sum of the three family lepton numbers $L = L_e + L_\mu + L_\tau$. The latter might be violated by instanton effects as in [81]. This would allow for the neutrino oscillations actually observed in experimental data[1], in contrast with the Standard Model Lagrangian analysed in the first chapter that does not contain the description of this phenomenon. In general the only gauge symmetry left is the hypercharge.

5.2 Tadpole and anomaly cancellation

As seen in the previous section, D-Branes act as sources for the RR-fields. The couplings arise from the Wess-Zumino term of the low-energy action and in particular it reveals that a Dp-Brane is always charged under a C_{p+1} RR-form. The D-Branes under consideration are wrapped on a compact internal manifold such as a T^6. From the physical point of view the Gauss's law in a compact space requires the total charge to vanish. This is a fundamental requirement for these

[1] Recall that these oscillations require Majorana masses for the neutrinos which do violate the conservation of L.

models that is related to the consistency of the equations of motion for the fields coupled to the charged objects. In order to see that this is indeed the case we will focus for instance on the intersecting D6-Branes picture [25]. A D6-Brane on a geometrical background given by $M^4 \times T^6$ will wrap a 3-cycle in the internal compact manifold, which will be indicated with Π_i for the i-th brane. The action for the RR-form C_7 coupled to the brane reads

$$\begin{aligned} S_{C_7} &= \int_{M^4\times T^6} H_8 \wedge *H_8 + \sum_i N_i \int_{M^4\times \Pi_i} C_7 \\ &= \int_{M^4\times T^6} C_7 \wedge *dH_2 + \sum_i N_i \int_{M^4\times T^6} C_7 \wedge \delta\left(\Pi_i\right) \end{aligned} \tag{5.2.11}$$

where $H_8 = dC_7$ is the field strength associated to the RR-form, H_2 is its Hodge dual and $\delta\left(\Pi_i\right)$ is a bump three form located on the cycle Π_i. It is easy to see that the equations of motion are

$$dH_2 = \sum_i N_i \delta\left(\Pi_i\right) \tag{5.2.12}$$

In the language of cohomology these equations are consistent only if the right hand side is an exact form as well as dH_2. Upon integration on T^6, in the language of homology, the consistency condition for the equations of motion of C_7 is

$$\sum_i N_i \left[\Pi_i\right] = 0 \tag{5.2.13}$$

where $[\Pi_i]$ is the homology cycle corresponding to the i-th wrapped D-Brane along the 3-cycle Π_i. This condition is the cancellation of the RR tadpoles for the intersecting branes model. The reason for this nomenclature is more visible from the point of view of the compactified theory [25].
The dimensional reduction of the RR-form C_7 yields a C_4 on the flat directions of M^4. In particular taking a basis of 3-cycles on T^6, $[\Sigma_a]$, these correspond to the zero-modes $C_{4,a} = \int_{[\Sigma_a]} C_7$. Being 4-forms, they do not have kinetic terms in their four dimensional action, and they only appear in linear tadpole terms of the form

$$S_{C_{4,a}} = [\Pi_i] \cdot [\Sigma_a] \int_{M^4} C_{4,a} \tag{5.2.14}$$

where in this context the dot denotes the intersection between the cycles $[\Pi_i]$ and $[\Sigma_a]$ as in Eq.(4.6.83). The equations of motion then imply that the coefficient of the tadpole must vanish, $[\Pi_i] \cdot [\Sigma_a] = 0$, which is not a condition on the field but a consistency condition for the model. For a complete basis of $[\Sigma_a]$'s this implies

the relation in Eq.(5.2.13).
In the magnetised brane worlds instead the presence of D9-Branes leads to a tadpole associated to the RR-form C_{10}. The cancellation of such tadpole is achieved by introducing orientifold planes in the model [20, 21]. This operation introduces mirror D-Branes with opposite magnetic fields turned on in their world volumes and, as a consequence, new twisted sectors appear in the model involving strings attached to these mirror branes. We will not enter the details of the spectrum of the orientifolded models for which a large amount of literature is available [25, 34, 27].
Returning to the intersecting D-Branes models one can show that the cancellation of the RR-tadpoles yields the cancellation of chiral four dimensional gauge anomalies in the low-energy effective action. Recall in fact that the cubic $SU(N_i)^3$ anomaly is proportional to the difference between the number of fundamental and anti-fundamental representations of $SU(N_i)$, which, as a consequence of Eq.(4.6.83), is given by $\sum_j I_{ij} N_j$. Then it is not difficult to show that the cancellation of the RR-tadpoles yields the cancellation of the cubic anomalies as

$$\sum_j I_{ij} N_j = [\Pi_i] \cdot \sum_j N_j \, [\Pi_j] = 0 \tag{5.2.15}$$

where in the last equation we have made use of the Eq.(5.2.13). Observe that the request of the cancellation of RR tadpoles is actually stronger than the cancellation of anomalies and the argument above shows that in consistent intersecting D-Branes models the number of fundamental and anti-fundamental representations of $SU(N_i)$ should coincide even for $N_i = 1, 2$ where no gauge anomalies exist at the level of the low-energy effective theory. In the model building, as a consequence, one has to insure that all of the $SU(N_i)$ gauge symmetries obey this constraint.
The mixed $U(1)_i$-$SU(N_j)^2$ anomalies also cancel as a result of the tadpole condition in Eq.(5.2.13). We will not enter the details of the calculation that shows how this cancellation works. We will only sketch that, differently to the case of the cubic anomalies, the cancellation here is a result of the combined effect of two types of diagrams. First we have to consider the usual triangle diagram as in Fig.2.1, involving two $SU(N_j)$ and a $U(1)_i$ external legs; second we must include the interaction, already discussed in the previous section, which couples the $U(1)_i$ with a RR two-form B_S, that in turns splits into two $SU(N_j)$ non abelian gauge bosons (Fig.5.1). Upon the cancellation of the RR tadpoles, the sum of these two diagrams is shown to vanish giving the expected cancellation of the mixed anomalies. This is known in the literature as generalised Green-Schwarz mechanism.
We will not consider here the question related to the cancellation of the NSNS

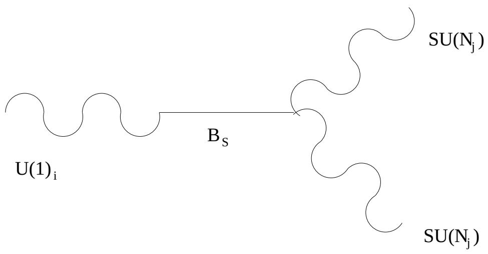


Fig. 5.1: Additional diagram for the cancellation of mixed $U(1)$ and $SU(N)$ anomalies

tadpoles [25] given by the fields associated to the background geometry (graviton, Kalb-Ramond field, dilaton, closed string moduli). Let us simply stress that unlike the case of the RR tadpoles, the cancellation of the NSNS tadpoles is not a consistency condition for the model. The NSNS fields in fact do have kinetic terms in their four dimensional action, hence any tadpole, upon the variation of the action, gives a condition on the fields and their equations of motion. These tadpoles imply that the background fields chosen do not really solve the equations of motion. However, as the complete equations for string theory are not known, it could still be that the solutions used are a good approximation of the complete unknown ones.

6. CLOSED STRINGS, D-BRANES AND BOUNDARY STATES

We have so far focused on the open string sector of the Brane World Models distinguishing between twisted open strings, which lead to the presence of four-dimensional chiral fermions in the toroidal compactifications considered, and untwisted open strings that describe the gauge sector of the models. We want to describe here the D-Branes involved in the construction of phenomenologically realistic models from the closed string point of view. As already seen in the previous chapter, D-Branes are coupled to the closed string sector of the theory, and the interactions involved are encoded in the low-energy effective description given by the combination of the DBI action and the Wess-Zumino term. In this chapter we will see that in fact a D-Brane can be described as a particular state in the closed string sector of the theory, which introduces a boundary in the closed strings world-sheet where left and right-handed closed string fields are suitably identified. This is known as the boundary state for the D-Brane under study. We will discuss the general properties of such states and we will follow the traditional procedure to determine the boundary state for a magnetised D-Brane in a generic toroidal compactification. However this method will not be appropriate to obtain all of the necessary prefactors to the boundary state, thus we will also introduce a different approach to the same computation that will allow us to obtain them explicitly. These factors will actually prove to be of fundamental importance in the computation of the Yukawa couplings in these models, as it will be discussed in the following chapters.

6.1 D-Branes as Boundary states

We have already described a D-Brane from the open string perspective in section 4.1, by studying the boundary conditions the strings have to satisfy on their end-points when they are attached to the Brane, as in Eq.(4.1.4). In this section we want to consider the description of the same system from a closed string point of view. In order to do this we can exploit the conformal invariance of the world-sheet theory defined by the actions (3.3.40) and (4.1.2). The relation between the open and closed string languages in describing strings interactions, in partic-

ular with D-Branes for the purpose of the present discussion, was traditionally associated to the study of the interaction between two D-Branes [20, 82]. We can imagine this interaction as the effect of the vacuum fluctuations of the open strings stretching between the branes, similarly to the interaction between two superconducting plates caused by the vacuum fluctuation of the electromagnetic field that gives rise to the Casimir effect [82]. Then this interaction is related to the one-loop open string partition function represented by the annulus diagram. The conformal transformation that exchanges of the roles of the two world-sheet variables, τ and σ, leads to the interpretation of the same diagram as a tree level interaction involving closed strings. This statement is actually very general and also holds when one takes into consideration more complicated interactions involving the presence of more than two boundaries in the strings world-sheet.
As discussed in section 4.1, the presence of a D-Brane, as a boundary for the open strings world sheet, induces some identifications on the left and right-handed fields that solve the free equations of motion (4.1.3). Such identifications follow from the boundary conditions imposed on these fields in Eq.(4.1.6). In order to exchange the roles of the two world-sheet variables and write these conditions in the closed string channel, it is convenient to map the upper half complex plane, where the variable z is defined, into the unit disk, through the conformal map

$$w = -\frac{(z - \mathrm{i})}{(z + \mathrm{i})} \tag{6.1.1}$$

so that z is mapped into w and $\bar{z}$ into $1/\bar{w}$. The boundary conditions (4.2.15) become identifications on the string coordinates that have to be satisfied by a boundary state $|B_i\rangle$ as follows

$$\left[\bar{w}\bar{\partial}x^M(w, \frac{1}{\bar{w}}) + (R_i)^M{}_N w\partial x^N(w, \frac{1}{\bar{w}})\right]\Bigg|_{\tau=0} |B_i\rangle = 0 \tag{6.1.2}$$

where the reflection matrix R_i corresponds to the i-th boundary of the world-sheet.
Differently to the case of the open strings, in which the boundary conditions give rise to twisted sectors with modified mode expansions, like in the particular example of the shifted modes discussed for a toroidal compactification with magnetised D-Branes in section 4.2, the closed strings spectrum is unaffected by the presence of the D-Brane. The reflection matrices R_i, that encode the information of the presence of such boundary, here only enforce an identification between the left and right handed oscillators *on* the boundary. Hence, recalling the results already discussed in Section 3, for the closed strings emitted by the brane the

mode expansion

$$x^M(z,\bar{z}) = \frac{1}{2}\left[X^M(z) + \tilde{X}^M(\bar{z})\right] \tag{6.1.3}$$

with

$$X^M(z) = x_0^M - \mathrm{i}\sqrt{2\alpha'}\alpha_0^M \ln(z) + \mathrm{i}\sqrt{2\alpha'}\sum_{m\neq 0}\frac{\alpha_m^M}{m}z^{-m} \tag{6.1.4}$$

as in Eqs.(3.3.41) and (3.3.42), sill holds and the geometrical background is still encoded in the Narain lattice

$$\begin{aligned}(\alpha_0)^M &= \frac{G^{MN}}{\sqrt{2}}\left[n_N + (G-B)_{NN'}m^{N'}\right]\\ (\tilde{\alpha}_0)^M &= \frac{G^{MN}}{\sqrt{2}}\left[n_N - (G+B)_{NN'}m^{N'}\right]\end{aligned} \tag{6.1.5}$$

as in Eq.(3.3.48). Upon canonical quantisation one also gets

$$\left[\alpha_m^M, \alpha_n^N\right] = m\delta_{n+m,0}G^{MN} \quad , \quad \left[x^M, \alpha_0^N\right] = \mathrm{i}\sqrt{2\alpha'}G^{MN} \tag{6.1.6}$$

and similar relations for the right moving sector. Thus at the level of closed string modes the identification (6.1.2) becomes

$$\left(\tilde{\alpha}_{-m}^M + (R_i)^M{}_N\alpha_m^N\right)|B_i\rangle = 0 \tag{6.1.7}$$

Observe that this is again consistent with the T-Duality transformations (3.3.71) and (4.1.12). Also, similarly to the open string case, the two copies of the Virasoro algebra encoded in the holomorphic and anti-holomorphic stress energy tensor (4.1.9) of the string CFT, are identified on the boundary and give rise to a unique Virasoro algebra (written in terms of only one set of oscillators), provided that the reflection matrices R_i's preserve the metric, namely that ${}^tR_iGR_i = G$.
Let us focus on the non-zero modes contribution only. It is easy to check that one possible solution to Eq.(6.1.7) is given by

$$|B_i\rangle|_{\text{n.z.m.}} = \sqrt{\mathrm{Det}(G+\mathcal{F}_i)}\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n}^M(GR_i)_{MN}\tilde{\alpha}_{-n}^N}|0\rangle \tag{6.1.8}$$

where $|0\rangle$ is the vacuum annihilated by the left and right moving oscillators α_n^M, $\tilde{\alpha}_n^M$, $\forall n > 0$. The normalisation factor, which can not be determined by means of the identifications encoded in Eq.(6.1.7), ensures that, when saturating the boundary state with closed string external states such as the dilaton, the graviton and the B-field, one recovers the effective action describing the D-Brane in the form of the Dirac-Born-Infeld action [83, 84]. By recalling the relations in

Eq.(6.1.5) and the fact that ${}^tR_iGR_i = R_i$, the relation (6.1.7) on the zero-modes can be rewritten using a matricial notation as

$$\begin{aligned} & \left[\left(1+GR_iG^{-1}\right)(\hat{n}-B\hat{m})-G\left(1-R_i\right)\hat{m}\right] \quad |B_i\rangle|_{\text{z.m.}} = \\ & = \left[\left(1+{}^tR_i^{-1}\right)(\hat{n}-B\hat{m})-G\left(1-R_i\right)\hat{m}\right] \quad |B_i\rangle|_{\text{z.m.}} = 0 \end{aligned} \tag{6.1.9}$$

where the hats simply refer to the operators reading the corresponding eigenvalues in the ket $|B_i\rangle$. Like in section 4.1, this is a general framework that allows for the description of any D-Brane from the closed strings point view. Specifying the explicit form of the reflection matrix R_i one can use the results found so far in the context of both magnetised and intersecting D-Branes.
Let us stress that in this approach the boundary state describing a D-Brane from the closed string point of view is obtained by using a linear relation such as (6.1.7), hence any prefactor, including the normalisation, has to be fixed by other means. The normalisation can actually be determined by comparison between the low energy action of the D-Brane and the contraction of the boundary state describing it with massless closed string external states, or by using the so-called Cardy conditions (see for instance [85]), but this still leaves the possibility of having an additional prefactor in the form of a non trivial phase. We will follow a different approach to show that this is indeed the case, for instance, for magnetised D-Branes.

6.1.1 The magnetised D-Brane example

Let us consider in more details the case of a magnetised D-Brane (see [86, 87, 88, 65] for works on this subject) on a generic T^{2d}, while the boundary state for a D-Brane at angle will be subsequently found by means of T-Duality, exploiting the result in Eq.(4.1.12).
The non-zero modes dependence of the boundary state has the same form as Eq.(6.1.8) with

$$R = (G-\mathcal{F})^{-1}(G+\mathcal{F}) \tag{6.1.10}$$

The expression in Eq.(6.1.9) can instead be reshuffled in the following fashion

$$\begin{aligned} \hat{n}|B\rangle &= \left\{B\hat{m}+\left[1+(G+\mathcal{F})(G-\mathcal{F})^{-1}\right]^{-1}G\left[1-(G-\mathcal{F})^{-1}(G+\mathcal{F})\right]\hat{m}\right\}|B\rangle \\ &= \left\{B\hat{m}+(G-\mathcal{F})\left[2G\right]^{-1}G(G-\mathcal{F})^{-1}\left[-2\mathcal{F}\right]\hat{m}\right\}|B\rangle \\ &= -F\hat{m}|B\rangle \end{aligned} \tag{6.1.11}$$

with F as in Eq.(3.2.29). The zero modes eigenvalues in the ket $|B\rangle$ contain the Kaluza-Klein momenta n_M and the winding numbers m^M of the closed string emitted by the magnetised D-Brane, namely $|B\rangle_{\text{z.m.}} \equiv |0; n_M, w^M\rangle$. Before introducing the identification above these are independent numbers and in particular, if the D-Brane has non-trivial wrappings w_M along the cycles of the torus, then $|B\rangle_{\text{z.m.}} = |0; n_M, w_M m^M\rangle$, where no sum is understood over the repeated index M. This can be intuitively justified by observing that the closed string emitted by the D-Brane must have endpoints identified on the brane world-volume, which is wrapped w_M times along the M-th direction of the torus. Thus it is necessary for the string winding numbers to be integer multiples of the wrapping of the brane. In the different approach we will adopt in the following sections we will be able to derive explicitly this result. The identification $\hat{n}_M = -F_{MN}\hat{m}^N$ then would naively yield $n_M = -F_{MN}(w_N m^N)$ in the boundary state, but, as both the Kaluza-Klein momenta and the winding numbers of the emitted closed string must be integer numbers, the equation resulting from the left-right zero-modes identification is a Diophantine equation. Given the form of the magnetic field as in (3.2.33), the solution to this equation becomes in matricial notation $n = -F \cdot (wm)$, where we have introduced the matrix

$$w = \begin{pmatrix} W_1 & 0 & \cdots \\ 0 & W_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \otimes 1_{2\times 2} \tag{6.1.12}$$

Again this result will be better justified in the alternative approach to the construction of the boundary state.
Hence the naive boundary state for a magnetised D-Brane reads

$$|B\rangle = \sum_{m\in\mathbb{Z}^{2d}} \sqrt{\text{Det}(G+\mathcal{F})} \prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n}\cdot(GR)\cdot\tilde{\alpha}_{-n}} |0; -F\cdot(wm), (wm)\rangle \tag{6.1.13}$$

Recall in fact that this form does not include yet the phase factor, dependent on the magnetic field and on the Kaluza-Klein and winding numbers of the emitted strings, which will be determined in the subsequent section devoted to an alternative approach to the construction of the boundary state.

6.2 Magnetised D-Branes boundary state: an alternative approach

We have stressed that Eq.(6.1.2), being a set of linear constraints, fixes the form of the boundary state up to an overall factor that can also depend on the Kaluza-Klein and winding operators. We will now show that the boundary state for a

magnetised D-Brane does indeed contain non-trivial phases that depend on the winding numbers and on the magnetic field F. Moreover it turns out that this phase will be strictly related to the phase that closed string states should acquire under T-Duality. In the discussion of the interaction vertex between closed strings in a compact space in the following chapter, we will in fact see that the naive T-Duality rules analysed in section 3.3 have to be generalised including some phases known as cocycle factors to obtain a consistent picture. These cocycles will actually depend on Kaluza-Klein and winding operators and we will show that the phase found in the present approach to a magnetised D-Brane boundary state is an example of the realisation of the new T-Duality rules we will introduce.
Following [89], we consider the gauge field contribution to the action (4.1.2) in the string path-integral as an interaction term that acts on the standard boundary state for an unmagnetised D-Brane wrapping the T^{2d}. The computation was performed in [89] in a Minkowski flat target space. The novelty of the present calculation, which is also performed in [88, 65], is the compactness of the torus wrapped by the D-Branes. Hence we will be able to reproduce the same result found in [89] as far as the non-zero modes dependence of the boundary state is concerned. The zero-modes analysis will reveal different characteristics of the boundary state with respect to [89], which will include the F-dependent phase related to the cocycle factors of T-Duality and the restrictions on the Kaluza-Klein and winding numbers for the emitted closed strings already introduced in the previous section.
We want to derive the dependence of $|B_F\rangle$ on the magnetic field by applying the usual path ordered Wilson loop operator to the unmagnetised D-Brane boundary state, namely

$$|B_F\rangle \sim \mathrm{P}[e^{\frac{\mathrm{i}}{2\pi\alpha'}\oint A dx}]|B_{F=0}\rangle \tag{6.2.14}$$

where the magnetic field will be taken in the form of Eq.(3.2.33) and the corresponding potential will be chosen in the following gauge

$$A_{M\equiv 2\alpha}(x) = \frac{p_\alpha}{W_\alpha} x_{2\alpha-1} + 2\pi\sqrt{\alpha'}C_{2\alpha-1} \quad \text{and} \quad A_{M\equiv 2\alpha-1}(x) = 2\pi\sqrt{\alpha'}C_{2\alpha} \tag{6.2.15}$$

$\forall\alpha = 1, \dots, d$. Notice that with respect to Eq.(3.2.34) we have also introduced non-zero Wilson lines $2\pi\sqrt{\alpha'}C_M$, with C_M adimensional and we have to promote the coordinates on which the gauge potential depends to closed string fields as in Eq.(6.1.3), evaluated on the boundary at $\tau = 0$. In this way the non-abelian gauge bundle describing the magnetised wrapped D-Brane is characterised by the holonomy matrices in Eq.(3.2.36). Let us start from the observation that the path ordered Wilson loop operator as written in the equation above is not complete as it is not invariant under gauge transformations. Consider in fact a path c connecting two points that are separated by the lattice vector $\sum_{L=1}^{2d} m_L a_L$, $m_L \in \mathbb{Z}$, which means that they are identified on the torus (see the conventions used in section 3.2). In order to be consistent with our convention for $\mathcal{F} = B + F$, this path will start from $x + 2\pi\sqrt{\alpha'}\sum_{L=1}^{2d} m_L a_L$ and end in x. Then it is clear that the naive path ordered Wilson loop operator transforms as

$$\mathrm{P}\left[e^{\frac{\mathrm{i}}{2\pi\alpha'}\int_c A dx}\right] \to \gamma(x)\ \mathrm{P}\left[e^{\frac{\mathrm{i}}{2\pi\alpha'}\int_c A dx}\right]\ \gamma^\dagger\left(x + 2\pi\sqrt{\alpha'}\sum_{L=1}^{2d} m_L a_L\right) \tag{6.2.16}$$

where $\gamma(x)$ is a unitary gauge transformation. Recalling the relation (3.2.39) that encodes the gauge transformation for the holonomy matrices associated to the description of a magnetised D-Brane as a gauge bundle, it is manifest that we can recover a gauge invariant object to be applied to the unmagnetised boundary state if we multiply the naive form of the Wilson loop operator by a sequence of holonomy matrices U which forms a discretised version of the path c. The consistency condition (3.2.37) allows us to collect together all shifts, induced by the holonomy matrices on the fields defined on the magnetised D-Brane world-volume as in Eqs.(3.2.35) and (3.2.38), along the direction $K = 1$, then those along $K = 2$ and so on. In formulae we can define

$$\mathcal{O}_A = \mathrm{Tr}\left\{\left[\prod_{K=1}^{2d}\prod_{m=0}^{\hat{m}_K - 1} U_K\left(x\big|_{\sigma,\tau=0} + \sum_{L=1}^{K-1} 2\pi\sqrt{\alpha'}\hat{m}_L a_L + 2\pi\sqrt{\alpha'} m a_K\right)\right] \mathrm{P}[e^{\frac{\mathrm{i}}{2\pi\alpha'}\int_c A dx}]\right\} \tag{6.2.17}$$

where the x-dependence in the operator has been promoted to become the string coordinate in Eq.(6.1.3) evaluated again on the boundary at $\tau = 0$ and taken also at $\sigma = 0$. This is due to the fact that each U_K in the trace shifts the string along a particular segment in the discretisation of the path c and in each segment the world-sheet spacial coordinate ranges between 0 and 2π. The choice $\sigma = 0$ corresponds to shifting the coordinates starting from the first end-point of the string. Finally the $\hat{m}_K$'s are the operators that read the winding numbers of the closed string states. Observe that only the eigenvalues $m_K \geqslant 1$ are relevant in the expression above as, if $m_K = 0$ for a certain K, then the corresponding U_K does not appear in the product. It is not difficult to check that the operator $\mathcal{O}_A$ is now invariant under unitary gauge transformations $\gamma(x)$.
In the following subsections we will use this procedure to first determine the boundary state of a wrapped D-Brane with non-trivial Wilson lines turned on, starting from the one of an unwrapped D-Brane as known from the literature [83]; then we will obtain the zero-modes dependence of the boundary state for a wrapped magnetised D-Brane and finally we will also reproduce the result of [89] related to the non-zero modes contribution.

