From Trees to Loops and Back

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based on hep-th/0510253 AB-Spence-Travaglini

and also
hep-th/0412108 Bedford-AB-Spence-Travaglini
hep-th/0410280 Bedford-AB-Spence-Travaglini
hep-th/0407214 AB-Spence-Travaglini

Outline

- **Goal**
- **Covariance of MHV Diagrams**
  - Tree/Loop Amplitudes from MHV Vertices
  - Feynman Tree Theorem
  - Proof of Covariance of Loop MHV Diagrams
- **Discontinuities**
- **Checking Singularities/Factorisation**
  - Collinear Limits
  - Soft Limits
- **Conclusions**

Goal

- The goal of this talk is to address the following question:

  Do MHV Diagrams provide a new, complete, perturbative expansion of Supersymmetric Yang-Mills ?

Evidence

- Tree Level Amplitudes ⟹ complete proof (CSW, Britto-Cachazo-Feng-Witten, Risager)
- One-Loop Amplitudes in (S)YM
  - MHV method at one-loop ⟹ Explicit calculation of complete MHV one-loop Amplitudes in N=4 SYM (AB-Spence-Travaglini)
  - Generalisation to N=1 SYM: MHV one-loop amplitudes (Bedford-AB-Spence-Travaglini, Quigley-Rozali)
  - Cut-