6.2.1 Wrapped D-Brane with non-trivial Wilson lines

The boundary state for a wrapped D-Brane is found from the one of an unwrapped brane by applying the same operator in Eq.(6.2.17) with the choice

$$A_M = 0 \quad \text{and} \quad U_K = 1_{w_1 \times w_1} \otimes \ldots \otimes P_{w_K \times w_K} \otimes 1_{w_{K+1} \times w_{K+1}} \otimes \ldots \times 1_{w_{2d} \times w_{2d}} \tag{6.2.18}$$

with U_K defined as in Eq.(3.2.28). In this case the trace in Eq.(6.2.17) reads

$$\mathrm{Tr}\left[\prod_{K=1}^{2d} U_K^{\hat{m}_K}\right] \tag{6.2.19}$$

and it is different from zero only when the windings of the emitted closed strings are integer multiples of the wrappings of the D-Brane on each cycle of the torus. Hence only these states couple to the wrapped D-brane, as expected, and we have

$$|B_{A=0}\rangle = \sqrt{\mathrm{Det}(G+B)} \sum_{m^M \in Z} \prod_{n=1}^{\infty} e^{-\frac{1}{n}\tilde{\alpha}_n^\dagger G R^{(0)}\, \alpha_n^\dagger} |0; w_M m^M\rangle \tag{6.2.20}$$

where there is no sum understood over the repeated index M; $R^{(0)} = (G - B)^{-1}(G+B)$ is the identification matrix between left and right moving oscillators and depends on the geometric background of the torus; finally the ket $|0; w_M m^M\rangle$ represents the closed string vacuum state with zero Kaluza-Klein momenta and winding numbers equal to $w_M m^M$ for $M = 1, 2, ..., 2d$.

It is then easy to turn on the Wilson lines and to keep vanishing magnetic fields on the D-brane world volume. We have just to isolate the Wilson line contribution to $\mathcal{O}_A$ in Eq.(6.2.17) when acting on $|B_{A=0}\rangle$, namely

$$|B_C\rangle = e^{\frac{\mathrm{i}}{\sqrt{\alpha'}}\int_c C\cdot dx}\, |B_{A=0}\rangle = \mathrm{e}^{2\pi \mathrm{i} C\cdot \hat{m}} |B_{A=0}\rangle \tag{6.2.21}$$

6.2.2 Zero-modes dependence of a wrapped magnetised D-Brane boundary state

In order to make use of the explicit expression of the operator (6.2.17) with holonomy matrices defined as in Eq.(3.2.36) we need to calculate the trace of the sequence of U's that appears in its definition. It is useful to observe that if $K = 2\alpha$, with $\alpha = 1, \ldots, d$, then the corresponding U_K is a constant matrix that does not have any dependence on the coordinates, while if $K = 2\alpha - 1$ then the only dependence is on the coordinate $x_{2\alpha}$. This means that in the computation of the trace we do not need to worry about the shifts in the coordinate dependence of the holonomy matrices which only involve the coordinates x_M in U_K with

$M \leqslant K$ as it is manifest in Eq.(6.2.17). Thus the computation of the trace in the operator $\mathcal{O}_A$ simply becomes

$$\mathrm{Tr}\left[\prod_{K=1}^{2d}\prod_{m=0}^{\hat{m}_K-1} U_K\left(x\big|_{\sigma,\tau=0}+\sum_{L=1}^{K-1}2\pi\sqrt{\alpha'}\hat{m}_L a_L+2\pi\sqrt{\alpha'}m a_K\right)\right]= \qquad (6.2.22)$$
$$\mathrm{Tr}\left[\prod_{\alpha=1}^{d}U_{2\alpha}^{\hat{m}_{2\alpha}}U_{2\alpha-1}^{\hat{m}_{2\alpha-1}}\right]=\mathrm{Tr}\left[\prod_{\alpha=1}^{d}P_{W_\alpha\times W_\alpha}^{\hat{m}_{2\alpha}}Q_{W_\alpha\times W_\alpha}^{p_\alpha\hat{m}_{2\alpha-1}}\right]e^{\frac{\mathrm{i}}{\sqrt{\alpha'}}\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\hat{m}_{2\alpha-1}x^{2\alpha}|_{\sigma,\tau=0}}$$

Hence when we apply the operator $\mathcal{O}_A$ to the boundary state (6.2.21) a first result we find is that the magnetised boundary state couples only to the closed strings whose winding numbers in the α-th T^2, as defined by the form of the magnetic field (3.2.33), are integer multiples of W_α. Introducing the same matrix w as in Eq.(6.1.12), we will indicate with wm the $2d$ column of the winding numbers of the closed strings emitted by the magnetised D-brane. We can also see how the action of $\mathcal{O}_A$ yields the relation between windings and Kaluza-Klein numbers

$$\hat{n}=-F\hat{m} \qquad (6.2.23)$$

This is achieved by focusing on the zero-modes contribution linear both in the position operator $\hat{q}^M = \left(x_0^M+\tilde{x}_0^M\right)/2$ and in the oscillators α_0 or $\tilde{\alpha_0}$ in the expansion (6.1.4) of the closed string coordinates that appear in the exponential in the final expression for the trace (6.2.22) and in the path-ordered Wilson loop in the operator $\mathcal{O}_A$ with gauge potential given in Eq.(3.2.34). We can evaluate this contribution as follows

$$\begin{aligned}|B_F\rangle &\sim e^{\frac{\mathrm{i}}{\sqrt{\alpha'}}\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\hat{m}_{2\alpha-1}\hat{q}^{2\alpha}}e^{-\mathrm{i}\frac{\sqrt{\alpha'}}{2\sqrt{2}}\sum_{\alpha=1}^{d}\int_0^{2\pi}d\sigma\frac{1}{2\pi\alpha'}\frac{p_\alpha}{W_\alpha}(x_0^{2\alpha-1}+\tilde{x}_0^{2\alpha-1})(\alpha_0^{2\alpha}-\tilde{\alpha}_0^{2\alpha})}|0;wm\rangle\\ &=e^{\mathrm{i}\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\hat{m}_{2\alpha-1}\frac{\hat{q}^{2\alpha}}{\sqrt{\alpha'}}}e^{-\mathrm{i}\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\frac{\hat{q}^{2\alpha-1}}{\sqrt{\alpha'}}\hat{m}_{2\alpha}}|0;wm\rangle=|-Fwm,wm\rangle\end{aligned} \qquad (6.2.24)$$

Notice that at this stage we can forget about the path-ordering in the Wilson operator $\mathcal{O}_A$ and explicitly perform the integration in the first line of the previous equation, as the zero-mode contributions of the string fields entering the Wilson loop in $\mathcal{O}_A$ commute with each other at different values of σ. Let us now consider the terms quadratic in the zero-modes α_0 and $\tilde{\alpha}_0$. It is easy to see that the exponential that results from the computation of the trace (6.2.22) does not contribute as its quadratic term in the zero-modes reads

$$e^{\mathrm{i}\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\hat{m}_{2\alpha-1}\hat{m}_{2\alpha}\sigma} \qquad (6.2.25)$$

and vanishes at $\sigma = 0$. Hence we will only consider the expansion of the string fields in the path-ordered Wilson loop in Eq.(6.2.17), namely

$$e^{-\frac{\mathrm{i}}{2}\sum_{\alpha=1}^{d}\int_0^{2\pi} d\sigma \frac{1}{2\pi}\frac{p_\alpha}{W_\alpha}\left(\alpha_0^{2\alpha-1}\alpha_0^{2\alpha}-\tilde{\alpha}_0^{2\alpha-1}\alpha_0^{2\alpha}-\alpha_0^{2\alpha-1}\tilde{\alpha}_0^{2\alpha}+\tilde{\alpha}_0^{2\alpha-1}\tilde{\alpha}_0^{2\mu}\right)\sigma}|-Fwm;wm\rangle =$$
$$= \mathrm{e}^{-\mathrm{i}\pi\sum_{\alpha=1}^{d} W_\alpha p_\alpha m_{2\alpha-1}m_{2\alpha}}|-Fwm;wm\rangle = \mathrm{e}^{-\mathrm{i}\pi\sum_{M<N}\hat{m}^M F_{MN}\hat{m}^N}|-Fwm;wm\rangle \tag{6.2.26}$$

As already stressed this is the phase depending on the magnetic field F and on the Kaluza-Klein and winding numbers of the closed strings that could not have been deduced just by looking at the constraints (6.1.2). Thus the zero-modes contribution to the expression for the boundary state describing a wrapped magnetised D-Brane is

$$|B_{F,C}\rangle\big|_{\text{z.m.}} = \sum_{m\in\mathbb{Z}^{2d}} e^{-\mathrm{i}\pi\sum_{M<N}\hat{m}^M F_{MN}\hat{m}^N}\, e^{2\pi\mathrm{i}C\hat{m}}|-Fwm,wm\rangle \tag{6.2.27}$$

As a final remark let us underline that even if the phase in Eq.(6.2.27) has been calculated for a block-diagonal F, it holds for a generic form of the magnetic field as in Eq.(3.2.29). Indeed, as we show in Appendix A.3.1, this phase factor is not affected by the change of the fundamental cell in the lattice torus performed in Eq.(3.2.32), a part from possible half integer shifts in the Wilson lines. Thus we will always refer to the magnetic field F in the phase factor as in Eq.(3.2.29) unless explicitly stated. This will be particularly useful in the computation of the Yukawa couplings in the following chapters where these phases will play a crucial role for the consistency of the model.

6.2.3 Cancellation of mixed contributions of zero and non-zero modes

In the previous subsection we focused on the zero-modes dependence from both the fields in the Wilson loop and from the trace in the operator (6.2.17). Here we show that the mixed contributions involving both zero and non-zero modes do not appear in $\mathcal{O}_A$. Let us concentrate on the term in the Wilson loop that arises from taking the zero-modes contribution in the coordinates $x^{2\alpha-1}$ in the gauge potential (6.2.15) and the non-zero modes in $dx^{2\alpha}$, $\alpha = 1,\ldots,d$, namely

$$\exp\left\{-\frac{\mathrm{i}}{4\pi}\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\sum_{m>0}\int_0^{2\pi} d\sigma\left[\left(\alpha_0^{2\alpha-1}-\tilde{\alpha_0}^{2\alpha-1}\right)\left(\alpha_m^{2\alpha}-\tilde{\alpha}_m^{\dagger 2\alpha}\right)e^{-\mathrm{i}m\sigma}+\right.\right. \tag{6.2.28}$$
$$\left.\left.\left(\alpha_0^{2\alpha-1}-\tilde{\alpha}_0^{2\alpha-1}\right)\left(\alpha_m^{\dagger 2\alpha}-\tilde{\alpha}_m^{2\alpha}\right)e^{\mathrm{i}m\sigma}\right]\sigma\right\}$$

Notice that it has been possible to rewrite the exponential in terms of a set of *commuting* variables [89], thus the integration can be performed explicitly to find

$$\exp\left\{-\sum_{\alpha=1}^{d}\frac{p_\alpha}{\sqrt{2}W_\alpha}\sum_{m>0}\frac{1}{m}\hat{m}^{2\alpha-1}\left(\alpha_m^{\dagger 2\alpha}-\tilde{\alpha}_m^{2\alpha}-\alpha_m^{2\alpha}+\tilde{\alpha}_m^{\dagger 2\alpha}\right)\right\} \tag{6.2.29}$$

The same expression with opposite sign in the exponent can be derived by considering the non-zero modes contribution in the $x^{2\alpha}$-dependence of the exponential in the trace (6.2.22), thus leading to the cancellation of this mixed term. The remaining mixed contributions involving, in the Wilson loop in $\mathcal{O}_A$, the zero-modes in $x^{2\alpha-1}$ and the oscillators in $x^{2\alpha}$ also vanish upon integration in σ, once we introduce again the same set of commuting variables.

6.2.4 Non-zero modes dependence and the complete wrapped magnetised D-Brane boundary state

The quadratic non-zero modes term arising from the Wilson loop in the operator (6.2.17) reads

$$\begin{aligned}\exp\Big\{-\tfrac{1}{4\pi}\textstyle\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}\sum_{m,n>0}\frac{1}{m}\int_0^{2\pi}d\sigma\,\Big[&\left(\alpha_m^{2\alpha-1}-\tilde{\alpha}_m^{\dagger 2\alpha-1}\right)\left(\alpha_n^{2\alpha}-\tilde{\alpha}_n^{\dagger 2\alpha}\right)e^{-\mathrm{i}(m+n)\sigma}\\ &-\left(\alpha_m^{\dagger 2\alpha-1}-\tilde{\alpha}_m^{2\alpha-1}\right)\left(\alpha_n^{2\alpha}-\tilde{\alpha}_n^{\dagger 2\alpha}\right)\;e^{\mathrm{i}(m-n)\sigma}\\ &+\left(\alpha_m^{2\alpha-1}-\tilde{\alpha}_m^{\dagger 2\alpha-1}\right)\left(\alpha_n^{\dagger 2\alpha}-\tilde{\alpha}_n^{2\alpha}\right)e^{-\mathrm{i}(m-n)\sigma}\\ &-\left(\alpha_m^{\dagger 2\alpha-1}-\tilde{\alpha}_m^{2\alpha-1}\right)\left(\alpha_n^{\dagger 2\alpha}-\tilde{\alpha}_n^{2\alpha}\right)\;e^{\mathrm{i}(m+n)\sigma}\;\Big]\Big\}\end{aligned} \tag{6.2.30}$$

As the combinations of the string oscillators in the previous expression commute with each other it is possible to compute explicitly the integral and find

$$e^{-\sum_{\alpha=1}^{d}\frac{p_\alpha}{2W_\alpha}\sum_{m>0}\frac{1}{m}\left[\left(\alpha_m^{2\alpha-1}-\tilde{\alpha}_m^{\dagger 2\alpha-1}\right)\left(\alpha_m^{\dagger 2\alpha}-\tilde{\alpha}_m^{2\alpha}\right)-\left(\alpha_m^{\dagger 2\alpha-1}-\tilde{\alpha}_m^{2\alpha-1}\right)\left(\alpha_m^{2\alpha}-\tilde{a}_m^{\dagger 2\alpha}\right)\right]} \tag{6.2.31}$$

By combining this contribution with the oscillator exponential in the unmagnetised D-Brane boundary state in Eq.(6.2.20) one obtains

$$e^{-\frac{1}{2}\sum_{m>0}\frac{1}{m}\left(\alpha_m-\tilde{\alpha}_m^\dagger\right)^M F_{MN}\left(\alpha_m^\dagger-\tilde{\alpha}_m\right)^N}e^{-\sum_{n>0}\frac{1}{n}\alpha_n^{\dagger M}(GR^{(0)})_{MN}\tilde{a}_n^{\dagger N}}|0\rangle \tag{6.2.32}$$

where we have made use of the form of the magnetic fields as in Eq.(3.2.33). In the following we will understand the oscillator indices and the sums over them, which will be reinstated in the final result. It turns out to be convenient to introduce an auxiliary oscillator A^M such as $\left[A^M, A^{\dagger N}\right]=G^{MN}$ and rewrite the expression above as

$${}_A\langle 0|e^{-\frac{1}{2}(\alpha-\tilde{\alpha}^\dagger)^M F_{MN}A^N}e^{(\alpha^\dagger-\tilde{\alpha})^M G_{MN}A^{\dagger N}}e^{-\alpha^{\dagger M}(GR^{(0)})_{MN}\tilde{a}^{\dagger N}}|0\rangle_A|0\rangle_{\alpha,\tilde{\alpha}} \tag{6.2.33}$$

Here the vacua subscripts indicate which oscillators annihilate them. The right-handed oscillator $\tilde{a}^M$ in the second exponential can annihilate its vacuum commuting with the operators in the last exponential to give

$$ {}_A\langle 0| e^{-\frac{1}{2}(\alpha-\tilde{\alpha}^\dagger)^M F_{MN} A^N} e^{[\alpha^{\dagger M} G_{MN} + \alpha^{\dagger M}(GR^{(0)})_{MN}]A^{\dagger N}} e^{-\alpha^{\dagger M}(GR^{(0)})_{MN}\tilde{a}^{\dagger N}} |0\rangle_{A,\alpha,\tilde{\alpha}} \tag{6.2.34} $$

Let us now introduce the so-called coherent states associated to the oscillators α^M and A^M

$$ |\lambda,\mu\rangle = e^{\lambda^M G_{MN}\alpha^{\dagger N} + \mu^M G_{MN} A^{\dagger N}} |0\rangle_{A,\alpha} \quad , \quad \langle\lambda,\mu| = {}_{A,\alpha}\langle 0| e^{\alpha^M G_{MN}\bar{\lambda}^N + A^M G_{MN}\bar{\mu}^N} \tag{6.2.35} $$

where λ and μ are complex numbers. These have the property

$$ \alpha^M|\lambda,\mu\rangle = \lambda^M|\lambda,\mu\rangle \quad \text{and} \quad A^M|\lambda,\mu\rangle = \mu^M|\lambda,\mu\rangle \tag{6.2.36} $$

It is possible to show that these states form a complete set as, using a matricial notation

$$ (\mathrm{Det}G)^2 \int d^2\lambda d^2\mu \; e^{-\frac{1}{2}(\bar{\lambda}G\lambda + \bar{\mu}G\mu)} |\lambda,\mu\rangle\langle\lambda,\mu| = 1 \tag{6.2.37} $$

Inserting this identity into Eq.(6.2.34) gives

$$ \begin{aligned} &(\mathrm{Det}G)^2 \int d^2\lambda d^2\mu \; e^{-\frac{1}{2}(\bar{\lambda}G\lambda + \bar{\mu}G\mu)} \; {}_A\langle 0| e^{-\frac{1}{2}(\lambda-\tilde{\alpha}^\dagger)F\mu} |\lambda,\mu\rangle \times \\ &\qquad \times \langle\lambda,\mu| e^{\bar{\lambda}G(1+R^{(0)})\bar{\mu} - \bar{\lambda}GR^{(0)}\tilde{\alpha}^\dagger} |0\rangle_{A,\alpha,\tilde{\alpha}} = \\ &(\mathrm{Det}G)^2 \int d^2\lambda d^2\mu e^{-\frac{1}{2}(\bar{\lambda}G\lambda + \bar{\mu}G\mu)} e^{-\frac{1}{2}(\lambda-\tilde{\alpha}^\dagger)F\mu} e^{\lambda G\alpha^\dagger} e^{\bar{\lambda}G(1+R^{(0)})\bar{\mu}} e^{-\bar{\lambda}GR^{(0)}\tilde{\alpha}^\dagger} |0\rangle_{\alpha,\tilde{\alpha}} \end{aligned} \tag{6.2.38} $$

We can now perform the resulting gaussian integral explicitly, considering λ, $\bar{\lambda}$, μ and $\bar{\mu}$ four real variables. Recalling that

$$ \int d\Lambda e^{-{}^t\Lambda H\Lambda + {}^t\Delta\Lambda} \sim \frac{1}{\sqrt{\mathrm{Det}H}} e^{\frac{1}{2}{}^t\Delta H^{-1}\Delta} \tag{6.2.39} $$

in the expression above we define the four-vector $\Lambda = \left\{\lambda, \bar{\lambda}, \mu, \bar{\mu}\right\}$ and identify

$$ \Delta = \begin{pmatrix} G\alpha^\dagger \\ -GR^{(0)}\tilde{\alpha}^\dagger \\ -\frac{1}{2}F\tilde{\alpha}^\dagger \\ 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} 0 & G & \frac{1}{2}F & 0 \\ G & 0 & 0 & -G(1+R^{(0)}) \\ -\frac{1}{2}F & 0 & 0 & G \\ 0 & -{}^t[G(1+R^{(0)})] & G & 0 \end{pmatrix} \tag{6.2.40} $$

hence it is possible to show (see Appendix A.2 for further details) that

$$ \frac{1}{2}\,{}^t\Delta H^{-1}\Delta = -\alpha^\dagger(GR)\tilde{\alpha}^\dagger \quad \text{and} \quad \frac{1}{\sqrt{\mathrm{Det}H}} = \frac{1}{(\mathrm{Det}G)^2}\frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})} \tag{6.2.41} $$

with

$$R = (G - \mathcal{F})^{-1}(G + \mathcal{F}) \tag{6.2.42}$$

Thus the final result of the gaussian integration, after reinstating the oscillator indices, reads

$$\prod_{n=1}^{\infty} \frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})} \prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_n^\dagger (GR)\tilde{\alpha}_n^\dagger} |0\rangle_{\alpha_n,\tilde{\alpha}_n} \tag{6.2.43}$$

The prefactor is obviously divergent, but can be rewritten as follows using the Riemann Zeta-function regularisation

$$\prod_{n=1}^{\infty} \frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})} = \left(\frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})}\right)^{\sum_{n=1}^{\infty} 1} = \left(\frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})}\right)^{\lim_{s\to 0}\sum_{n=1}^{\infty} n^{-s}}$$
$$= \left(\frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})}\right)^{\zeta(0)} = \left(\frac{\mathrm{Det}(G+B)}{\mathrm{Det}(G+\mathcal{F})}\right)^{-\frac{1}{2}} \tag{6.2.44}$$

This expression has to be combined with the determinant in the prefactor of the unmagnetised wrapped D-Brane boundary state in Eq.(6.2.20) and with the zero-modes dependence of the magnetised boundary state in Eq.(6.2.27) to determine the final form for the full magnetised wrapped D-Brane boundary state, namely

$$|B_{F,C}\rangle = \sqrt{\mathrm{Det}(G+\mathcal{F})} \;\; \sum_{m\in\mathbb{Z}^{2d}} \; \mathrm{e}^{-\mathrm{i}\pi\sum_{M<N}\hat{m}^M F_{MN}\hat{m}^N} \; \mathrm{e}^{2\pi\mathrm{i}C\hat{m}} \tag{6.2.45}$$
$$\times \;\; \left[\prod_{n=1}^{\infty} \mathrm{e}^{-\frac{1}{n}\alpha_n^\dagger GR\,\tilde{\alpha}_n^\dagger}\right] |-Fwm, wm\rangle$$

Observe that the effect of the non-zero modes contribution has been the replacement of B with $\mathcal{F} = B + F$ in the unmagnetised D-Brane boundary state as discussed in [89].

6.3 T-Duality and the boundary state for D-Branes at angles

Let us first analyse the effect of a T-Duality transformation, encoded in an $O(2d, 2d, \mathbb{Z})$ matrix of the type in Eq.(3.3.54) with integer $2d \times 2d$ blocks, on the Kaluza-Klein and winding numbers of a closed string. This follows from the duality transformation acting on the string oscillators as in the relations (3.3.71), combined with the definition of the closed string zero-modes in terms of Kaluza-Klein and winding numbers (6.1.5), namely

$$(G^{-1})[\hat{n} - (G+B)\hat{m}] = T_{-}(G')^{-1}[\hat{n}' - (G+B)'\hat{m}'] = (G^{-1})\,{}^{t}T_{-}^{-1}[\hat{n}' - (G+B)'\hat{m}'] \tag{6.3.46}$$

where in the last equation we have made use of the duality transformation acting on the background metric as in Eq.(3.3.68). Recalling also the transformation of the background fields in Eqs.(3.3.63) and the constraints on the $2d \times 2d$ entries of the duality matrix (3.3.56), one can simplify the r.h.s. of the previous relation, using that

$$ {}^tT_-^{-1}(G+B)' = [-(G+B)\,{}^tc + {}^td][a(G+B)+b][c(G+B)+d]^{-1} = [(G+B)\,{}^ta - {}^tb] \tag{6.3.47} $$

to find

$$ \hat{n} - (G+B)\hat{m} = [-(G+B)\,{}^tc + {}^td]\hat{n}' - [(G+B)\,{}^ta - {}^tb]\hat{m}' \tag{6.3.48} $$

which implies

$$ \hat{n} = {}^td\hat{n}' + {}^tb\hat{m}' \ , \quad \hat{m} = {}^tc\hat{n}' + {}^ta\hat{m}' \quad \Longleftrightarrow \quad \hat{n}' = a\hat{n} + b\hat{m} \ , \quad \hat{m}' = c\hat{n} + d\hat{m} \tag{6.3.49} $$

Notice that the inverse relations can be determined by using again the constraints (3.3.57).
A T-Duality performed on the wrapped magnetised D-Brane boundary state (6.2.45) generically yields a new magnetised D-Brane boundary state with

$$ F' = (aF - b)(-cF + d)^{-1} \quad \text{and} \quad R' = T_-^{-1} R T_+ \tag{6.3.50} $$

The transformation of the reflection matrix R has already been determined in the discussion of the duality on twisted open strings, in Eq.(4.1.12), while the novelty here is the transformation of the magnetic field F. This is a consequence of the duality relations in Eq.(6.3.49). In fact by acting with the Kaluza-Klein and winding number operators on a magnetised D-Brane boundary state one finds

$$ n' = (-aF + b)m \quad \text{and} \quad m' = (-cF + d)m \tag{6.3.51} $$

and the relation between n' and m' interpreted as in Eq.(6.2.23) leads to the form of the magnetic field upon duality transformation. It is not difficult to check that F' is an antisymmetric matrix as the combination ${}^tF' + F'$ can be shown to vanish using the constraints on the $2d \times 2d$ blocks of the T-Duality matrix.
If $(-cF + d)$ is not invertible, then the transformed D-Brane will have some directions with Dirichlet boundary conditions, where there is no winding number allowed. For instance, we can check that any magnetised brane can be easily related to a lower dimensional D-Brane at angle by means of T-Duality. In particular taking the magnetic field F in the form of Eq.(3.2.33), one can T-dualise the even direction of each of the T^2's defined by F in the T^{2d}, choosing

an $O(2d,2d,\mathbb{Z})$ matrix with entries as in Eq.(4.3.43). Then, for instance, the prefactor $\sqrt{\mathrm{Det}(G+\mathcal{F})}$ in Eq.(6.2.45) transforms into the one expected for the boundary state of a D-Brane at angle, once one reinstates the g_s dependence. In fact the rows of the $O(2d,2d,\mathbb{Z})$ duality matrix can be reordered in such a way that in a and d the only non vanishing entry will be the first $d\times d$ block, given by the identity matrix in the dualised directions of the torus which will be referred to as $A,B=1,\dots,d$, while in b and c the second $d\times d$ diagonal block will be the identity matrix in the other directions identified with $I,J=1,\dots,d$. By recalling the Eq.(3.3.63) it is possible to show [90] that $\mathrm{Det}(G+B+F)$ becomes

$$\mathrm{Det}\begin{bmatrix} \mathrm{E}_{AB}-\mathrm{E}_{AI}\mathrm{E}^{IJ}\mathrm{E}_{JB} & \mathrm{E}_{AK}\mathrm{E}^{KJ}+\tan(\pi\theta_{AJ}) \\ -\mathrm{E}^{IK}\mathrm{E}_{KB}-\tan(\pi\theta_{BI}) & \mathrm{E}^{IJ} \end{bmatrix} \tag{6.3.52}$$

where $\mathrm{E}=G+B$, upper indices indicate inverse matrices and $\tan(\pi\theta_{AI})$ is the tangent of the angle formed by the dualised D-Brane in the (A,I)-plane of the dual T^{2d}. The determinant above can be checked (see the Appendix A.2) to give $\mathrm{Det}(G'+B')\times\mathrm{Det}[(G+B)^{-1}_{IJ}]$ where the first factor contains, as expected, the pull-back of the metric and Kalb-Ramond field of the torus on the D-Brane at angle, while the second determinant is canceled against the duality transformation of the dilaton in g_s

$$e^{-\phi}\to\frac{e^{-\phi}}{\mathrm{Det}\left[(G+B)^{-1}_{IJ}\right]} \tag{6.3.53}$$

In the non-zero modes exponential in Eq.(6.2.45) the reflection matrix R simply becomes the one that encodes the identifications between left and right-handed oscillators induced by the presence of a D-Brane at angle, while the zero-modes dependence in $|-Fwm,wm\rangle$ becomes

$$|n'_{2\alpha-1},n'_{2\alpha},m'_{2\alpha-1},m'_{2\alpha}\rangle=|-p_\alpha m_{2\alpha},W_\alpha m_{2\alpha},W_\alpha m_{2\alpha-1},p_\alpha m_{2\alpha-1}\rangle \tag{6.3.54}$$

as a consequence of Eq.(6.3.49). This means that closed strings emitted by a D-Brane at angle are characterised by having winding numbers which are parallel to the tilted cycle wrapped by the D-Brane, given by $m_{2\alpha-1}(W_\alpha,p_\alpha)$, and momenta which are orthogonal to such cycle, $m_{2\alpha}(-p_\alpha,W_\alpha)\ \forall\alpha=1,\dots,d$. Also the same relation shows explicitly that the Wilson lines in Eq.(6.2.45) are related to the positions of the dualised D-Brane in the directions with Dirichlet boundary conditions, while they remain as residual Wilson lines along the cycle in the Neumann directions. Finally the phase (6.2.26) can be rewritten as

$$e^{-\mathrm{i}\pi\sum_{\alpha=1}^{d}\frac{p_\alpha}{W_\alpha}(W_\alpha^2)m_{2\alpha-1}m_{2\alpha}}=e^{-\mathrm{i}\pi\sum_{\alpha=1}^{d}m'_{2\alpha}n'_{2\alpha}} \tag{6.3.55}$$

Apparently this term does not have any interpretation in the boundary state of a D-Brane at angle. As already mentioned, we will be able to show that indeed this phase has to be incorporated into the definition of the T-Duality rules as a cocycle factor. In fact in order to obtain a consistent picture, whilst considering the interaction of three closed strings emitted by three magnetised D-Brane boundary states and sewn by an interaction vertex which will be defined in the following chapter, we will see that the duality rule used so far that relates closed string states with non-vanishing Kaluza-Klein or winding numbers has to be generalised (see also [91]) to

$$|n, m\rangle \to e^{\mathrm{i}\pi\left[\sum_{M<N}\left((\hat{n}')_M (d^t c)^{MN} (\hat{n'})_N + (\hat{m}')^M (b^t a)_{MN} (\hat{m}')^N\right) + {}^t\hat{m}' b^t c \hat{n}'\right]} |n', m'\rangle \quad (6.3.56)$$

The cocycle introduced exactly resembles the phase found above which is then interpreted as the one that a boundary state should acquire in the duality that relates a D-Brane at angle (where the boundary state does not have any non-trivial phase) to a magnetised D-Brane.
Let us lastly mention that it is also possible to transform any magnetized D-Brane into a D-Brane with Dirichlet boundary conditions along all the coordinates of the torus. In this case, the matrices c and d defining the T-duality are related to the magnetic field $F = c^{-1}d$. When F is block-diagonal as in Eq.(3.2.33), one can choose $c = w$ given by Eq.(6.1.12) and easily build integer matrices a (antisymmetric) and b (symmetric) satisfying the constraints in Eqs.(3.3.56) and (3.3.57) using the fact that the wrappings W_α and the Chern numbers p_α are co-prime and thus the system of Diophantine equations $-awF + bw = 1$ admits infinite solutions (see Appendix A.1). Exactly as in the previous example, the phase of the magnetised boundary state cancels against the phase generated by the T-duality transformation Eq.(6.3.56). Thus one recovers the standard form of a Dirichlet boundary state, where the identification matrix is simply $R = -1$. Notice that when the D-Brane is transformed into a point in the compact space, then all the $2d$ components of the Wilson lines are geometrised into $2d$ positions of the dual D-Brane in the internal directions of the torus.

7. THE CLOSED STRINGS INTERACTION VERTEX AND GENERALISED T-DUALITY RULES

In what follows we will review the fundamental characteristics of the tree-level interaction vertex among N off-shell closed strings (Reggeon vertex) for the computation of tree-level closed strings amplitudes, as an alternative approach to the use of on-shell vertex operators. We will briefly recall how to build such an object in string theory and then focus in particular on its zero-modes dependence. As in the construction of the boundary state for a magnetised D-Brane wrapped on a T^{2d}, we will see that the compactness of the target manifold indeed affects only the zero-modes dependence of the vertex and makes the naive generalisation of the Reggeon vertex defined on an uncompact space, as found in the literature, effectively inconsistent unless some phase factors, also known as "cocycles" are appropriately introduced [65]. The origin of these phases, which will be related to the ones already determined for the magnetised D-Brane boundary state in Eq.(6.2.26), is well-known: in toroidal compactifications the logarithmic branch cut of the bosonic Green function can sometimes become visible and adding a suitable cocycle factor to the string vertices is necessary to compensate for this. The presence of these cocycles has some consequences also on the precise formulation of the T-Duality transformations. In fact these transformations should preserve both the spectrum and the interactions, including, in the latter case, the phases needed for ensuring the locality of the interactions. This does not happen with the naive version of the T-Duality rules usually written, and we will show that it is necessary to introduce some cocycle phases also in the T-Duality transformations to recover a consistent picture.

7.1 The tree-level closed string Reggeon vertex

The Reggeon vertex is a bra-like object that describes the emission of a virtual (off-shell) closed string from the world-sheet of a second closed string propagating in the target space, as in Fig.7.1. The main difference with the usual approach to string interactions using vertex operators is that the emitted string can be off-shell, while a vertex operator describes the same situation depicted in Fig.7.1

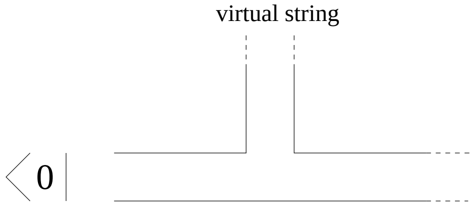


Fig. 7.1: The closed string Reggeon vertex

where the emitted string is pictorially pulled away to infinity to become a puncture in the world sheet of the propagating string. In this case the corresponding operator carries the on-shell quantum numbers of the emitted string and the oscillators of the propagating one. We then expect the Reggeon vertex to have two separate sets of oscillators related to the virtual, which will be denoted with a v superscript, and propagating strings. We start from the 3-Reggeon vertex in the Della Selva-Saito form (see [92, 93] for the original works on the subject and [94] for a more recent review)

$$\mathbf{V} = \langle 0| : e^{\oint_0 dz X^v(1-z)\partial X(z)} : \tag{7.1.1}$$

This operator has the property that, when it is saturated with a highest weight state $|s\rangle$ it yields the corresponding primary field $\mathcal{V}_s$ calculated at $z = 1$ in the space of the virtual string oscillators, i.e.

$$\mathbf{V}|s\rangle = \mathcal{V}_s^v(1) \tag{7.1.2}$$

which can be easily checked in some explicit examples and follows in general from the fact that

$$\partial X(z)\mathcal{V}_s(X(w)) = \frac{1}{z-w}\frac{\delta}{\delta X}\mathcal{V}_s(X(w)) \tag{7.1.3}$$

The vertex that encodes the interaction among $N = g+1$ closed strings should have the property that

$$\mathbf{V}_{g+1}\prod_{i=0}^{g}|c_i\rangle = \langle 0|\mathcal{V}_{c_0}(z_0)\dots\mathcal{V}_{c_g}(z_g)|0\rangle \tag{7.1.4}$$

where $\mathcal{V}_{c_i}$, $i = 0,\dots,g$, are closed string vertex operators (primary fields) corresponding to the $g+1$ highest weight states $|c_i\rangle$. It can be determined from the object in Eq.(7.1.1) by first transporting the argument of the $g+1$ primary fields from $z = 1$ to arbitrary points z_i through conformal transformations of the type

$$\hat{\gamma}_i(z) = V_i(1-z) \quad \text{such that} \quad V_i^{-1}(z_i) = 0 \tag{7.1.5}$$

and then by gluing $N = g+1$ vertices (7.1.1) of the same kind in the following fashion

$$\mathbf{V}_{g+1} = {}_v\langle 0| \prod_{i=0}^{g} \hat{\gamma}_i \mathbf{V}_i \hat{\gamma}_i^{-1} |0\rangle_v \tag{7.1.6}$$

The reason why we have rewritten $N = g+1$ will be shortly manifest when considering tree-level closed string amplitudes involving such vertex. We will not reproduce in detail the steps mentioned above, which can be found in the literature, but only state the final result, that reads [64]

$$\begin{aligned}\mathbf{V}_{g+1} \sim & \prod_{i=0}^{g} \left[\sum_{\hat{n}^i, \hat{m}^i} \langle \hat{n}^i, \hat{m}^i; 0| \right] \exp\left[- \sum_{j>i=0}^{g} \sum_{k,l=0}^{\infty} \hat{a}_k^i D_{kl} \left(\Gamma V_i^{-1} V_j \right) G \hat{a}_l^j \right] \\ & \times \exp\left[- \sum_{j>i=0}^{g} \sum_{k,l=0}^{\infty} \hat{\tilde{a}}_k^i D_{kl} \left(\Gamma \bar{V}_i^{-1} \bar{V}_j \right) G \hat{\tilde{a}}_l^j \right] \delta\left(\sum_{i=0}^{g} \hat{a}_0^i \right) \delta\left(\sum_{j=0}^{g} \hat{\tilde{a}}_0^j \right)\end{aligned} \tag{7.1.7}$$

A few comments are in order now.

In the operator formalism all the basic building blocks to compute tree-level amplitudes like string vertices and propagators are written in terms of the Schottky group (for some details see Appendix B). This group is generated by 2×2 matrices that act in the complex plane as follows

$$\gamma(z) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{pz+q}{rz+s} \,, \quad \text{with} \quad ps - qr = 1 \tag{7.1.8}$$

In the string vertices the bosonic oscillators $a_n = \alpha_n/\sqrt{n}$ or $\tilde{a}_n = \tilde{\alpha}_n/\sqrt{n}$ are naturally contracted with the following infinite dimensional representation of the Schottky group (here $n, m \neq 0$)

$$\begin{aligned} D_{mn}(\gamma) &= \frac{1}{m!}\sqrt{\frac{m}{n}} \partial_z^m \gamma^n \Big|_{z=0} \quad , \quad D_{00} = -\ln s \ , \\ D_{n0} &= \frac{1}{\sqrt{n}} \gamma^n(0) \quad , \quad D_{0m} = \frac{\sqrt{m}}{2m!} \partial_z^m \ln \gamma' \Big|_{z=0} \end{aligned} \tag{7.1.9}$$

Strictly speaking this is not a true representation when the zero-modes are included. In fact the product law is

$$D_{nm}(\gamma_1\gamma_2) = \sum_{l=1}^{\infty} D_{nl}(\gamma_1) D_{lm}(\gamma_2) + D_{n0}(\gamma_1)\delta_{m0} + D_{0m}(\gamma_2)\delta_{n0} \tag{7.1.10}$$

These matrices appear in the vertex with a dependence on elements of the Schottky group which correspond to the V_i's introduced earlier as local coordinates around each puncture placed at $V_i(0) = z_i$, while Γ is the inversion $\Gamma(z) = 1/z$. The right-moving coordinates are just the complex conjugate of the left-moving ones. Finally the δ's should be interpreted as Kronecker delta's imposing the separate conservations of Kaluza-Klein and winding numbers. As it is evident, in fact, from the bra in the vertex where $\hat{n}^i$ and $\hat{m}^i$ are respectively the operators that read the Kaluza-Klein and winding numbers of the i-th closed string, we have already considered the case of the vertex in toroidal compactifications. However in order to describe the interaction of closed strings propagating on a compact manifold the expression (7.1.7) is not consistent from the zero-modes dependence point of view. In the following section we will take care of this issue and introduce the appropriate phase factors into the vertex to recover a consistent operator.

7.2 Zero-modes, cocycle factors and T-Duality

Let us start from the example of the tree-level Reggeon vertex for the emission of three strings on a torus. Since there is no preferred ordering of three points on a sphere, the vertex must be invariant under the action of the permutation group exchanging any of the punctures. The usual emission vertex valid in the uncompact space does not have this property when it is naively generalised to the T^{2d} case as in Eq.(7.1.7). This is a well known issue, related to the compactness of the target space, and it is a consequence of the logarithmic branch cut of the bosonic Green function. Similar problems arise also in the usual formalism of the vertex operators describing the emission of particular on-shell string states (see, for instance, Sect. 8.2 in [20] and [91]). In order to eliminate this branch cut one has to add suitable cocycle factors to the usual expression of the vertex operators. Here we tackle this issue exactly in the same way by generalising these cocycle factors to the Reggeon vertex formalism.

In order to see that the usual 3-string vertex for closed strings is not invariant under the permutation of the external states when the target space is a torus, it is sufficient to focus on the zero-modes contribution

$$\exp\left[-\sum_{j>i=0}^{2}\alpha_0^i D_{00}\left(\Gamma V_i^{-1}V_j\right)G\alpha_0^j-\sum_{j>i=0}^{2}\tilde{\alpha}_0^i D_{00}\left(\Gamma\bar{V}_i^{-1}\bar{V}_j\right)G\tilde{\alpha}_0^j\right] \qquad (7.2.11)$$

where the upper index identifies one of the three external states, and all space-times indices have been suppressed. Recalling that $V_i(0) = z_i$ and $\Gamma(z) = 1/z$, one can write the $SL(2,\mathbb{C})$ matrices associated to the Schottky group elements in the D_{00} dependence as

$$V_i = \begin{pmatrix} 1 & z_i \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.2.12}$$

from which $D_{00} = -\ln s = -\ln(z_j - z_i)$. Thus the zero-modes dependence in the vertex (7.1.7) becomes

$$\exp\left[\sum_{j>i=0}^{2} \alpha_0^i \ln(z_j - z_i) G \alpha_0^j + \sum_{j>i=0}^{2} \tilde{\alpha}_0^i \ln(\bar{z}_j - \bar{z}_i) G \tilde{\alpha}_0^j\right] \tag{7.2.13}$$

where the z_i $(i = 0, 1, 2)$ are the positions of the three punctures on the sphere (which is represented as the compactified complex plane). The oscillator part of the 3-string vertex is invariant under the exchange of the strings $1 \leftrightarrow 2$, while the zero-mode contribution (7.2.13) gets a phase given by

$$\exp\left[-\mathrm{i}\pi \left(\alpha_0^1 G \alpha_0^2 - \tilde{\alpha}_0^1 G \tilde{\alpha}_0^2\right)\right] = \exp\left[-\mathrm{i}\pi\left(\hat{n}_1 \hat{m}_2 + \hat{m}_1 \hat{n}_2\right)\right] \tag{7.2.14}$$

where we have used Eq.(6.1.5). By exploiting this result, we can build a new invariant vertex with a cocycle factor that compensates for the phase (7.2.14). A possible choice for this cocycle factor is

$$\mathbf{V}_3^c = \mathbf{V}_3 \exp\left[\frac{\mathrm{i}\pi}{2}\left(\hat{n}_1 \hat{m}_2 - \hat{m}_1 \hat{n}_2\right)\right] \tag{7.2.15}$$

From the conservation of the Kaluza-Klein and winding numbers of the emitted strings follows that the vertex (7.2.15) is now easily shown to be invariant under the full permutation group acting on the three punctures.
It is not difficult to generalise the analysis above to the case of a Reggeon vertex describing the interaction of many closed strings. This can be obtained just by gluing together the 3-point vertices in Eq.(7.2.15) and the result is

$$\mathbf{V}_{g+1}^c = \mathbf{V}_{g+1} \exp\left[\frac{\pi \mathrm{i}}{2} \sum_{j>i=1}^{g} \left(\hat{n}_i \hat{m}_j - \hat{m}_i \hat{n}_j\right)\right] \tag{7.2.16}$$

The cocycle factor added, however, seems to break the vertex invariance under T-Duality transformations. For instance, there is a particular T-duality transformation that exchanges the Kaluza-Klein and winding operators. If its effect could be simply written as $\hat{n} \leftrightarrow \hat{m}$, as it is usually done, then we could have that

$\mathbf{V}^c_{g+1} \to -\mathbf{V}^c_{g+1}$ for certain external states. In order to be more precise recalling Eq.(6.3.49) the generic T-Duality transformation for the vertex (7.2.16) is

$$\mathbf{V}^c_{g+1} = \mathbf{V}^{c'}_{g+1} \exp\left\{ \mathrm{i}\pi \sum_{j>i=1}^{g} \left[{}^t\hat{n}'_i d\, {}^t c \hat{n}'_j + {}^t\hat{m}'_i b\, {}^t a \hat{m}'_j + {}^t\hat{n}'_i d\, {}^t a \hat{m}'_j + {}^t\hat{m}'_i b^t c \hat{n}'_j - {}^t\hat{n}'_i \hat{m}'_j \right] \right\} \tag{7.2.17}$$

where a, b, c and d are the $2d \times 2d$ integer entries of the T-Duality matrix written in the form of Eq.(3.3.54), which obey the constraints in Eq.(3.3.56). Thus the vertex (7.2.15) is fully symmetric under permutations of the external states, but is not invariant under T-Duality.

It is interesting to notice that the phase (which is actually just a sign) generated by the duality transformations can always be written as a product of $N = g+1$ terms each one depending only on a single external state. In order to show that this is indeed the case let us for instance consider in detail the first term in the exponential above.

$$\sum_{j>i=1}^{g} {}^t\hat{n}'_i d\, {}^t c \hat{n}'_j = \sum_{j>i=1}^{g} \left[\sum_{M<N} \hat{n}'_{iM} (d\, {}^t c)_{MN} \hat{n}'_{jN} - \sum_{M<N} \hat{n}'_{jM} (d\, {}^t c)_{MN} \hat{n}'_{iN} \right] \tag{7.2.18}$$

The conservation of the total Kaluza-Klein momentum in the expression on the r.h.s. of this expression yields

$$\begin{aligned}
&\sum_{M<N} \left[\sum_{i=1}^{g-1} \sum_{j=0}^{i} \hat{n}'_{iM} (d\, {}^t c)_{MN} \hat{n}'_{jN} + \sum_{i=2}^{g} \sum_{j=i}^{g} \hat{n}'_{iM} (d\, {}^t c)_{MN} \hat{n}'_{jN} - \sum_{j=2}^{g} \hat{n}'_{jM} (d\, {}^t c)_{MN} \hat{n}'_{0N} \right] \\
&= \sum_{M<N} \left[\sum_{i=2}^{g-1} \sum_{j=0}^{g} \hat{n}'_{iM} (d\, {}^t c)_{MN} \hat{n}'_{jN} + \sum_{i=2}^{g-1} \hat{n}'_{iM} (d\, {}^t c)_{MN} \hat{n}'_{iN} + \right. \\
&\left. + \sum_{j=0}^{1} \hat{n}'_{1M} (d\, {}^t c)_{MN} \hat{n}'_{jN} + \hat{n}'_{gM} (d\, {}^t c)_{MN} \hat{n}'_{gN} - \sum_{j=2}^{g} \hat{n}'_{jM} (d\, {}^t c)_{MN} \hat{n}'_{0N} \right] \\
&= \sum_{M<N} \sum_{i=0}^{g} \hat{n}'_{iM} (d\, {}^t c)_{MN} \hat{n}'_{iN}
\end{aligned} \tag{7.2.19}$$

Observe that it is possible to freely change the sign of each term as $e^{\mathrm{i}\pi k} = e^{-\mathrm{i}\pi k}$, $\forall k \in \mathbb{Z}$. With similar manipulations on the remaining contributions in the phase of the transformed vertex one can determine the following final expression

$$\mathbf{V}_{g+1} = \mathbf{V}^{c'}_{g+1} \prod_{j=0}^{g} \left\{ e^{\mathrm{i}\pi \left[\sum_{M<N} \left((\hat{n}'_j)_M (d\, {}^t c)^{MN} (\hat{n}'_j)_N + (\hat{m}'_j)^M (b\, {}^t a)_{MN} (\hat{m}'_j)^N \right) + {}^t\hat{m}'_j b\, {}^t c \hat{n}'_j \right]} \right\} \tag{7.2.20}$$

This means that the invariance of the vertex (7.2.16) under T-Duality can be restored, provided that we introduce the appropriate cocycle also in the T-duality transformations, as a generalisation of the standard rules discussed in section 3.3 (see also [91]). In fact it is sufficient to postulate that the closed string states transform according to Eq.(6.3.49) *and* also acquire the same phase in the curly brackets of (7.2.20)

$$|n, m\rangle \to \mathrm{e}^{\mathrm{i}\pi\left[\sum_{M<N}\left((\hat{n}')_M (d^t c)^{MN} (\hat{n})_N + (\hat{m}')^M (b^t a)_{MN} (\hat{m}')^N\right) + {}^t\hat{m}' b^t c \hat{n}'\right]} \, |n', m'\rangle \qquad (7.2.21)$$

This is in perfect agreement with what has already been found in the discussion of a magnetised D-Brane boundary state in Eq.(6.3.56). In fact boundary states for purely geometrical configurations of D-Branes, like D-Branes intersecting at angles, do not contain any non trivial phase depending on the emitted closed strings zero-modes. Performing a T-Duality that relates such configurations to wrapped magnetised D-Branes, the boundary state acquires a phase given by Eq.(6.2.26) that resembles the one determined in the relation above.

8. TWISTED PARTITION FUNCTION ON T^{2D} IN THE CLOSED STRING CHANNEL

We will consider the main features of the computation of the tree-level closed string amplitude involving $g+1$ closed strings emitted by wrapped magnetised D-Branes (6.2.45) and interacting through the vertex (7.2.16) determined in the previous chapters [64] (similar calculations were also performed with unmagnetised D-Branes in [95]). The resulting string world-sheet in the amplitude is a Riemann surface with $g+1$ borders (coinciding with the D-Branes boundary states) and no handles or crosscaps. Now the nomenclature used so far becomes clear if it is recalled that thanks to the conformal properties of string theory the same diagram can be reinterpreted from the open string point of view as the g-loop partition function involving twisted strings, i.e. open strings attached to magnetised D-Branes with different magnetic fields turned on in their world-volumes. In the open string channel, indeed, one of the D-branes, whose identification matrix is indicated with R_0, is singled out as the external border of the diagram. Thus it is natural to introduce, for the open strings stretched between such border and the remainders, the monodromy matrices (4.1.13) $\mathcal{S}_\mu \equiv R_0^{-1} R_\mu$, with $\mu = 1, \ldots, g$, whose eigenvalues are $e^{\pm 2\pi \mathrm{i} \theta_\mu^\alpha}$, $(\alpha = 1, 2, .., d)$. The only assumption we will make on the monodromy matrices $\mathcal{S}_\mu$ is that they commute with each other, namely that

$$[\mathcal{S}_\mu, \mathcal{S}_\nu] = 0 \tag{8.0.1}$$

Notice that this does not imply that the identification matrices R_i with $i = 0, \ldots, g$ also commute with each other. Of course the converse holds and Eq.(8.0.1) is implied by the requirement that $[R_i, R_j] = 0$. However, while Eq.(8.0.1) is invariant under the T-Duality, the constraint among the identification matrices R is not, as they do not transform by a similarity transformation (see Eq.(4.1.12)). By following the classification of [48] notice that configurations with *parallel* magnetic fluxes are characterised by commuting F_i's. The presence of the B-field in the definition of the identification matrices is such that the requirement $[R_i, R_j] = 0$ is already more general and allows for configurations with *oblique* fluxes. Here we will consider the even more general oblique class of configurations satisfying

Eq.(8.0.1). In this case, it is convenient to perform a T-Duality and transform the zeroth D-Brane into a purely Dirichlet D-Brane, i.e. with $R_0 = -1$. Like in the discussion of the T-Duality transformations for D-Branes boundary states, this can be achieved by means of a duality specified by an $O(2d, 2d, \mathbb{Z})$ matrix of the type in Eq.(3.3.54) with $c^{-1}d = F_0$. With this choice we have $\mathcal{S}_\mu = -R_\mu$ and the commutator above can be rewritten as

$$\left[G^{-1}\mathcal{F}_\mu, G^{-1}\mathcal{F}_\nu\right] = 0 \tag{8.0.2}$$

Moreover it implies:

$$\left[G^{-1}(F_{\hat{\mu}} - F_g), G^{-1}(F_{\hat{\nu}} - F_g)\right] = 0, \qquad \forall \hat{\mu}, \hat{\nu} = 1, \ldots, g-1 \tag{8.0.3}$$

as a consequence, for a generic G one can deduce that a fundamental cell of the lattice torus exists where all the field differences $(F_{\hat{\mu}} - F_g)$ are simultaneously block diagonal.
At any time, we can use again the T-duality rules discussed in the previous chapters and go back to the original system with all magnetized D-branes.

We will start from briefly reviewing the results found in [64] where the sewing of the vertex (7.2.16) with $g+1$ closed string propagators ending on magnetised D-Branes boundary states is performed without introducing the cocycle and phase factors either in the vertex (7.2.16) or in the boundary states (6.2.26). Also the Wilson line dependence in the boundary states is neglected there. We will use these results as a starting point to discuss in more details the subtleties that arise once we consider the introduction of such phase factors to recover, as we have discussed in the previous chapter, a fully consistent picture in a toroidal compactification [65] (i.e. ensuring both the invariance of the vertex under permutations of the emitted closed strings and the invariance of the whole amplitude under T-Duality transformations). The following two sections are devoted to the separate discussion of the zero and non-zero modes contributions to the amplitude.

8.1 Non-zero modes contribution

If we start from an off-shell closed string vertex in the z coordinates as in Eq.(7.1.7) describing the emission of $g+1$ states and insert a first boundary state, then we get a disk parametrised as the upper half complex plane, where the real axis represents the boundary we have just inserted. Each insertion of the remaining g boundary states cuts out a small disk in the upper half complex plane plus an image in the lower part and the circle and its image are identified. These non overlapping disks are completely specified by the 2×2 matrices (see

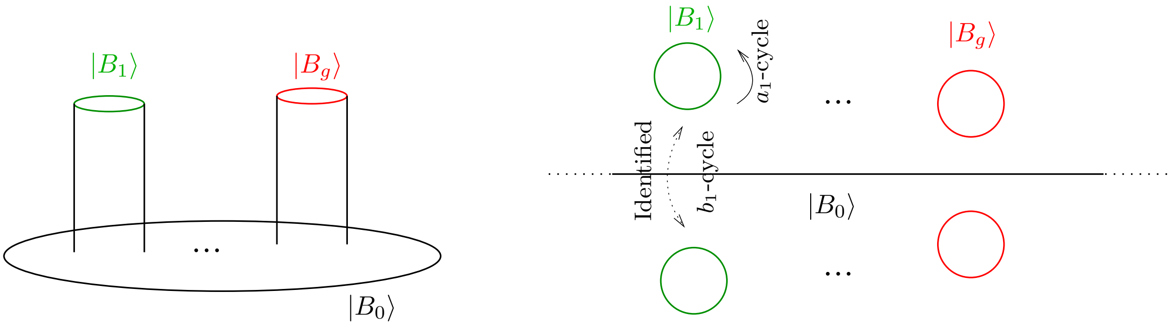


Fig. 8.1: 8.1a represents the partition function under study from a space-time point of view; 8.1b is a representation from the world-sheet point of view: the world-sheet is the upper half part of the complex plane that is outside all disks.

Appendix B)

$$S_\mu^{\rm cl} = \begin{pmatrix} a_\mu^{\rm cl} & b_\mu^{\rm cl} \\ c_\mu^{\rm cl} & d_\mu^{\rm cl} \end{pmatrix} \equiv \frac{1}{2\sqrt{q_\mu}(\chi_\mu - \bar{\chi}_\mu)} \begin{pmatrix} \chi_\mu - q_\mu \bar{\chi}_\mu & -|\chi_\mu|^2(1-q_\mu) \\ (1-q_\mu) & q_\mu \chi_\mu - \bar{\chi}_\mu \end{pmatrix} \quad (8.1.4)$$

with $\mu = 1, \ldots, g$. Observe that in the closed string channel the representation of the Schottky group is such that χ and $\bar{\chi}$ are complex conjugate to one another and have the same role of the real parameters η_μ and ξ_μ defined in the appendix. The multiplier q_μ is also a real number. Thus we get a world-sheet parametrisation as in Fig.8.1b, with $g+1$ borders which are the real axis plus the g circles specified above. The real multipliers q_μ's and the complex fixed points χ_μ's yield $3g$ real parameters. As usual three real parameters among the χ_μ's can be fixed arbitrarily thanks to the $SL(2,\mathbb{R})$ invariance, that in this context is simply the freedom to change all $S_\mu^{\rm cl}$ by a similarity transformation with a $SL(2,\mathbb{R})$ matrix. So we get the correct dimension $(3g-3)$ for the moduli space of a disk with $g+1$ boundaries. A generic element of the Schottky group for the disk with $g+1$ boundaries will be indicated by $T_\alpha^{\rm cl}$ and it has the same form of the $S_\mu^{\rm cl}$'s above, so for each $T_\alpha^{\rm cl}$ we can derive a real q_α and a complex χ_α which are calculable functions of the moduli q_μ and χ_μ. In the operator formalism the interaction vertices (7.1.7) are written in terms of a simple representation of the Schottky group. Sewing together vertices and boundary states amounts to multiply together representations of Schottky elements so it comes as no surprise that the result for the partition function is given in terms of sums and products over the Schottky group. The sewing procedure also carries a dependence on the space-times R_i matrices contained in the boundary states. Since in our case all the $\mathcal{S}_\mu$'s can be diagonalised simultaneously as a consequence of Eq.(8.0.1), the non-zero mode contribution to the partition function can be written in terms of the eigenvalues of $T_\alpha^{\rm cl}$ and those of $\mathcal{S}_\mu$. The first ones are basically the multipliers q_α, while the second ones are just $e^{\pm 2\pi {\rm i}\theta_\mu^a}$, for $\mu = 1, \ldots, g$. Notice that from the world-sheet point of view the eigenvalues of $\mathcal{S}_\mu^{\pm 1}$ are simply parameters that twist the periodicity conditions of the string coordinates along the $b^{\rm cl}$ cycles, that are those connecting each circle in Fig.8.1b with its own image. In formulae (using the operator formalism developed in [96, 97], see also [98]) the integrand for the twisted partition function can be written in terms of the g-loop untwisted result times d factors of $\left[\mathcal{R}^a(\vec{\theta}^a)\right]_g^{\rm cl}$ defined as follows

$$\left[\mathcal{R}^a(\vec{\theta}^a)\right]_g^{\rm cl} = \frac{\prod_\alpha' \prod_{n=1}^\infty (1-q_\alpha^n)^2}{\prod_\alpha' \prod_{n=1}^\infty \left(1 - e^{-2\pi {\rm i}\vec{\theta}^a\cdot\vec{N}_\alpha} q_\alpha^n\right)\left(1 - e^{2\pi {\rm i}\vec{\theta}^a\cdot\vec{N}_\alpha} q_\alpha^n\right)} \tag{8.1.5}$$

In the vector $\vec{\theta}^a$ we have collected the θ_μ^a, $\mu = 1, \ldots, g$; also $\vec{N}_\alpha$ is a vector with g integer entries: the μ-th entry counts how many times the Schottky generator $S_\mu^{\rm cl}$ enters in the element of the Schottky group $T_\alpha^{\rm cl}$, whose multiplier is q_α. The primed product over the Schottky group means product over primary classes, that is one has to take only the elements that cannot be written as power of another element of the group; moreover one has to take only one representative for each conjugacy class (class of elements that are related by cyclic permutations of their constituent factors). Thus the oscillator modes provide the following contribution to the twisted partition function

$$Z_g^{\rm cl}(F) = \left(\prod_{i=0}^{g} \sqrt{{\rm Det}(1 - G^{-1}\mathcal{F}_i)}\right) \int [dZ]_g^{\rm cl} \prod_{a=1}^{d} \left[\mathcal{R}^a(\vec{\theta}^a)\right]_g^{\rm cl} \tag{8.1.6}$$

where $[dZ]_g^{\rm cl}$ is the integrand of the usual ($F_i = 0$) partition function in Minkowski space. Notice that the overall coefficient is just the usual Born-Infeld Lagrangian rescaled by a factor of $\sqrt{G}$, since all the F_i-dependent normalisations are included in $[dZ]_g^{\rm cl}$. The equation above provides a direct generalisation of the 1-loop result [99] to the multiloop case. In fact in this case there is only one conjugacy class and the product $\prod_\alpha'$ is absent. Thus

$$[\mathcal{R}^a(\theta^a)]_{g=1}^{\rm cl} = -2\sin(\pi\theta^a)\frac{\eta^3(\tau^{\rm cl})}{\vartheta_{11}(\theta^a, \tau^{\rm cl})} \tag{8.1.7}$$

where η and ϑ_{11} are the usual Dedekind and odd Theta function with $q = e^{2\pi {\rm i}\tau^{\rm cl}}$

8.2 Zero-modes contribution

8.2.1 Result in absence of phases and cocycle factors

The zero-modes contribution in sewing the off-shell closed string vertex with the magnetised D-Brane boundary states is explicitly computed in [64], but still in absence of all the phases and cocycle factors we have discussed in the previous two chapters, appearing in particular in Eqs.(6.2.21),(6.2.26) and (7.2.16). The final result there is

$$\mathcal{A}^{(0)} = \sum \Delta \; \exp\left\{\pi \mathrm{i} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} \alpha_0^{\hat{\mu}} G \mathcal{S}_{\hat{\mu}}^{-1/2} \boldsymbol{D}_{\hat{\mu}\hat{\nu}} \mathcal{S}_{\hat{\nu}}^{1/2} \; \alpha_0^{\hat{\nu}} \right\} \tag{8.2.8}$$

where $\boldsymbol{D}_{\hat{\mu}\hat{\nu}}$ is a space-time matrix determined by the $\mathcal{S}_\mu$'s and the sum is over all the winding numbers that satisfy the Kronecker's deltas representing the identification Eq.(6.2.23) for each boundary state and the Kaluza-Klein and winding conservations (recall that closed strings emitted by the D-Brane with $R_0 = -1$ are characterized by unconstrained Kaluza-Klein momenta and no winding numbers):

$$\Delta = \left[\prod_{\mu=1}^{g} \delta\left(\hat{n}_\mu + F_\mu \hat{m}_\mu\right)\right] \delta\left(\sum_{i=0}^{g} \hat{n}_i\right) \delta\left(\sum_{\mu=1}^{g} \hat{m}_\mu\right) \tag{8.2.9}$$

The explicit form of the matrix $\boldsymbol{D}$, which depends on the moduli of the Riemann surface, can be found in [64]. We do not need the precise form of this ingredient, but only some properties that we will recall later in this section. Notice that the classical contribution Eq.(8.2.8) is nontrivial only for $g \geqslant 2$: for the annulus we have $\mathcal{A}^{(0)} = 1$ and the full partition function contains only the non-zero modes contribution as in the previous section [100]. By using Eqs.(6.1.5), (6.2.23) and (8.0.2), and the various conservation laws we can rewrite it as follows

$$\exp\left\{\frac{\pi \mathrm{i}}{2} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}} G \left[1 - \left(G^{-1}\mathcal{F}_{\hat{\mu}}\right)^2\right]^{\frac{1}{2}} \boldsymbol{D}_{\hat{\mu}\hat{\nu}}(\mathcal{S}) \left[1 - \left(G^{-1}\mathcal{F}_{\hat{\nu}}\right)^2\right]^{\frac{1}{2}} \hat{m}_{\hat{\nu}} \right\} \tag{8.2.10}$$

where we explicitly remind that $\boldsymbol{D}$ is a function of the space-time matrices $\mathcal{S}$.
It is convenient to rewrite Eq.(8.2.10) in a particular complex basis defined by a complex vielbein $\mathcal{E}$ as in Eq.(4.2.18). In fact in the presence of a constant antisymmetric two-form $\mathcal{F}$ there is a particular complex structure that plays a special role. In a real cartesian basis

$$\mathcal{F}_c = {}^tE^{-1} \mathcal{F} E^{-1} = E G^{-1} \mathcal{F} E^{-1} \tag{8.2.11}$$

having used Eq.(3.2.26), and it is related by a similarity transformation to the combination $G^{-1}\mathcal{F}$. The antisymmetric matrix $\mathcal{F}_c$ can be reduced to a block-diagonal form $\mathcal{F}_{\rm b.d.}$ by means of an orthogonal rotation O_f (where the subscript is just to recall that this transformation depends in general on $\mathcal{F}$)

$$\mathcal{F}_{\rm b.d.} = \begin{pmatrix} 0 & \mathfrak{f}_d \\ -\mathfrak{f}_d & 0 \end{pmatrix} = {}^tE_f^{-1}\mathcal{F}E_f^{-1} = E_f G^{-1}\mathcal{F}E_f^{-1} \tag{8.2.12}$$

where $\mathfrak{f}_d$ is a $d \times d$ diagonal matrix with real entries $\mathfrak{f}_{aa}$. The vielbein matrix $E_f = O_f E$ transforms at the same time the metric G into the identity and $\mathcal{F}$ into the block-diagonal matrix (8.2.12). Of course we can use the vielbein matrix E_f to introduce a particular set of complex coordinates, specified by $\mathcal{E}_f = SE_f$ with S as in Eq.(4.2.18), which diagonalises $G^{-1}\mathcal{F}$

$$\mathcal{F}^{(d)} = \mathcal{E}_f G^{-1}\mathcal{F}\mathcal{E}_f^{-1} = \begin{pmatrix} -\mathrm{i}\mathfrak{f}_d & 0 \\ 0 & \mathrm{i}\mathfrak{f}_d \end{pmatrix} = \mathcal{G}\, {}^t\mathcal{E}_f^{-1}\mathcal{F}\mathcal{E}_f^{-1} \tag{8.2.13}$$

where ${}^t\mathcal{E}_f^{-1}\mathcal{F}\mathcal{E}_f^{-1}$ is a block-diagonal matrix. From Eq.(8.2.13) it is easy to see that, in this complex basis, $\mathcal{F}$ is a $(1,1)$-form. In the following we will always use, as Cartesian basis, the one defined by the vielbeins E_f or $\mathcal{E}_f$, thus we will drop the subscripts without risk of ambiguities.
In this basis Eq.(8.2.10) reads

$$\exp\left\{\frac{\pi\mathrm{i}}{2}\sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}} \left[{}^t\mathcal{E}\mathcal{G}\sqrt{\left(1-(\mathcal{F}^{(d)}_{\hat{\mu}})^2\right)\left(1-(\mathcal{F}^{(d)}_{\hat{\nu}})^2\right)} \begin{pmatrix} \mathcal{D}_{\hat{\mu}\hat{\nu}}(\theta) & 0 \\ 0 & \mathcal{D}_{\hat{\mu}\hat{\nu}}(-\theta) \end{pmatrix} \mathcal{E}\right] \hat{m}_{\hat{\nu}}\right\} \tag{8.2.14}$$

where now each $\mathcal{D}_{\hat{\mu}\hat{\nu}}$ is $d \times d$ diagonal matrix that depends on the eigenvalues of the $\mathcal{S}_\mu$'s.
The square parenthesis in Eq.(8.2.14) is contracted with a symmetric combination of $\hat{m}$, so we can symmetrise it. Then, by using Eq.(4.2.19), one can easily check that Eq.(8.2.14) is equal to

$$\exp\left\{\frac{\pi\mathrm{i}}{2}\sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}} \left[{}^t\mathcal{E}\mathcal{G}\sqrt{\left(1-(\mathcal{F}^{(d)}_{\hat{\mu}})^2\right)\left(1-(\mathcal{F}^{(d)}_{\hat{\nu}})^2\right)} \begin{pmatrix} \hat{\tau}_{\hat{\mu}\hat{\nu}} & 0 \\ 0 & \hat{\tau}_{\hat{\nu}\hat{\mu}} \end{pmatrix} \mathcal{E}\right] \hat{m}_{\hat{\nu}}\right\} \tag{8.2.15}$$

where the $d \times d$ diagonal matrix $\hat{\tau}_{\hat{\mu}\hat{\nu}}$ is given by

$$\hat{\tau}_{\hat{\mu}\hat{\nu}} \equiv \frac{1}{2}\left[\mathcal{D}_{\hat{\mu}\hat{\nu}}(\theta) + \mathcal{D}_{\hat{\nu}\hat{\mu}}(-\theta)\right] \tag{8.2.16}$$

Notice that $\hat{\tau}$ has the same role as the τ of [101].
Expressing $\mathcal{E}$ in terms of the real vielbein E we can go to the real basis, where

Eq.(8.2.15) becomes

$$\exp\left\{\frac{\pi \mathrm{i}}{2}\sum_{\hat\mu,\hat\nu=1}^{g-1}{}^t\hat m_{\hat\mu}\left[{}^tE\sqrt{\left(1-(\mathcal{F}^{(d)}_{\hat\mu})^2\right)\left(1-(\mathcal{F}^{(d)}_{\hat\nu})^2\right)}\begin{pmatrix}\hat\tau^S & \mathrm{i}\hat\tau^A\\ -\mathrm{i}\hat\tau^A & \hat\tau^S\end{pmatrix}_{\hat\mu\hat\nu}E\right]\hat m_{\hat\nu}\right\} \tag{8.2.17}$$

where $\hat\tau^S$ and $\hat\tau^A$ are the symmetric and the antisymmetric part of $\hat\tau$, in the exchange of $\hat\mu$, $\hat\nu$. As $\hat\tau$ is purely imaginary and $\mathrm{Im}\,\hat\tau^S$ is positive definite because of the Riemann bilinear identities [101] and moreover $\left(\sqrt{1-(\mathcal{F}^{(d)}_{\hat\mu})^2}\right)_{ab} = \delta_{ab}\mid\left(1-\mathcal{F}^{(d)}_{\hat\mu}\right)_{aa}\mid$, the convergence of the series in Eq.(8.2.8) is assured.
Following [76] we can rewrite the Born-Infeld square roots above in yet another way by using another important consequence of the Riemann bilinear identities [101], namely

$$C_{\hat\mu\hat\nu}=C_{\hat\nu\hat\mu}\equiv\frac{1}{2}\left[\mathcal{D}_{\hat\mu\hat\nu}(\theta)-\mathcal{D}_{\hat\nu\hat\mu}(-\theta)\right]=\mathrm{i}\frac{\sin(\pi\theta_{\hat\mu})\sin(\pi\theta_{\hat\nu}-\pi\theta_g)}{\sin(\pi\theta_g)},\quad \hat\nu\geqslant\hat\mu \tag{8.2.18}$$

where also $C_{\hat\mu\hat\nu}$ and the sines are $d\times d$ diagonal matrices whose entries depend on the different values of θ. The $2d\times 2d$ matrix diag $\{\sin(\pi\theta_\mu),\sin(\pi\theta_\mu)\}$ can be written as[1]

$$\begin{pmatrix}\sin(\pi\theta_\mu) & 0\\ 0 & \sin(\pi\theta_\mu)\end{pmatrix}=\sqrt{\frac{1}{1-(\mathcal{F}^{(d)}_\mu)^2}} \tag{8.2.19}$$

This can be checked by rewriting the sine in terms of exponentials which are directly related to the components of $\mathcal{S}$ in the complex basis:
$\sin(\pi\theta^\alpha_\mu)=[\sqrt{2-\mathcal{S}^{-1}_\mu-\mathcal{S}_\mu}\,]_{\alpha\alpha}/2$, $\alpha=1,2,..,d$. Also, using the same procedure, we have

$$\begin{pmatrix}\sin(\pi\theta_{\hat\nu}-\pi\theta_g) & 0\\ 0 & \sin(\pi\theta_{\hat\nu}-\pi\theta_g)\end{pmatrix}=\frac{\begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}(\mathcal{F}^{(d)}_g-\mathcal{F}^{(d)}_{\hat\nu})}{\sqrt{\left(1-(\mathcal{F}^{(d)}_g)^2\right)\left(1-(\mathcal{F}^{(d)}_{\hat\nu})^2\right)}} \tag{8.2.20}$$

with $\mathcal{F}^{(d)}_\mu$ defined as in Eq.(8.2.13). From Eq.(8.2.19) and Eq.(8.2.20) one can see that

$$C_{\hat\mu\hat\nu}=C_{\hat\nu\hat\mu}=\frac{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\mathcal{F}^{(d)}_{\hat\nu}-\mathcal{F}^{(d)}_g\right)}{\sqrt{\left(1-(\mathcal{F}^{(d)}_{\hat\mu})^2\right)\left(1-(\mathcal{F}^{(d)}_{\hat\nu})^2\right)}}\quad,\quad\text{for}\quad\hat\nu\geqslant\hat\mu \tag{8.2.21}$$

[1] We will not keep track of the sign choices for the square roots: they clearly depend on whether each θ_μ is negative or positive.

Thus we can eliminate the square roots in Eq.(8.2.15) in favor of C. Then it is convenient to decompose $\hat{\tau}$ into its symmetric ($\hat{\tau}^S$) and the antisymmetric ($\hat{\tau}^A$) parts and use Eq.(8.2.13) to rewrite the diagonal fields $\mathcal{F}^{(d)}$ in terms of the $\mathcal{F}$'s. Taking also advantage of the identity $\mathcal{F}_\mu - \mathcal{F}_\nu = F_\mu - F_\nu$, we can write the final expression for Eq.(8.2.8)[2]

$$\mathcal{A}^{(0)} = \sum \Delta \exp\left\{ \frac{\pi \mathrm{i}}{2} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}} \, {}^t\mathcal{E} \left[\begin{pmatrix} \frac{\hat{\tau}^S}{C} & 0 \\ 0 & -\frac{\hat{\tau}^S}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} + \begin{pmatrix} \frac{\hat{\tau}^A}{C} & 0 \\ 0 & \frac{\hat{\tau}^A}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} \right] {}^t\mathcal{E}^{-1} F_{\hat{\mu}\hat{\nu}} \hat{m}_{\hat{\nu}} \right\} \tag{8.2.22}$$

where

$$F_{\hat{\mu}\hat{\nu}} = F_{\hat{\nu}\hat{\mu}} = F_{\hat{\nu}} - F_g \quad , \quad \text{for } \hat{\nu} \geqslant \hat{\mu} \tag{8.2.23}$$

8.2.2 Inclusion of the phase and cocycle factors

In this section we complete Eq.(8.2.22) to include also the effects of the cocycle phases, multiple wrappings and open string moduli (Wilson lines and/or D-brane positions). The classical contribution is the only part of the partition function that is affected by this generalisation, as it is clear from the form of the interaction vertex (7.2.16) and of the magnetised D-Branes boundary states (6.2.45). Basically we need to include in the sewing procedure the cocycle factor in (7.2.16) and the phases in (6.2.45). It is then clear that, in the expression for $\mathcal{A}$, we will have the same exponential of Eq.(8.2.22) multiplied by some additional factors related to cocycles. So let us consider these new contributions: by using Eqs.(7.2.16), (6.2.23) and (6.2.45), one can see that all the phases from the cocycles and the Wilson lines yield

$$\exp\left\{ 2\pi \mathrm{i} \left[\sum_{\mu=1}^{g} C_\mu \hat{m}_\mu + Y_0 \hat{n}_0 \right] \right\} \times \exp\left\{ \pi \mathrm{i} \sum_{\mu=1}^{g} \sum_{M<N} \hat{m}_\mu^M (F_\mu)_{MN} \, \hat{m}_\mu^N \right\}$$
$$\times \ \exp\left\{ \frac{\pi \mathrm{i}}{2} \sum_{\nu>\mu=1}^{g} (\hat{m}_\mu F_\mu \hat{m}_\nu + \hat{m}_\mu F_\nu \hat{m}_\nu) \right\} \tag{8.2.24}$$

Recall that the zeroth D-Brane has been chosen to have Dirichlet boundary conditions along all of the directions of the torus, hence Y_0 encodes its position in the torus directions. By using Eq.(8.2.9), we can eliminate $\hat{m}_g$ from these sums. Then it is easy to see that we can rewrite the dependence on the open string

[2] The inverse of C must be meant only with respect to the Lorentz indices, at fixed $\hat{\mu}$ and $\hat{\nu}$.

moduli as follows:

$$\exp\left\{2\pi\mathrm{i}\sum_{\hat\mu=1}^{g-1}\left[C_{\hat\mu}-C_g+Y_0(F_{\hat\mu}-F_g)\right]\hat m_{\hat\mu}\right\}\equiv e^{2\pi\mathrm{i}\sum\limits_{\hat\mu=1}^{g-1}{}^t\rho_{\hat\mu}\hat m_{\hat\mu}} \tag{8.2.25}$$

The second exponential in (8.2.24) comes from the boundary states and we can rewrite it as follows

$$\sum_{\mu=1}^{g}\sum_{M<N}\hat m_\mu^M(F_\mu)_{MN}\hat m_\mu^N=\sum_{\hat\mu=1}^{g-1}\sum_{M<N}\hat m_{\hat\mu}^M(F_{\hat\mu})_{MN}\hat m_{\hat\mu}^N-\sum_{\hat\mu,\hat\nu=1}^{g-1}\sum_{M<N}\hat m_{\hat\mu}^M(F_g)_{MN}\hat m_{\hat\nu}^N \tag{8.2.26}$$

We can combine this contribution with the last exponent in Eq.(8.2.24) and use (8.2.9). Here it is convenient to introduce the notation in which $F^{(u)}$ is the upper triangular matrix whose non-zero entries coincide with the ones of F, and $F^{(s)}$ is the symmetric matrix such that $2F^{(u)}=F+F^{(s)}$. Thus we have

$$\begin{aligned}
&-\sum_{\hat\mu}\hat m_{\hat\mu}F_{\hat\mu}^{(u)}\hat m_{\hat\mu}+\sum_{\hat\mu,\hat\nu}\hat m_{\hat\mu}F_g^{(u)}\hat m_{\hat\nu}+\frac12\sum_{\hat\nu>\hat\mu}\hat m_{\hat\mu}F_{\hat\mu}\hat m_{\hat\nu}-\frac12\sum_{\hat\mu,\hat\nu}\hat m_{\hat\mu}F_{\hat\mu}\hat m_{\hat\nu}+\\
&\frac12\sum_{\hat\nu>\hat\mu}\hat m_{\hat\mu}F_{\hat\nu}\hat m_{\hat\nu}=-\frac12\sum_{\hat\mu}\hat m_{\hat\mu}F_{\hat\mu}^{(s)}\hat m_{\hat\mu}+\frac12\sum_{\hat\mu,\hat\nu}\hat m_{\hat\mu}F_g^{(s)}\hat m_{\hat\nu}-\sum_{\hat\mu>\hat\nu}\hat m_{\hat\mu}F_{\hat\mu}^{(s)}\hat m_{\hat\nu}\\
=&\ -\frac12\sum_{\hat\mu}\hat m_{\hat\mu}F_{\hat\mu}^{(s)}\hat m_{\hat\mu}+\frac12\sum_{\hat\mu,\hat\nu}\hat m_{\hat\mu}F_g^{(s)}\hat m_{\hat\nu}-\frac12\sum_{\hat\mu>\hat\nu}\hat m_{\hat\mu}F_{\hat\mu}^{(s)}\hat m_{\hat\nu}-\frac12\sum_{\hat\nu>\hat\mu}\hat m_{\hat\mu}F_{\hat\nu}^{(s)}\hat m_{\hat\nu}\\
=&\ -\frac12\sum_{\hat\mu,\hat\nu}\hat m_{\hat\mu}(F_{\hat\mu}^{(s)}-F_g^{(s)})\hat m_{\hat\nu}\quad\text{for}\quad\hat\mu\geqslant\hat\nu
\end{aligned} \tag{8.2.27}$$

where in the first line we have changed the sign of the combination $\hat m_\mu F_\mu^{(u)}\hat m_\mu$ as each of the terms of the understood sum over the Lorentz indices is integer (see Appendix A.3.1) and $e^{\mathrm{i}\pi n}=e^{-\mathrm{i}\pi n}$, $\forall n\in\mathbb{Z}$. Observe that the matrix in the last line is symmetric under the exchange of $\hat\mu$ and $\hat\nu$.
The final result for the F-dependent phase factors reads

$$\exp\left[-\pi\mathrm{i}\sum_{\hat\mu,\hat\nu=1}^{g-1}\sum_{M<N}\hat m_{\hat\mu}^M(F_{\hat\mu\hat\nu})_{MN}\hat m_\nu^N\right] \tag{8.2.28}$$

where

$$F_{\hat\mu\hat\nu}=F_{\hat\nu\hat\mu}=F_{\hat\nu}-F_g\quad,\quad\text{for }\hat\nu\geqslant\hat\mu. \tag{8.2.29}$$

Thus the total contribution from the various phase factors is just the product of Eq.(8.2.25) and Eq.(8.2.28). This expression has no dependence on the metric of the torus and, in particular, Eq.(8.2.28) provides just some relative signs between contributions related to different values of $\hat{m}$. Moreover it depends only on the differences $(F_{\hat{\nu}} - F_g)$; therefore, thanks to Eq.(8.0.3), we can always consider $F_{\hat{\mu}\hat{\nu}}$ block diagonal as in Eq.(3.2.33).

8.2.3 The classical contribution to the twisted partition function

Combining the expressions found in Eqs.(8.2.22), (8.2.25) and (8.2.28) we finally get the classical contribution to the twisted partition function describing wrapped D-branes on an generic T^{2d}

$$\mathcal{A} = \sum \Delta \exp\left\{\pi \mathrm{i} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} \sum_{M<N} {}^t\hat{m}_{\hat{\mu}}^M (F_{\hat{\mu}\hat{\nu}})_{MN} \hat{m}_{\hat{\nu}}^N \right\} e^{2\pi \mathrm{i} \sum_{\hat{\mu}=1}^{g-1} {}^t\rho_{\hat{\mu}} \hat{m}_{\hat{\mu}}} \tag{8.2.30}$$
$$\times \exp\left\{ \frac{\pi \mathrm{i}}{2} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}} {}^t\mathcal{E} \left[\begin{pmatrix} \frac{\hat{\tau}^S}{C} & 0 \\ 0 & -\frac{\hat{\tau}^S}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} + \begin{pmatrix} \frac{\hat{\tau}^A}{C} & 0 \\ 0 & \frac{\hat{\tau}^A}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} \right] {}^t\mathcal{E}^{-1} F_{\hat{\mu}\hat{\nu}} \hat{m}_{\hat{\nu}} \right\}$$

where $F_{\hat{\mu}\hat{\nu}}$ is the matrix defined in Eq.(8.2.23).
If we restrict ourselves to the case of a factorizable torus $T^{2d} = (T^2)^d$, then Eq.(8.2.30) agrees[3] with the results of [101]. In order to make contact with their setup it is first useful to perform a T-Duality in such a way that the singled-out boundary with identifications encoded in the R_0 matrix is transformed back to a magnetized D-Brane. From Eq.(6.3.50) we can compute the transformation of the difference between two gauge field strengths

$$F'_i - F'_j = (F_j\, {}^t c + {}^t d)^{-1} (F_j - F_i)(cF_i - d)^{-1} \tag{8.2.31}$$

In particular the duality we performed to put $R_0 = -1$ had $d = cF_0$. Then by combining Eq.(8.2.31) with the transformation of the winding numbers (6.3.49), the amplitude (8.2.30) reduces to the following product of terms, each one related to a single T^2:

$$\mathcal{A}^w_{(T^2)^d} = \sum \Delta \exp\left\{\pi \mathrm{i} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} \sum_{M<N} {}^t\hat{m}_{\hat{\mu}}^M (F_{\hat{\mu}\hat{\nu}})_{MN} \hat{m}_{\hat{\nu}}^N \right\} e^{2\pi \mathrm{i} \sum_{\hat{\mu}=1}^{g-1} {}^t\rho'_{\hat{\mu}} \hat{m}_{\hat{\mu}}} \tag{8.2.32}$$
$$\times \prod_{\alpha=1}^{d} \exp\left\{ \frac{\pi}{2} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}}^\alpha \left[\left(\frac{\tau_S(\epsilon^\alpha)}{C(\epsilon^\alpha)}\right)_{\hat{\mu}\hat{\nu}} \mathcal{I}^{(\alpha)} + \mathrm{i} \left(\frac{\tau_A(\epsilon^\alpha)}{C(\epsilon^\alpha)}\right)_{\hat{\mu}\hat{\nu}} \right] F^{(\alpha)}_{\hat{\mu}\hat{\nu}} \hat{m}_{\hat{\nu}}^\alpha \right\}$$

where now

$$F_{\hat{\mu}\hat{\nu}} = F_{\hat{\nu}\hat{\mu}} = (F_0 - F_{\hat{\mu}})(F_0 - F_g)^{-1}(F_{\hat{\nu}} - F_g) \quad , \quad \text{for } \hat{\nu} \geqslant \hat{\mu} \tag{8.2.33}$$

[3] Apart from some factors of two.

and $\mathcal{I}^{(\alpha)}$ is the complex structure of each of the T^2's defined from Eq.(4.2.17) as

$$\mathcal{I}_M{}^N = \left({}^t\mathcal{E}\right)_M{}^a \mathcal{I}_a{}^b \left({}^t\mathcal{E}^{-1}\right)_b{}^N \tag{8.2.34}$$

Observe that the Wilson lines contribution has also been transformed under the duality resulting in

$${}^t\rho'_{\hat{\mu}}\hat{m}_{\hat{\mu}} = \left[C'_{\hat{\mu}} - C'_g(F_0 - F_g)^{-1}(F_0 - F_{\hat{\mu}}) + C_0(F_0 - F_g)^{-1}(F_{\hat{\mu}} - F_g)\right]\hat{m}_{\hat{\mu}} \tag{8.2.35}$$

having used the transformations

$$C_\mu = C'_\mu(F_0 - F_\mu)^{-1}c^{-1} \quad \text{and} \quad Y_0 = C_0{}^t c \tag{8.2.36}$$

together with Eq.(8.2.31).
The final result above can be compared with Eq.(A.28) of [101]. The norm of the vector v_i appearing there is related to the Born-Infeld square roots: $|v_i U|/\sqrt{U_2 T_2} = w_{\hat{\mu}}\sqrt{\left(1 - (\mathcal{F}^{(d)}_{\hat{\mu}})^2\right)}$. One can use this in (8.2.21) and (9.2.11) together with the explicit expression for the T^2 complex structure (9.2.10) to check that the two results are related by a further T-duality that exchanges $T \leftrightarrow -1/U$.

9. THE TWISTED PARTITION FUNCTION IN THE OPEN STRING CHANNEL AND YUKAWA COUPLINGS

In this chapter the phenomenological relevance of the twisted partition function previously analysed will be investigated. First of all following [64] we will rewrite the twisted partition function, obtained by combining the non-zero modes contribution in Eq.(8.1.6) with the classical contribution in Eq.(8.2.30), in the open string channel, describing a g-loop twisted open strings partition function. This can be achieved performing a modular transformation on the world-sheet moduli represented in the previous chapter as the eigenvalues of the generators of the Schottky group for the description of a disk with $g+1$ boundaries, namely the q_μ's. Then we will specify to the $g=2$ case and study its degeneration limit [65] in which all of the open string propagators become long and thin, hence only the low-lying states of the twisted open string spectrum will circulate in the loops. Using the unitarity of the theory it is not difficult to see that the resulting 2-loop diagram in the open string channel can be factorised into the product of three propagators connecting two vertices that couple three twisted open strings. The couplings emerging from the factorisation procedure can then be interpreted as the Yukawa couplings for the states of the low-energy spectrum of twisted open strings in the context of the magnetised brane worlds. In this computation we will mainly consider the classical contribution to the twisted partition function as this provides in the degeneration limit the classical contribution to the correlator of three twist fields. This is actually the limit we are most interested in as it is the classical contribution to the correlator that enters the sought effective action which reproduces the physics of the brane world models in the low energy approximation. The quantum part of the couplings is explicitly computed by means of the factorisation of the non-zero modes contribution of the twisted partition function in the open string channel in [64] and here it will be reinstated at the end of the main calculation. This contribution represents in fact a well-known result in the literature since [59], where it was determined in the context of closed strings orbifolds. Finally observe that we will only discuss the bosonic open string partition function. Indeed, as it has already been noticed in the analysis of the spectrum of twisted open strings in the brane worlds, the major difficulties in

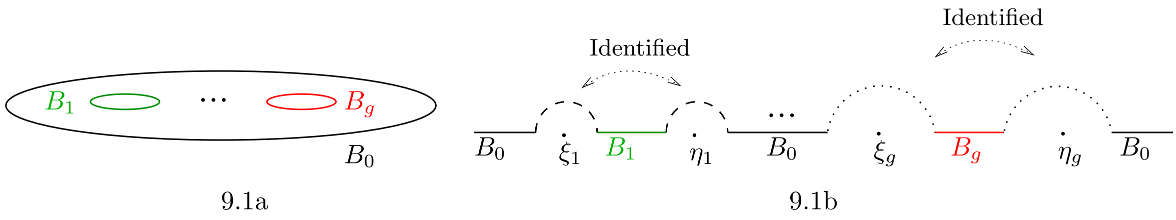


Fig. 9.1: 9.1a is a space-time representation of the same partition function of Fig.8.1 in the open string channel; 9.1b is the corresponding world-sheet surface in the Schottky parametrisation.

computing correlation functions involving twist fields arise in the bosonic sector, in which the conformal operators (4.2.30) do not have any description in terms of free fields. This is the reason why we have mainly focused on the study of the bosonic contribution of the boundary states and vertices sawn in the previous chapter. The fermionic contribution to the twist field correlation function can be separately computed via bosonisation and gives only some selection rules for non-vanishing amplitudes.

9.1 The twisted partition function in the open string channel

Following the results found in [64], the aim of this section is to briefly discuss the world-sheet modular transformation which allows to rewrite the result in Eq.(8.1.6) in the open string channel. Pictorially this map transforms the world-sheet of Fig.8.1 into the one of Fig.9.1. Again the Schottky parametrisation of the world-sheet in Fig.9.1b is completely specified in terms of g 2×2 matrices[1]

$$S_\mu = \begin{pmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{pmatrix} \equiv \frac{1}{\sqrt{k_\mu}(\eta_\mu - \xi_\mu)} \begin{pmatrix} \eta_\mu - k_\mu \xi_\mu & -\eta_\mu \xi_\mu (1 - k_\mu) \\ (1 - k_\mu) & k_\mu \eta_\mu - \xi_\mu \end{pmatrix} \qquad (9.1.1)$$

where the fixed points η, ξ are all real (see Appendix B). The open string world-sheet is the upper half part of the complex plane that is outside all the circles defined by the S_μ's, which are identified pairwise (see Fig.9.1b).
As usual, the modular transformation on the world-sheet in is non-analytic in the Schottky parameters. This is already manifest in the one loop case where one has $\ln q = 4\pi^2 / \ln k$. In order to circumvent this technical problem in [64] the authors first rewrite the products over the Schottky group in terms of genus g Theta functions and other geometrical objects. This can be done thanks to

[1] We have endowed the quantities in the closed string channel with a suffix "cl"; those without any suffix will refer to the open string channel, if not otherwise specified.

the identities derived in [96] which are consequence of the equivalence between fermionic and bosonic theories in two dimensions. In this description one can perform explicitly the modular transformation that exchanges the a and the b-cycles[2] as follows: $a^{\rm cl}_\mu = b_\mu$ and $b^{\rm cl}_\mu = a^{-1}_\mu$. After this map one has a world-sheet surface that looks like Fig.9.1a, then by using the identities discussed in [96] one rewrites the result in terms of the open string Schottky group generated by the S_μ's in Eq.(9.1.1). In the non-compact case, this computation was also performed in [97, 102] and the result for the partition function in the open string channel is

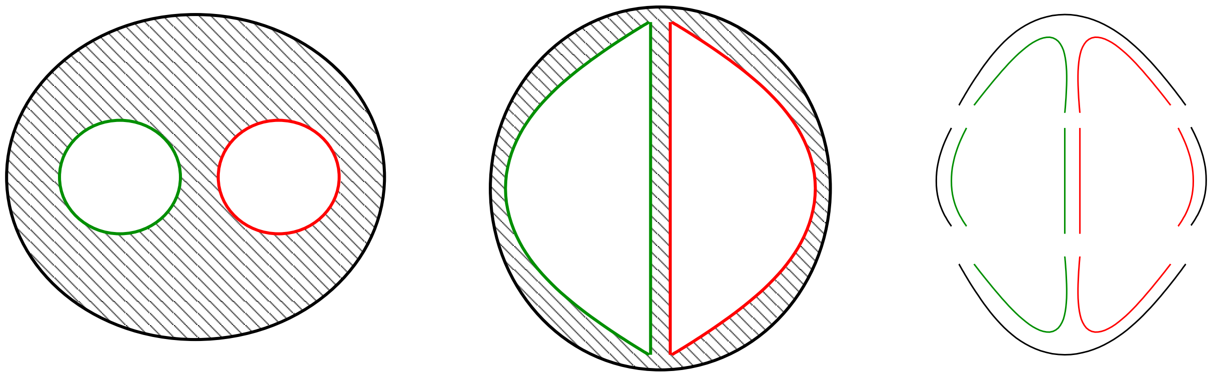
$$Z_g(F)|_{\rm unc.} = \left[\prod_{i=0}^{g} \sqrt{{\rm Det}\,(1-G^{-1}\mathcal{F}_i)}\right] \int [dZ]_g \prod_{a=1}^{d} \left[{\rm e}^{-{\rm i}\pi\vec{\theta^a}\cdot\tau\cdot\vec{\theta^a}} \, \frac{\det{(\tau)}}{\det{\left(T_{\vec{\theta}^a}\right)}} \, \mathcal{R}_g\left(\vec{\theta^a}\cdot\tau\right)\right] \tag{9.1.2}$$

Here $\tau_{\mu\nu}$ is the usual period matrix (written in the open string channel) and *det* is the determinant over the "loop" indices $\mu\,,\ \nu = 1,2,...,g$; $T_{\vec{\theta}^a}$ is a twisted generalisation of the period matrix, defined in [64]. We will not need the explicit form of such object in the most general case in what follows.

The zero-modes contribution instead in Eq.(8.2.30) has exactly the same form but the explicit world-sheet moduli dependence of the matrix $\hat{\tau}$ in Eq.(8.2.16) and of $\mathcal{D}_{\hat{\mu}\hat{\nu}}$ of course changes in the open and closed string channels. So far we have never investigated such dependence in detail. Nevertheless following [64] we will be able to write the explicit form of these matrices in the case we are most interested in, namely for $g = 2$. Thus collecting all of the contributions the final expression for the bosonic twisted partition function in the open string channel reads

$$\begin{aligned} Z_g(F) \;&=\; \left[\prod_{i=0}^{g} \sqrt{{\rm Det}\,(1-G^{-1}\mathcal{F}_i)}\right] \int [dZ]_g \prod_{a=1}^{d} \left[{\rm e}^{-{\rm i}\pi\vec{\theta^a}\cdot\tau\cdot\vec{\theta^a}} \, \frac{\det{(\tau)}}{\det{\left(T_{\vec{\theta}^a}\right)}} \, \mathcal{R}_g\left(\vec{\theta^a}\cdot\tau\right)\right] \\ &\times\; \exp\left\{ \frac{\pi{\rm i}}{2} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} {}^t\hat{m}_{\hat{\mu}}\, {}^t\mathcal{E} \left[\begin{pmatrix} \frac{\hat{\tau}^S}{C} & 0 \\ 0 & -\frac{\hat{\tau}^S}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} + \begin{pmatrix} \frac{\hat{\tau}^A}{C} & 0 \\ 0 & \frac{\hat{\tau}^A}{C} \end{pmatrix}_{\hat{\mu}\hat{\nu}} \right] {}^t\mathcal{E}^{-1} F_{\hat{\mu}\hat{\nu}} \hat{m}_{\hat{\nu}} \right\} \\ &\times\; \sum \Delta \exp\left\{ \pi{\rm i} \sum_{\hat{\mu},\hat{\nu}=1}^{g-1} \sum_{M<N} {}^t\hat{m}^M_{\hat{\mu}} (F_{\hat{\mu}\hat{\nu}})_{MN} \hat{m}^N_{\hat{\nu}} \right\} e^{2\pi{\rm i}\sum_{\hat{\mu}=1}^{g-1} {}^t\rho_{\hat{\mu}}\hat{m}_{\hat{\mu}}} \end{aligned} \tag{9.1.3}$$

[2] In the closed string parametrization, Fig.8.1b, we define the $b^{\rm cl}_\mu$-cycles to be the segments between w and $S^{\rm cl}_\mu(w)$, while the $a^{\rm cl}_\mu$-cycles are the contours around the repulsive fixed points $\bar{\chi}_\mu$ clockwise oriented. In a similar way, in the open string parametrisation, the b-cycles are the segments between w and $S_\mu(w)$, while the a-cycles are the contours around the ξ_μ's clockwise oriented.



9.2a 9.2b 9.2c

Fig. 9.2: 9.2a represents the twisted 2-loop partition function in the open string channel in a generic point of the world-sheet moduli space: on the three borders there are different magnetic fields F_i; 9.2b is the degeneration limit we are interested in, which is obtained by focusing on the corner of the world-sheet moduli in which the Schwinger times for the three propagators are taken to infinity; 9.2c is the factorization of 9.2b into two twisted 3-string vertices and three propagators: this is obtained by focusing only on the leading term in the expansion of the previous point .

9.2 Factorisation of the $g = 2$ case and Yukawa couplings

In this section we will focus on the vacuum diagram with three boundaries (i.e. $g = 2$) and study the degeneration limit where all three open string propagators in Fig.9.2 become long and thin. In this situation the partition function factorises in two tree-level 3-point correlators between twist fields. We will focus in particular on the classical contribution to the partition function essentially given by the last two lines of Eq.(9.1.3). As already mentioned, indeed, the factorisation of the non-zero modes dependence yields the quantum part of the twists fields correlation function, as shown in [64], which will be reinstated at the end of the present computation.

For $g = 2$ the only non-vanishing entry for $\hat{\tau}$ is clearly $\hat{\tau}^S_{11}$. In the degeneration limit under study we have [64] that $\mathcal{D}_{11}(\theta) \to 0$ from which

$$\left(\frac{\hat{\tau}^s(\theta)}{C(\theta)}\right)_{11} \to -1_{d\times d} \tag{9.2.4}$$

thus the exponential in the second line of Eq.(9.1.3) becomes

$$\exp\left\{\frac{\pi}{2}{}^t\hat{m}_1\mathcal{I}(F_2 - F_1)\hat{m}_1\right\} \tag{9.2.5}$$

where we have introduced the complex structure of the torus in the integral basis as in Eq.(8.2.34). In order to write the final form of the amplitude as a sum over unconstrained integers, it is necessary to solve the conservations (8.2.9), which, in the case under study, become

$$\hat{n}_1 = -F_1 \hat{m}_1 \;\; , \;\; \hat{n}_2 = -F_2 \hat{m}_2 \;\; , \;\; \hat{m}_1 + \hat{m}_2 = 0 \tag{9.2.6}$$

Of course the solutions must have integer Kaluza-Klein and windings numbers, so there must exist a minimal[3] integer invertible matrix H such that $F_1 H$ and $F_2 H$ are integer matrices and the solution can be written as $\hat{m}_1 = Hh$, with $h \in \mathbb{Z}^{2d}$. Then we define $\mathcal{I}' = {}^t\!H \mathcal{I}\, {}^t\!H^{-1}$, which is still a complex structure, and $F \equiv {}^t\!H(F_2 - F_1)H$; so the degeneration limit of the amplitude (9.1.3) in our $g = 2$ case is

$$\mathcal{A} = \sum_{h \in \mathbb{Z}^{2d}} \exp\left\{ \frac{\pi}{2} \left[{}^t\!h \mathcal{I}' F h + 2\mathrm{i} \sum_{M<N} h^M F_{MN} h^N \right] \right\} \times \exp\left\{ 2\pi \mathrm{i}\, {}^t\!\rho_1 H h \right\} \tag{9.2.7}$$

with a possible half-integer shift of ρ_1 (see Appendix A.3.1 for further details). By unitarity it must be possible to rewrite the result in Eq.(9.2.7) as a sum where each term is the product of two functions that are one the complex conjugate of the other. Each function represents the classical contribution to the coupling among three twist fields. The presence of the sum is due to the fact that the vacuum describing an open string stretched between two magnetised D-branes has a finite degeneracy in compact spaces, as discussed in the chapter devoted to the analysis of the spectrum of twisted open strings. So each string state has a number of replica and the various terms in the sum describe the couplings between these different copies of the twist fields. Observe again that these replica have the nice phenomenological interpretation of reproducing the families of the Standard Model in Brane Worlds, hence here the sum will be related to Yukawa couplings for different families.

9.2.1 The two-torus example

Let us start from the analysis of the simple case of a two-dimensional torus. For a generic tilted torus the metric can be written as a function of the Kähler and complex structure moduli

$$G = \frac{T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} = {}^t\mathcal{E} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{E} \tag{9.2.8}$$

[3] By minimal we mean that any other matrix with the same property is an integer multiple of H.

having defined the complex structure as $U = U_1 + \mathrm{i}U_2$ and the Kähler form as $T = T_1 + \mathrm{i}T_2$. Thus the complex vielbein defined in Eq.(8.2.13) reads

$$\mathcal{E} = \sqrt{\frac{T_2}{2U_2}} \begin{pmatrix} 1 & U \\ 1 & \bar{U} \end{pmatrix} \tag{9.2.9}$$

from which one can write the explicit form of the T^2 complex structure in the integral basis

$$\mathcal{I} = {}^t\mathcal{E} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} {}^t\mathcal{E}^{-1} = -\frac{1}{U_2} \begin{pmatrix} U_1 & -1 \\ |U|^2 & -U_1 \end{pmatrix} \tag{9.2.10}$$

The magnetic fields on the two magnetised D-Branes are identified by the Chern numbers p_i and the products of the wrappings along the two cycles of the torus W_i:

$$F_i = \begin{pmatrix} 0 & \frac{p_i}{W_i} \\ -\frac{p_i}{W_i} & 0 \end{pmatrix}, \qquad i = 1, 2 \tag{9.2.11}$$

One can easily check that the matrix H is simply proportional to the 2×2 identity, namely $H = W_1 W_2/\delta \otimes 1_{2\times 2}$, where $\delta = \text{G.C.D}\,\{W_1, W_2\}$. This implies that Eq.(9.2.7) can be put in the following form[4]

$$\sum_{h_1, h_2 \in \mathbb{Z}} \exp\left\{ -\frac{\pi}{2} \frac{I}{U_2} \left[h_1^2 + |U|^2 h_2^2 + 2U h_1 h_2 \right] + 2\pi\mathrm{i} \frac{1}{\delta} \mathcal{C}_M h^M \right\} \tag{9.2.12}$$

with

$$I = \frac{W_1^2 W_2^2}{\delta^2} \left(\frac{p_2}{W_2} - \frac{p_1}{W_1} \right) = I_{21} \frac{W_1 W_2}{\delta^2} \tag{9.2.13}$$

where we introduced the intersection numbers

$$I_{ij} = p_i W_j - p_j W_i \tag{9.2.14}$$

and

$$\mathcal{C}_M = W_1 W_2 \left[(F_1 - F_2) Y_0 + (C_1 - C_2) \right]_M \tag{9.2.15}$$

Recall that the twisted partition function in the closed string channel has been discussed in the case in which the zeroth D-Brane has Dirichlet boundary conditions along all of the directions of the compactification torus. In order to compare our results with the existing literature on the computation of Yukawa couplings in magnetised D-Brane world models, we perform a T-Duality in such a way that

[4] In the configuration of [64] one gets $I > 0$, otherwise one should write $|I|$, because of the note before Eq.(8.2.19).

also the zeroth D-Brane is magnetised. The intersection numbers are invariant under this operation being physical quantities that count the multiplicity of the twisted strings states, while the form of the matrix I is modified due to the transformation of the wrapping numbers. Indeed the T-Duality relating a Dirichlet to a magnetised brane is encoded in an $O(2,2,\mathbb{Z})$ matrix of the type in Eq.(3.3.54) with[5]

$$d = \begin{pmatrix} 0 & -p_0 \\ p_0 & 0 \end{pmatrix} \quad \text{and} \quad c = \tilde{w}_0 \equiv \tilde{W}_0 \otimes 1_{2\times 2} \tag{9.2.16}$$

in such a way that $F = c^{-1}d$ in Eq.(6.3.50). It is possible to show, combining the invariance of I_{21} with Eq.(8.2.31), that $W_1 \to I_{01}$ and $W_2 \to I_{20}$. Thus

$$I = \frac{I_{01}I_{21}I_{20}}{\delta^2} \tag{9.2.17}$$

with $\delta = \text{G.C.D.}\,\{I_{20}, I_{10}\} = \text{G.C.D.}\,\{I_{20}, I_{10}, I_{21}\}$, since we can make use of the property $I_{21}\tilde{W}_0 + I_{20}\tilde{W}_1 + I_{01}\tilde{W}_2 = 0$. Under the same duality the open string moduli transform as follows

$$C_\mu = \frac{\tilde{C}_\mu}{\tilde{w}_0(F_0 - F_\mu)} \quad \text{and} \quad Y_0 = \tilde{w}_0\tilde{C}_0 \tag{9.2.18}$$

hence

$$\mathcal{C}_M = \tilde{W}_0 I_{12}\tilde{C}_M^{(0)} + \tilde{W}_1 I_{20}\tilde{C}_M^{(1)} + \tilde{W}_2 I_{01}\tilde{C}_M^{(2)} \tag{9.2.19}$$

where the superscript (i) indicates the three boundaries and the subscript M is the Lorentz index.

The configurations studied [64] had δ and all $\tilde{W}_i$ equal to 1. In this case I is always an even number, since it is the product of three integers that sum to zero. Thus the contribution of the cocycles in Eq.(9.2.7) is irrelevant and Eq.(9.2.12) can be rewritten as it was done in [64]. We choose not to do that here, because it is easier to deal always with Eq.(9.2.12) without treating the case $\tilde{W}_i = 1$ separately.

In order to factorise the amplitude above and find the Yukawa couplings corresponding to the states of the open strings stretched between pairs of D-Branes with different magnetic fields on their world volume, it is necessary to first perform a Poisson resummation on the integer h_1

$$\sum_{h_1=-\infty}^{+\infty} \mathrm{e}^{-\pi A h_1^2 + 2\pi h^1 A s} = \frac{1}{\sqrt{A}}\mathrm{e}^{\pi A s^2} \sum_{h_1=-\infty}^{+\infty} \mathrm{e}^{-\pi\frac{h_1^2}{A} - 2\pi i h_1 s}\,, \qquad A > 0. \tag{9.2.20}$$

[5] We indicate with a tilde the quantities in the picture with a magnetised zeroth D-Brane.

This yields a new form of the same amplitude which is easy to factorise once we introduce a new pair of integers, r and k, through the relations

$$h_1 = rI + l = I\left(r + \frac{l}{I}\right) \quad \text{and} \quad h_2 = k - r \tag{9.2.21}$$

where $l = 1, \ldots, I$. It is manifest that in this way both the former and the latter pair of integers range in the whole $\mathbb{Z}$. Notice that this is ensured by summing over the additional integer l as well. Simple algebraic manipulations then lead to the product of two Jacobi Theta-functions as follows

$$\mathcal{A} = \sqrt{\frac{2U_2}{I}} \sum_{l=1}^{I} \vartheta \begin{bmatrix} \frac{l}{I} - \frac{1}{I}\frac{\mathcal{C}_1}{\delta} \\ \frac{\mathcal{C}_2}{\delta} \end{bmatrix} (0|IU) \times \vartheta \begin{bmatrix} \frac{l}{I} - \frac{1}{I}\frac{\mathcal{C}_1}{\delta} \\ -\frac{\mathcal{C}_2}{\delta} \end{bmatrix} (0|-I\bar{U}) \tag{9.2.22}$$

This result generalises the one of [64] and is in agreement[6] with the Section 3.1.3 of [61], as we find that, if the intersection numbers I_{ij} are not co-prime, one has $I_{20}I_{01}I_{21}/\delta^2$ non vanishing Yukawa couplings, labeled by the integer l. Indeed, in the dual picture involving intersecting D-Branes, every open string living in the intersection between two fixed D-Branes, say for instance i and j, will only couple to $|I_{jk}I_{ik}|/\delta^2$ strings from the intersections between the D-Brane k and the D-Branes i and j [61].

9.2.2 Twist fields couplings on a generic T^{2d} compactification

In order to factorise the classical contribution to the partition function (9.2.7) in the most general case and read the corresponding twist field couplings, we need to use the properties of the complex structure. Let us first decompose $\mathcal{I}'$ in terms of its $d \times d$ blocks

$$\mathcal{I}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{9.2.23}$$

Hence it is simple to check that $\mathcal{I}'^2 = -1$ yields

$$AB = -BD \quad , \quad \text{and} \quad A^2 = -(1 + BC) \tag{9.2.24}$$

Then we choose a basis for the torus lattice where the combination ${}^tH(F_2 - F_1)H$ takes the following form

$$F = \begin{pmatrix} 0 & \hat{F} \\ -\hat{F} & 0 \end{pmatrix} \tag{9.2.25}$$

[6] The apparent mismatch related to the presence of the wrapping numbers of the D-Branes in the Wilson lines dependence of the couplings is resolved by checking that $W_iC_1^{(i)}$, $W_iC_2^{(i)} \in [0, 1]$ as well as the parameters ϵ_i and θ_i defined in [61].

This can be done by putting the matrix F in the form of Eq.(3.2.33) (thanks again to the result of the Appendix A.3.1) and by a suitable relabeling of the rows and the columns. Notice that this relabeling does not affect the form of the amplitude to be factorised. For the sake of brevity, we also introduce the $2d$-components vector $\beta = {}^tH\rho_1$. Then the general amplitude to be factorised has the following form[7]

$$\sum_{h_i\in\mathbb{Z}^d} \exp\left\{\frac{\pi}{2}\left[\begin{pmatrix}{}^th_1 & {}^th_2\end{pmatrix}\begin{pmatrix} -B\hat{F} & (i+A)\hat{F} \\ (i-D)\hat{F} & C\hat{F}\end{pmatrix}\begin{pmatrix} h_1 \\ h_2\end{pmatrix}\right] + 2\pi\mathrm{i}\,{}^t\beta_1 h_1 + 2\pi\mathrm{i}\,{}^t\beta_2 h_2\right\} \tag{9.2.26}$$

As next step we need to perform a Poisson resummation

$$\sum_{h_1\in\mathbb{Z}^d} \mathrm{e}^{-\pi\,{}^th_1\mathcal{B}h_1 + 2\pi\,{}^th_1\mathcal{B}s} = \frac{1}{\sqrt{\mathrm{Det}\mathcal{B}}}\mathrm{e}^{\pi\,{}^ts\mathcal{B}s}\sum_{h_1\in\mathbb{Z}^d}\mathrm{e}^{-\pi\,{}^th_1\mathcal{B}^{-1}h_1 - 2\pi\mathrm{i}\,{}^th_1 s} \tag{9.2.27}$$

on the first d components of h which we indicate with h_1. So in our case we have

$$\mathcal{B} = \frac{1}{2}\hat{F}\,{}^tB \quad \text{and} \quad s = \frac{1}{2}\mathcal{B}^{-1}(A+i)\hat{F}h_2 + \mathrm{i}\mathcal{B}^{-1}\beta_1 \tag{9.2.28}$$

The first exponential in the r.h.s. of the Poisson resummation formula (9.2.27) yields a quadratic term in the vector h_2, which can be combined with a similar contribution present in the initial expression (9.2.26)

$$\begin{aligned} &\frac{\pi}{4}\,{}^th_2\hat{F}(i+{}^tA)\,{}^t\mathcal{B}^{-1}(i+A)\hat{F}h_2 + \frac{\pi}{2}\,{}^th_2 C\hat{F}h_2 = \\ &= \frac{\pi}{2}\,{}^th_2\left[\hat{F}(i+{}^tA)\hat{F}^{-1}B^{-1}(i+A)\hat{F} + C\hat{F}\right]h_2 \end{aligned} \tag{9.2.29}$$

Recalling that $\mathcal{I}'\hat{F}$ is a symmetric matrix, it is not difficult to see that $A\hat{F} = -\hat{F}\,{}^tD$. Some algebraic manipulations involving these identities simplify the previous expression into

$$\mathrm{i}\pi\,{}^th_2 B^{-1}(i+A)\hat{F}h_2. \tag{9.2.30}$$

Hence the Poisson resummation performed on Eq.(9.2.26) gives

$$\begin{aligned} \frac{1}{\sqrt{\mathrm{Det}(2B\hat{F})}} \sum_{h_i\in\mathbb{Z}^d} \exp\Big\{\mathrm{i}\pi\Big[&{}^th_2B^{-1}(i+A)\hat{F}h_2 + 2\mathrm{i}\,{}^t\beta_1\,{}^tB^{-1}\hat{F}^{-1}\beta_1 + \\ &+{}^th_2\hat{F}(i+{}^tA)\,{}^t\hat{F}^{-1}B^{-1}\beta_1 + {}^t\beta_1\hat{F}^{-1}B^{-1}(i+A)\hat{F}h_2 \\ &+2\,{}^t\beta_2h_2 + 2\mathrm{i}\,{}^th_1\,{}^tB^{-1}\hat{F}^{-1}h_1 - 2\,{}^th_2B^{-1}(i+A)h_1 - \\ &4\mathrm{i}\,{}^th_1\,{}^tB^{-1}\hat{F}^{-1}\beta_1\Big]\Big\} \end{aligned} \tag{9.2.31}$$

[7] In the following i indicates a $d\times d$ imaginary matrix: $i \equiv \mathrm{i}\,1_{d\times d}$.

In the following manipulations we will focus on the expression in the square brackets only. It is useful to observe that the by redefining

$$\gamma_1 = \hat{F}^{-1}\beta_1 \quad \text{and} \quad k = \hat{F}^{-1}h_1 \tag{9.2.32}$$

and making use again of the identities mentioned earlier, involving the entries of the complex structure and of $\hat{F}$, one can rewrite the content of the square brackets above as

$$\begin{aligned}&{}^t h_2 B^{-1}(i+A)\hat{F}h_2 + 2\mathrm{i}\,{}^t\gamma_1 B^{-1}\hat{F}\gamma_1 + {}^t h_2 B^{-1}(i+A)\hat{F}\gamma_1 + \\ &+ {}^t\gamma_1 B^{-1}(i+A)\hat{F}h_2 + 2\,{}^t\beta_2 h_2 + 2\mathrm{i}\,{}^t k B^{-1}\hat{F}k - \\ &-2\,{}^t h_2 B^{-1}(i+A)\hat{F}k - 4\mathrm{i}\,{}^t k B^{-1}\hat{F}\gamma_1\end{aligned} \tag{9.2.33}$$

As k in general is no longer a column of integers, it is convenient to write it distinguishing its integer part from the remainder

$$k = r + \hat{F}^{-1}l \tag{9.2.34}$$

with $r \in \mathbb{Z}^d$ and $l_\alpha \in [1, \hat{F}_{\alpha\alpha}]$. Thus the expression above reads

$$\begin{aligned}&2\,{}^t\!\left(r + \hat{F}^{-1}l - \gamma_1\right)\mathrm{i}B^{-1}\hat{F}\left(r + \hat{F}^{-1}l - \gamma_1\right) + {}^t h_2 B^{-1}(i+A)\hat{F}h_2 + \\ &-2\,{}^t h_2 B^{-1}(i+A)\hat{F}\left(r + \hat{F}^{-1}l - \gamma_1\right) + 2\,{}^t\beta_2 h_2\end{aligned} \tag{9.2.35}$$

Finally, defining $s = r - h_2 \in \mathbb{Z}^d$, one has

$$\begin{aligned}&{}^t\!\left(r + \hat{F}^{-1}l - \gamma_1\right)B^{-1}(i-A)\hat{F}\left(r + \hat{F}^{-1}l - \gamma_1\right) + 2\,{}^t\!\left(r + \hat{F}^{-1}l - \gamma_1\right)\beta_2 + \\ &{}^t\!\left(s + \hat{F}^{-1}l - \gamma_1\right)B^{-1}(i+A)\hat{F}\left(s + \hat{F}^{-1}l - \gamma_1\right) - 2\,{}^t\!\left(s + \hat{F}^{-1}l - \gamma_1\right)\beta_2\end{aligned}$$

Thus the factorised amplitude reads

$$\begin{aligned}\mathcal{A} = \sum_{l_\alpha=1}^{\hat{F}_{\alpha\alpha}} \frac{1}{\sqrt{\mathrm{Det}(2B\hat{F})}} \quad &\vartheta\begin{bmatrix}\hat{F}^{-1}(l-\beta_1)\\ \beta_2\end{bmatrix}\left(0\middle|B^{-1}(i-A)\hat{F}\right)\times \\ &\vartheta\begin{bmatrix}\hat{F}^{-1}(l-\beta_1)\\ -\beta_2\end{bmatrix}\left(0\middle|B^{-1}(i+A)\hat{F}\right)\end{aligned} \tag{9.2.36}$$

written in terms of d-dimensional Riemann Theta-functions

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(\nu|\tau) = \sum_{h\in\mathbb{Z}^d}\exp\left[\pi\mathrm{i}\,{}^t(h+a)\tau(h+a) + 2\pi\mathrm{i}\,{}^t(\nu+b)(h+a)\right] \tag{9.2.37}$$

Notice that the function in the second line of the Eq.(9.2.36) is indeed the complex conjugate of the one in the first line, as $\hat{F}$, A and B are real $d \times d$ matrices since $\mathcal{I}$ in Eq.(8.2.34) is real. This function is to be interpreted as the classical contribution to the Yukawa couplings for three twisted states arising in a generic T^{2d} compactification of string theory with magnetised space filling D-Branes. The sum in front of the couplings reveals their multiplicity, given by $\mathrm{Det}\hat{F} = \prod_{\alpha=1}^{d} \hat{F}_{\alpha\alpha}$.

Finally for the sake of completeness let us write the expression for the correlator between three twist fields (fixing one l, i.e. one particular coupling) including also its quantum contribution in the configuration $\theta_1^\alpha + \theta_2^\alpha + \theta_3^\alpha = 1$, which, in the dual language of intersecting branes, corresponds to the request that the three D-Branes form a triangle.

The complete coupling reads

$$
\begin{aligned}
\langle \sigma_{\theta_1} \sigma_{\theta_2} \sigma_{\theta_3} \rangle &= \prod_{i=1}^{3} \prod_{\alpha=1}^{d} \left[\frac{\Gamma(1-\theta_i^\alpha)}{\Gamma(\theta_i^\alpha)} \right]^{\frac{1}{4}} (\mathrm{Det}(2B))^{-\frac{1}{4}} \\
&\times \; \vartheta \begin{bmatrix} \hat{F}^{-1}(l-\beta_1) \\ \beta_2 \end{bmatrix} \left(0 \middle| B^{-1}(i-A)\hat{F} \right) \qquad (9.2.38)
\end{aligned}
$$

We can check that this result is in agreement with the literature considering in particular the multiplicity of the Yukawa couplings in the case of *parallel* fluxes, i.e. when all of the boundaries are magnetised D-Branes with magnetic fields of the type (3.2.33) put in the form (9.2.25). Recall that this setup can always be T-dualised into a configuration of intersecting D-Branes on a $2d$-dimensional torus that is not geometrically a direct product of d T^2's. However in the counting of the non vanishing 3-point correlators between twisted states the metric of the torus is not involved, and thus we expect that this multiplicity is equal to the one already calculated [61] in fully factorisable models of D-Branes at angles on $(T^2)^d$. Indeed following the steps of the previous subsection it is not difficult to see that here $H = w_1 w_2/\delta$, δ being a diagonal matrix whose eigenvalues are the G.C.D.'s of the entries of w_1 and w_2 as in Eq.(6.1.12). Then by generalizing also the definition (9.2.14) into a diagonal matrix, with the same structure and the intersection numbers as entries, one has $\hat{F} = I_{21} w_1 w_2/\delta^2$. Upon the straightforward generalisation of the T-Duality in Eq.(9.2.16) $\hat{F} \to I_{21} I_{20} I_{01}/\delta^2$ where δ now contains the G.C.D.'s of the entries of the three intersection numbers and

$$
\mathrm{Det}\hat{F} = \prod_{\alpha=1}^{d} \frac{(p_{2\alpha} W_{0\alpha} - p_{0\alpha} W_{2\alpha})(p_{0\alpha} W_{1\alpha} - p_{1\alpha} W_{0\alpha})(p_{2\alpha} W_{1\alpha} - p_{2\alpha} W_{2\alpha})}{\delta_{\alpha\alpha}^2} \qquad (9.2.39)
$$

This number agrees with the product of the multiplicity of Yukawa couplings in each of the T^2's inside the T^{2d} defined by the form of the magnetic fields (see [61]).

10. CONCLUSIONS

The results found in the previous chapter represent to the best of our knowledge the first attempt to give a complete description and to determine the most generic moduli dependence of the Yukawa couplings among three twisted states in the magnetised D-Branes models in toroidal compactifications. Indeed Eq.(9.2.38) contains both the closed and the open string moduli dependence of such couplings in the most general toroidal compactification, with the only assumption that the fluxes on the magnetised D-Branes are chosen to satisfy the commutation relation in Eq.(8.0.1). As already stressed, these configurations are slightly more general than the fully geometrisable ones involving parallel fluxes, characterised by commuting reflection matrices R_i on different branes, and realised for instance by choosing all of the background magnetic fields in the block-diagonal form (3.2.33). Nevertheless, even restricting to these more specific cases, the result in Eq.(9.2.38) still provides a generalisation of the moduli dependence of the Yukawa couplings found in the existing literature, since the background geometry of the compactification torus is not factorisable into the product of orthogonal two-dimensional tori. The consequence of this more general set-up resides in the fact that higher-dimensional Riemann Theta-functions arise as opposed to the product of Jacobi Theta-functions found for fully factorisable geometries. Hence the moduli dependence of the Yukawa couplings can be much richer than previously expected. In particular in the context of the magnetised branes models the only results available in the literature are computed in [67] in the low-energy field theory limit, neglecting stringy corrections and still lacking a complete string-theoretic derivation. Indeed in [67] the authors determine the shape of the wave-functions for charged fermions propagating on magnetised tori, as eigenfunctions of the internal Dirac operator with a magnetic field background. The overlap of three such functions yields the expression for the classical contribution to the Yukawa couplings of twisted states, which is computed in [67] for a two-dimensional torus or a factorised $(T^2)^d$. In configurations preserving $\mathcal{N} = 1$ supersymmetry the effective action that reproduces the tree-level amplitudes of the interactions of twisted and untwisted strings in the brane worlds should resemble an $\mathcal{N} = 1$ Super Yang-Mills action, completely characterised by three moduli dependent parameters [46]: the Kähler potential for the matter, the gauge kinetic function and the superpotential. The latter in particular has the property of being a holomorphic function of the background moduli. This implies that any non-holomorphic dependence on the moduli of the computed Yukawa couplings should derive from the normalisation of the kinetic term of the interacting twisted fields, namely from their Kähler metric, as in the following relation

$$Y_{ijk} = (K_{i\bar{i}} K_{j\bar{j}} K_{k\bar{k}})^{-\frac{1}{2}} e^{K/2} W_{ijk}$$

where Y_{ijk} is the Yukawa coupling between three twisted states i, j and k, K is the Kähler potential, whose derivatives $K_{i\bar{i}} = \partial_{i\bar{i}} \bar{\partial}_{i\bar{i}} K$ are the kinetic terms of the chiral fields and W_{ijk} is the trilinear coupling of the superpotential. Using this relation it is discussed in [67] that the Kähler metric for the twisted states in the magnetised brane worlds should depend on the background Wilson lines. Indeed a non-holomorphic Wilson lines dependence in the Yukawa couplings results from the overlap of three eigenfunctions of the Dirac operator and, in the particular factorised geometry considered in [67], this non-holomorphic part can be rewritten as the product of three contributions which are then reabsorbed in the Kähler metrics of the three interacting states. The expression in Eq.(9.2.38), which generalises the discussion in [67] and is derived in a fully string-theoretic context, still has a non-holomorphic dependence on the background Wilson lines, however a factorisation of such non-holomorphicity to be reabsorbed in the kinetic terms of the interacting states has not been attempted yet. We hope that a detailed analysis of the richer moduli dependence found in Eq.(9.2.38) may actually shed some light on the precise relationship between the terms of the low-energy $\mathcal{N} = 1$ Super Yang-Mills Lagrangian associated to the magnetised brane world models under study.

In the context of the intersecting D-Brane models instead a considerable amount of work has been produced to compute the Yukawa couplings for the twisted strings living in the intersection of three D-branes at angles. Most of the results determined are again valid for fully factorisable geometries only. The classical contribution to the Yukawa couplings has been investigated in detail for instance in [63, 61] and it has been shown to arise from world-sheet instantons. Indeed the three twist field amplitude in such models can be thought of as factorised in the following fashion

$$\mathcal{A} = \sum_{\langle \mathcal{Z}^i_{\rm cl} \rangle} e^{-S_{\rm cl}} \mathcal{A}_{\rm qu}$$

where the sum is over $\mathcal{Z}^i_{\rm cl}$ that must satisfy the classical equations of motion and possess the correct asymptotic behaviour near the brane intersections, the exponential represents the classical contribution and $\mathcal{A}_{\rm qu}$ contains the quantum part of the correlation function. In [63] using the OPEs

$$\begin{aligned} \partial \mathcal{Z}^i(z)\sigma_{\theta_i}(w,\bar{w}) &\sim (z-w)^{-(1-\theta_i)}\tau_{\theta_i}(w,\bar{w}) \\ \partial \bar{\mathcal{Z}}^i(z)\sigma_{\theta_i}(w,\bar{w}) &\sim (z-w)^{-\theta_i}\tau'_{\theta_i}(w,\bar{w}) \end{aligned}$$

where τ_{θ_i} and τ'_{θ_i} are excited twist fields, and the monodromies of the holomorphic fields $\mathcal{Z}^i(z)$ and $\bar{\mathcal{Z}}^i(z)$ as discussed here in section 4.2, it is explicitly shown that the contribution of the classical action $S_{\rm cl}$ is nothing else than the area of the minimal surface (which becomes the sum of the areas of $2d$ triangles for a T^{2d} compactification) formed by the three D-Branes at angles, in whose intersections live the three interacting states. Indeed in the interaction the three strings overlap in such a way that the world-sheet has to cover the mentioned triangle. In [63] this result is found by explicitly integrating the world-sheet action evaluated on the classical solution $\mathcal{Z}^i_{\rm cl}$ (and its conjugate), which has in turn the holomorphic dependence fixed by the OPEs above and the over-all factor determined by means of the global monodromy conditions, i.e. by considering the transformation of the field as it is transported around more than one twist operator such that the net twist is zero. In [61] assuming that the classical (world-sheet instanton) contribution to the Yukawa couplings is given by the sum of the areas of the $2d$-triangles in each of the T^2's formed by the three branes in whose intersections live the interacting states, the authors show that the final form of such a contribution can be expressed in terms of (products of) Jacobi Theta-functions of the type found also here in Eq.(9.2.22) modulo the T-Duality which connects the magnetised and intersecting branes descriptions. With respect to [63] in which the computation is mainly performed in the case of a square two-torus, they also consider in fully factorisable geometries the full moduli dependence of the Yukawa couplings, both on the closed and on the open string moduli. We have already discussed for instance the slightly different convention in our Eq.(9.2.22) for the parametrisation of the Wilson lines that appear in the characteristics of the Jacobi Theta-functions with respect to [61]. The other properties of the couplings found in Eq.(9.2.22) perfectly match (upon T-Duality) the results of [61, 63]. Nevertheless, the limitation of such computations again resides in the fully factorisable geometry of the compactification torus. In particular the analysis of [61] is not easily generalisable when a clear geometrical picture of the triangles involved in the world-sheet instantons is lacking.

The generalisation to non factorisable geometries is not only of fundamental theoretical importance to achieve the description of the most generic possible models, but it also has phenomenological implications. On one side the knowledge of the whole spectrum of all the possible brane world models would not be of secondary interest in order to shed some light on the counting of phenomenological vacua in the so-called string Landscape (see for instance [103] and references therein). On the other side the possibility of exploring more general (and still exactly solvable) compactification geometries turns out to have at least one fundamental desirable effect from the phenomenological point of view, i.e. the realisation of a non trivial CKM mixing matrix. As it was already noticed in [61, 104] the Yukawa couplings arising from fully factorisable models are factorised themselves as

$$Y_{ij} = a_i b_j$$

where i and j here indicate two given flavours and a_i and b_j are two vectors. The matrix Y_{ij} has rank one and as a consequence only the third generation of particles acquires a non-vanishing mass and the corresponding CKM matrix is trivially the identity which does not yield any mixing. Some effort has been devoted to overcome this difficulty, for instance in [104, 105]. In both the latter works the background geometry is still factorisable, but the Yukawa couplings receive one-loop corrections that spoil the rank one characteristic above. For instance in [104] the authors examine the combination of the tree-level Yukawa couplings with the one-loop interaction involving a chirality changing four fermion amplitude to construct a three external states coupling, and they show that this second diagram can contribute to give a non-trivial mixing and mass generation for the fundamental particles of the model. Similarly in [105] the tree-level coupling is accompanied by a one-loop interaction involving the so-called E2-Branes, i.e. instantons which can be thought of as D-Branes that have only three Neumann boundary conditions in the internal compactified directions of the target space and Dirichlet conditions everywhere else. The possible role of these instantons has been recently explored in phenomenological models also to discuss the mass generation for the neutrinos, by means of the so-called see-saw mechanism [106, 80, 81] introducing otherwise forbidden Majorana mass terms for these particles. Observe accidentally that the techniques analysed in the present work can also be used in this context where the main ingredients for the computations to be performed are again correlation functions of (three) twist fields corresponding to strings stretched between magnetised or intersecting D-Branes together with E2-instantons. As far as the non trivial mixing given by the CKM matrix is concerned, the general result found in Eq.(9.2.38) already provides a rank three matrix Y_{ijk} at tree level. This is the effect of the non factorisability of the background geometry, which yields a coupling in the form of a higher-dimensional Riemann Theta-function instead of a product of Jacobi Theta-functions. The rank one problem for the tree-level Yukawa couplings in factorised models indeed is due to the fact that left and right-handed fermions in all of the known explicit models arise in different orthogonal two-dimensional tori.

Finally let us briefly comment on the derivation of the quantum contribution to the correlation functions of three bosonic twist fields, which, using the techniques presented here, namely the factorisation of the full twisted partition function in the open string channel, is explicitly performed in [64]. The developed method represents a valid alternative to the traditional factorisation of the four point amplitude involving twist operators. Originally this procedure was exploited in the context of closed strings propagating in orbifolds in [59, 66]. One can draw in fact an analogy between twisted closed strings living in an orbifold and twisted open strings stretched between D-Branes with different magnetic fields on them, or, via T-Duality, intersecting at angles. Indeed the monodromy behaviour for the meromorphic fields in Eq.(4.1.13) exactly resembles the one of closed twisted strings in an orbifold, whose endpoints are identified up to the action of a generic orbifold group element g

$$X(\tau, \sigma + 2\pi) = g \cdot X(\tau, \sigma)$$

Clearly if $g \equiv R$, where R is the monodromy matrix in Eq.(4.1.13), the match between the two different types of strings becomes manifest. Exploiting this analogy the quantum contribution to the four point function involving bosonic twist operators was determined in [62, 60, 63] mostly in the context of the intersecting brane worlds. Obviously since the same conformal twist fields enter the description of twisted strings in the magnetised brane worlds, the same results also apply there. From the more easily computed four-point functions, by taking the limit in which two punctures are taken close to each other it is possible to obtain the three-point function [62, 60, 63] directly related to the quantum contribution to the Yukawa couplings.

Summarising, the new techniques presented in this work provide a unified method to explicitly compute both the quantum and the classical part of the Yukawa couplings. Being applicable in particular to the magnetised branes models, as we have already stressed, they represent the first complete attempt to determine in a purely string-theoretic context the moduli dependence of the classical contribution to such couplings, generalising the results found in the existing literature to any, non necessarily factorisable, toroidal compactifications of the theory.

APPENDIX

A. MATHEMATICAL TOOLS

A.1 Linear Diophantine equations

We want to provide a proof for the Bézout's lemma which states that if a and b are non-zero integers with greatest common divisor d, then the diophantine equation

$$ax + by = d \tag{A.1.1}$$

admits infinite solutions for x, y in $\mathbb{Z}$. It is easy to see that the solution, if it exists, cannot be unique as if the pair (x, y) satisfies the equation above, then

$$\left\{ \left(x + \frac{kb}{\text{G.C.D.}\{a,b\}}, y - \frac{ka}{\text{G.C.D.}\{a,b\}} \right) \middle| k \in \mathbb{Z} \right\} \tag{A.1.2}$$

is an infinite set of solutions of the same equation.
In order to prove that there is at least one pair (x, y) which satisfies the equation, one can start, without any loss of generality, taking $a, b > 0$. Then

$$Z = \left\{ am + bn \in \mathbb{N}^+,\ m, n \in \mathbb{Z} \right\} \tag{A.1.3}$$

is a non-empty set and, given that $\mathbb{N}^+$ is an ordered set, it must contain in particular a least member $d = ax + by$. We want to show that indeed $d = \text{G.C.D.}\{a, b\}$. Let us write $a = qd + r$ where $q \in \mathbb{N}^+$ and $0 \leqslant r < d$. Then

$$r = a - qd = a - q(ax + by) = a(1 - qx) + b(-qy) \tag{A.1.4}$$

This implies that either $r = 0$ or $r \in Z$ contradicting the fact the d is the least member in Z. Hence $a = qd$ and d is a divisor of a. The same arguments apply to b, thus d has to be a common divisor of a and b. If c is another common divisor of a and b then c is a divisor of $ax + by = d$ and so d must be the greatest common divisor of a and b.
Observe that if c is a multiple of the greatest common divisor of a and b, then the equation

$$ax + by = c \tag{A.1.5}$$

admits again infinite solutions, while in all of the other cases the diophantine equation does not have any solutions. In fact one can always rewrite the equation as

$$d(\frac{a}{d}x + \frac{b}{d}y) = c \tag{A.1.6}$$

where $d = \text{G.C.D.}\{a, b\}$. Thus $(a/d)x + (b/d)y \in \mathbb{Z}$ and c has to be multiple of d.

A.2 Determinant and inverse of a generic $2d \times 2d$ matrix

We will first of all consider how to calculate the determinant of a generic $2d \times 2d$ matrix written in terms of $d \times d$ blocks

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.2.7}$$

where the D block is an invertible square matrix. It is convenient to factorise M as follows

$$M = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} \tag{A.2.8}$$

In this way both the matrices in the product are written in a block-diagonal form and in particular the second one is also unimodular. Hence

$$\text{Det}M = \text{Det}(A - BD^{-1}C)\text{Det}D = \text{Det}(AD - BD^{-1}CD) \tag{A.2.9}$$

Observe that, if A in M is instead an invertible $d \times d$ block, then a second different factorisation is possible

$$M = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix} \tag{A.2.10}$$

and the determinant of M can be written as

$$\text{Det}M = \text{Det}(D - CA^{-1}B)\text{Det}A = \text{Det}(AD - ACA^{-1}B) \tag{A.2.11}$$

In the two equations (A.2.8) and (A.2.10) $S_A = D - CA^{-1}B$ and $S_D = A - BD^{-1}C$ are respectively the Schur's complements of the elements A and D in M.
It is not difficult to see that, if both the blocks A and D are invertible, then, by using the factorisations (A.2.8) and (A.2.10), the inverse of M reads

$$M^{-1} = \begin{pmatrix} S_D^{-1} & -A^{-1}BS_A^{-1} \\ -D^{-1}CS_D^{-1} & S_A^{-1} \end{pmatrix} = \tag{A.2.12}$$

$$\begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

A.3 Smith normal form for an antisymmetric integer matrix

A generic integer $2d \times 2d$ antisymmetric matrix M can always be put in a block-diagonal form by means of a transformation of the type $M \to {}^tOMO$, O being a unimodular integer matrix.

In order to show that this is indeed the case one can observe that any anti-symmetric matrix:

$$M = \begin{pmatrix} A & B \\ -{}^tB & C \end{pmatrix} \tag{A.3.13}$$

where A (invertible) and C are $2k \times 2k$ and $2(d-k) \times 2(d-k)$ antisymmetric matrices and B is a rectangular $2k \times 2(d-k)$ matrix, can be block diagonalized by

$$O = \begin{pmatrix} 1_{2k} & -A^{-1}B \\ 0 & 1_{2(d-k)} \end{pmatrix} \tag{A.3.14}$$

getting:

$${}^tOMO = \begin{pmatrix} A & 0 \\ 0 & {}^tBA^{-1}B + C \end{pmatrix} \tag{A.3.15}$$

Notice that each block of ${}^tOMO$ is also an antisymmetric matrix and that the determinant of the matrix O is one. Since the form of ${}^tOMO$ in Eq.(A.3.15) is independent of the choice of k, one can use an iterative procedure (if all the sub-blocks "A" are invertible) to obtain the final block diagonal form always choosing $k = 1$. Using this procedure $d-1$ times one finds

$${}^tOMO = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & a_2 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_d \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.3.16}$$

where: $O = O_1 O_2 ... O_{d-1}$. By a suitable permutation of the rows and of the columns of the matrix above one can rewrite the transformed matrix as

$$M' = \begin{pmatrix} 0 & \tilde{M} \\ -\tilde{M} & 0 \end{pmatrix} \tag{A.3.17}$$

where $\tilde{M} = \mathrm{diag}\{a_1, a_2, \ldots\, a_d\}$.

If the matrix M has integer elements, the transformed matrix will be integer only if $\mathrm{Det}A = 1$. We will now show that it is always possible to reduce to this case at any step of the iterative procedure. Let us consider in particular $a_1 \neq 1$, as the generalisation of what follows for the other a_α's is straightforward. If $\mathrm{Det}A \neq 1$

we can distinguish three cases. First there is a row with two elements that are co-prime. By exchanging rows and columns among themselves it is possible to put these elements (a and b) in the first row

$$M \equiv M_1 = \begin{pmatrix} 0 & a & b & \cdots \\ -a & 0 & c & \cdots \\ -b & -c & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.18}$$

Then it is not difficult to see that the unimodular integer matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & x & -b & 0 & \cdots \\ 0 & y & a & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.19}$$

transforms

$$M_1 \to {}^tQM_1Q = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & c & \cdots \\ 0 & -c & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.20}$$

as a consequence of the Diophantine equation $ax + by = 1$ which has infinite solutions (see Appendix A.1).
If, instead, a and b have a common factor of d, then $ax + by = d$ has infinite solutions (see Appendix A.1), hence one has to define

$$Q' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & x & -\frac{b}{d} & 0 & \cdots \\ 0 & y & \frac{a}{d} & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.21}$$

This yields

$$M_1 \to {}^tQ'M_1Q' = \begin{pmatrix} 0 & d & 0 & \cdots \\ -d & 0 & c & \cdots \\ 0 & -c & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.22}$$

One has to apply this procedure till the matrix is reduced to one of the following cases: either in the first row of the transformed $M \equiv M_1$ there are two co-prime non-vanishing entries, then one can use the matrix Q obtain $a_1 = 1$; or all of the non-zero elements there coincide with d. This is the case if for instance in the original matrix M_1 the first row contained elements which were all multiples of d. If there is any other row in the transformed matrix with two different non-zero elements $a \neq b$ for which d is not a divisor, then, by exchanging rows and columns among themselves, it is possible to bring this as the first row and reapply the transformations encoded in Q or Q'. Otherwise one can have two possible forms for the transformed matrix. One possibility is to have that all of the elements of $M \equiv M_1$ are integer multiples of d. In this case the common divisor d can be factored out to reduce to one of the cases already analysed. The other possibility is that the matrix has diagonal blocks, in which all the elements are multiple of different integers d_i. In this second scenario let us consider an explicit example

$$M_1 = \begin{pmatrix} 0 & d_1 & 0 & 0 & \cdots \\ -d_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & d_2 & \cdots \\ 0 & 0 & -d_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.23}$$

Although this already looks like the final form we are after, recall that the normal form of the initial matrix (A.3.16), as discussed in [74], has the further property that $a_{\alpha+1}/a_\alpha \in \mathbb{N}$, $\forall\alpha$. In order to achieve this (even if it is not strictly necessary for the computations considered here) one can use the following transformation

$$Q'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A.3.24}$$

that mixes the d_i's and gives back a form that can be reduced by means of either Q or Q'. Following this procedure, it is possible to convince oneself that the final matrix (A.3.16) entries satisfy the property mentioned above, since one actually ends the repeated application of Q, Q', and Q'' only if in the first 2×2 block there is a one, or if the matrix is proportional to an integer as a whole.

A.3.1 Applications

We will now show that the phase factor in the boundary state (6.2.45) is not affected by the change of the fundamental cell in the lattice torus

$$\hat{p} = \omega \times F \to {}^t\!O\hat{p}O = \omega\, {}^t\!OFO = \omega F_{\text{block}} \tag{A.3.25}$$

performed in Eq.(3.2.32). It reads

$$\text{Ph} = \exp\left[\mathrm{i}\pi \sum_{M<N} \hat{m}^M F_{MN} \hat{m}^N\right] = \exp\left[\frac{\mathrm{i}\pi}{\omega} \sum_{M<N} \hat{m}^M \hat{p}_{MN} \hat{m}^N\right] \tag{A.3.26}$$

where $\hat{p}$ is an integer matrix, that we write in the form of Eq.(A.3.13), and F is given as in Eq.(3.2.29). In order to write it as a block diagonal matrix, we use the techniques just discussed, focusing at first on the simplest case with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.3.27}$$

Let us consider how the phase Ph (A.3.26) transforms under the substitution $\hat{m} = O\hat{m}'$, with O as in Eq.(A.3.14). One gets:

$$\begin{aligned} \text{Ph} = \exp\Bigg[\frac{\mathrm{i}\pi}{\omega}\Bigg(& \hat{m}'_1 A_{12} \hat{m}'_2 + \sum_{j=3}^{2d} B_{2j} \hat{m}'_j A_{12} \hat{m}'_2 - \hat{m}'_1 A_{12} \sum_{j=3}^{2d} B_{1j} \hat{m}'_j \\ & - \sum_{j,k=3}^{2d} B_{2j} \hat{m}'_j A_{12} B_{1k} \hat{m}'_k + \sum_{j=3}^{2d} \hat{m}'_1 B_{1j} \hat{m}'_j + \sum_{j,k=3}^{2d} B_{2j} \hat{m}'_j B_{1k} \hat{m}'_k \\ & + \sum_{j=3}^{2d} \hat{m}'_2 B_{2j} \hat{m}'_j - \sum_{j,k=3}^{2d} B_{1j} \hat{m}'_j B_{2k} \hat{m}'_k + \sum_{k>j=3}^{2d} \hat{m}'_j C_{jk} \hat{m}'_k \Bigg)\Bigg] \end{aligned} \tag{A.3.28}$$

By remembering that all the winding numbers $\hat{m}^N$ must be integer multiples of the corresponding wrapping numbers w_N, one can check that, in spite of the denominator ω, all the terms in the exponent are integer multiples of $\mathrm{i}\pi$. In fact combining the form of the matrices (A.3.14) and (3.2.31), one finds the following expressions for $\hat{m} = O\hat{m}'$

$$\begin{aligned} \hat{m}_1 &= \hat{m}'_1 + \frac{\omega}{w_2} \sum_{i=3}^{2d} p_{2i} \frac{\hat{m}'_i}{w_i} \\ \hat{m}_2 &= \hat{m}'_2 - \frac{\omega}{w_1} \sum_{i=3}^{2d} p_{1i} \hat{m}'_i \\ \hat{m}'_i &= \hat{m}_i \end{aligned} \tag{A.3.29}$$

In this case, in order to have the matrix A in the form (A.3.27), it is necessary that $p_{12} = 1$ and $\omega = w_1 w_2$, hence, since from the last line of the previous equation $\hat{m}'_i/w_i$ must be integer, it is also true, in the first and second line, that the transformed winding numbers $\hat{m}'^N$ are integer multiples of the wrapping numbers w_N. Thus each of the terms in the sum (A.3.28) is an integer number. In order to show that this is the case we can for instance consider the first term of the second line in Eq.(A.3.28) writing $\hat{m}'^N = m'^N w_N$ with $m'^N \in \mathbb{Z}^{2d}$

$$\frac{1}{\omega} B_{2j} \hat{m}'_j A_{12} B_{1k} \hat{m}'_k = \frac{1}{\omega} \omega \frac{p_{2j}}{w_2 w_j} m'_j w_j \omega \frac{p_{1k}}{w_1 w_k} m'_k w_k = p_{2j} m'_j p_{1k} m'_k \in \mathbb{Z} \quad \text{(A.3.30)}$$

So we can freely change the sign of each term in Eq.(A.3.28), obtaining

$$\begin{aligned} \text{Ph} &= \exp\left[\mathrm{i}\pi \left(\sum_{M<N} \hat{m}'^M ({}^t OFO)_{MN} \hat{m}'^N + \frac{1}{\omega} \sum_{j=3}^{2d} B_{1j} B_{2j} \hat{m}'^2_j \right) \right] = \\ &= \exp\left[\mathrm{i}\pi \left(\sum_{M<N} \hat{m}'^M ({}^t OFO)_{MN} \hat{m}'^N + \sum_{j=3}^{2d} p_{1j} p_{2j} \frac{\hat{m}'_j}{w_j} \right) \right] \end{aligned} \quad \text{(A.3.31)}$$

where we have used the explicit expression of B_{1j} and B_{2j} in terms of the Chern numbers p_{1j} and p_{2j} and of the winding numbers; moreover we have taken into account the fact that $\left(\hat{m}'_j/w_j\right)^2$ has the same parity (even/odd) as the integer number $\hat{m}'_j/w_j$. Thus the phase factor can be written in terms of the transformed field ${}^t OFO$ and of the transformed winding numbers $\hat{m}'$'s with the same functional form as the original one (A.3.26), with a half-integer shift of the Wilson line when $p_{1j} p_{2j}$ is odd.

If ${}^t OFO$ is already block diagonal, we have ended our job, otherwise we have to repeat the procedure. In an analogous fashion, if the entries of A are not equal to one, one can check that the transformations related to the matrices in Eqs.(A.3.19) and (A.3.21), involved in reducing A to the form considered in the previous example, also preserve the form of the phase factor up to half-integer Wilson lines. In this case it also possible to show that again each term of the sum in Eq.(A.3.31) is integer. Considering in fact the example of the transformation (A.3.21), one has $a = \omega p_{12}/(w_1 w_2)$ and $b = \omega p_{13}/(w_1 w_3)$. By defining $d_1 =$ G.C.D.$\{p_{12}, p_{13}\}$, $d_2 =$ G.C.D.$\{\omega/(w_1 w_2), \omega/(w_1 w_3)\}$ and $w_{23} =$ m.c.m.$\{w_2, w_3\}$, it follows that $p_{1i} = d_1 p'_{1i}$, with p'_{1i} integer, and $\omega = d_2 w_1 w_{23}$ in order for a and b to have $d = d_1 d_2$ as the only common factor. Thus, if $\hat{m} = Q' \hat{m}'$, then $\hat{m}'_2 = (a\hat{m}_2 + b\hat{m}_3)/d = w_{23}(p'_{12}\hat{m}_2/w_2 + p'_{13}\hat{m}_3/w_3)$ which means that $\hat{m}'_2$ must be multiple of both w_2 and w_3. Furthermore from

$$\begin{aligned} \hat{m}_2 &= x\hat{m}'_2 - \frac{b}{d}\hat{m}'_3 \\ \hat{m}_3 &= y\hat{m}'_2 + \frac{a}{d}\hat{m}'_3 \end{aligned}$$

one can see that $(b/d)\hat{m}'_3$ is multiple of w_2 and $(a/d)\hat{m}'_2$ of w_3. This guarantees that with the transformation $\hat{m} = Q'\hat{m}'$ each term of the sum in (A.3.31) is an integer number similarly to the case discussed before. Observe finally that the same arguments can be used to show that Q in Eq.(A.3.19) transforms the phase factor into the form (A.3.31) with integer terms in the sum: it is sufficient in fact to take $d_1 = d_2 = d = 1$ and follow the same steps above.
With similar manipulations it is also possible to prove, in a basis in which $(F_2 - F_1)$ is block-diagonal, that the phase factors in Eq.(9.2.7) follow from those in Eq.(8.2.30). As usual, one has to introduce $h \in \mathbb{Z}^{2d}$ by using $\hat{m}_1 = Hh$; then it is possible to check that the combination $\sum_{M<N}(Hh)^M(F_2-F_1)_{MN}(Hh)^N$ is equal, modulus two, to $\sum_{M<N} h^M \left[H(F_2 - F_1)H\right]_{MN} h^N$, apart from terms quadratic in h^M that can be reabsorbed into a half-integer shift of the Wilson lines.

B. SCHOTTKY REPRESENTATION OF A RIEMANN SURFACE

Let us first introduce the notion of the Schottky group [94]. Given a set of g independent projective transformations of the complex plane S_μ, $\mu = 1, \ldots, g$, the Schottky group S is the group freely generated by the S_μ's. Therefore a generic element $T_\alpha \in S$, except for the identity, can be written in the following form

$$T_\alpha \equiv S_{\mu_1}^{n_1} S_{\mu_2}^{n_2} \ldots S_{\mu_r}^{n_r}\,, \quad r = 1, 2, \ldots\,; \quad n_i \in \mathbb{Z}/\{0\}\,, \quad \mu_i \neq \mu_{i+1} \tag{B.0.1}$$

The number of generators, or their inverse, in the element above is called its order n_α and it is given by

$$n_\alpha = \sum_{i=1}^{r} |n_i| \tag{B.0.2}$$

A primitive element of the Schottky group is an element which cannot be written as an integer power of other elements, i.e.

$$T_\alpha \neq (T_{\alpha'})^n \quad \text{with} \quad n \in \mathbb{N}_+ \Leftrightarrow T_\alpha \text{ is primitive} \tag{B.0.3}$$

Two elements of S, T_α and $T_{\alpha'}$, are said to belong to the same conjugacy class if they simply differ by a cyclic permutation of their constituent factors.
The Schottky group is specified by giving the parameters of the g generators S_μ's. This can be done either by writing

$$S_\mu(z) = \begin{pmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{a_\mu z + b_\mu}{c_\mu z + d_\mu} \tag{B.0.4}$$

with $a_\mu d_\mu - b_\mu c_\mu = 1$, or, more conveniently, by introducing for each generator, the multiplier k_μ and the fixed points ξ_μ and η_μ, defined by

$$\frac{S_\mu(z) - \eta_\mu}{S_\mu(z) - \xi_\mu} = k_\mu \frac{z - \eta_\mu}{z - \xi_\mu} \quad |k_\mu| \leqslant 1 \tag{B.0.5}$$

Since for any z we have

$$\lim_{n\to\infty} S_\mu^n(z) \equiv \eta_\mu \quad , \quad \lim_{n\to\infty} S_\mu^{-n}(z) \equiv \xi_\mu \tag{B.0.6}$$

η_μ is called attractive fixed point and ξ_μ is called repulsive fixed point.
Simple algebra reveals that the relation between the parameters a_μ, b_μ, c_μ and d_μ and the multiplier k_μ and the fixed points η_μ and ξ_μ is given by

$$\begin{aligned} a_\mu &= \frac{\eta_\mu - k_\mu \xi_\mu}{\sqrt{|k_\mu|}|\eta_\mu - \xi_\mu|} \quad ; \quad b_\mu = -\frac{\xi_\mu \eta_\mu (1-k_\mu)}{\sqrt{|k_\mu|}|\eta_\mu - \xi_\mu|} \\ c_\mu &= \frac{1-k_\mu}{\sqrt{|k_\mu|}|\eta_\mu - \xi_\mu|} \quad ; \quad d_\mu = \frac{k_\mu \eta_\mu - \xi_\mu}{\sqrt{|k_\mu|}|\eta_\mu - \xi_\mu|} \end{aligned} \tag{B.0.7}$$

It is quite clear that if we apply to the generators S_μ and their inverse a similarity transformation generated by a fixed projective map A, the resulting Schottky group S' will be essentially equivalent to the original one, because every element $T \in S$ will just transform into $T' = ATA^{-1} \in S'$. Obviously T' has the same multiplier as T, since it is related to the eigenvalues of the S_μ's (which are $\{\sqrt{|k_\mu|}, 1/\sqrt{|k_\mu|}\}$), but its fixed points will be $A(\eta)$ and $A(\xi)$. In particular this means that by exploiting this freedom (which is actually the overall projective invariance of the complex plane where the transformations are defined), we can always fix to arbitrary values three of the $2g$ fixed points of the generators and therefore inequivalent Schottky groups will be effectively described by g multipliers and $2g-3$ fixed points, which is also the number of parameters needed to specify inequivalent Riemann surfaces of genus g.
The g generators S_μ and their inverse identify in the extended complex plane $2g$ circles, $\mathcal{C}_\mu$ and $\mathcal{C}'_\mu$ for $\mu = 1, \ldots, g$, called isometric circles and defined respectively by

$$\left|\frac{dS_\mu}{dz}\right|^{-1/2} = |c_\mu z + d_\mu| = 1 \quad ; \quad \left|\frac{dS_\mu^{-1}}{dz}\right|^{-1/2} = |c_\mu z - a_\mu| = 1 \tag{B.0.8}$$

Their radii, $\mathcal{R}_\mu$ and $\mathcal{R}'_\mu$, are given by

$$\mathcal{R}_\mu \equiv \mathcal{R}'_\mu = \sqrt{|k_\mu|}\frac{|\xi_\mu - \eta_\mu|}{|1-k_\mu|} \tag{B.0.9}$$

while their centres are

$$-\frac{d_\mu}{c_\mu} = \frac{\xi_\mu - k_\mu \eta_\mu}{1-k_\mu} \quad , \quad \frac{a_\mu}{c_\mu} = \frac{\eta_\mu - k_\mu \xi_\mu}{1-k_\mu} \tag{B.0.10}$$

It easy to verify that the projective transformation S_μ maps the circle $\mathcal{C}_\mu$ onto the circle $\mathcal{C}'_\mu$ and that of course the inverse of the generator S_μ^{-1} maps the circle $\mathcal{C}'_\mu$ back onto the circle $\mathcal{C}_\mu$. Moreover, any point outside the circle $\mathcal{C}_\mu$ will be mapped by S_μ into a point inside the circle $\mathcal{C}'_\mu$, whereas any point outside the circle $\mathcal{C}'_\mu$

will be mapped by S_μ^{-1} into a point inside the circle $\mathcal{C}_\mu$. In particular this implies that the attractive fixed point η_μ is inside the circle $\mathcal{C}'_\mu$, while the repulsive fixed point ξ_μ is inside the circle $\mathcal{C}_\mu$.
From these considerations one can deduce that the fundamental region of the Schottky group S is precisely the part of the extended complex plane which is exterior to all the circles $\mathcal{C}_\mu$'s and $\mathcal{C}'_\mu$'s. It is intuitively clear that, if we identify in couples these circles, g handles are formed and we obtain a Riemann surface of genus g, Σ_g. More precisely we have

$$\Sigma_g = \frac{\mathbb{C} \cup \{\infty\} - \Lambda(S)}{S} \tag{B.0.11}$$

where $\Lambda(S)$ is the limit set of the Schottky group, that is the set of accumulation points of its orbits.
Going around a cycle a_μ of the canonical homology basis of the Riemann surface corresponds in the Schottky representation to going around $\mathcal{C}_\mu$ or $\mathcal{C}'_\mu$, while moving on a path that brings from a point z of $\mathcal{C}_\mu$ to the point $S_\mu(z)$ of $\mathcal{C}'_\mu$ corresponds to going around a b_μ cycle.

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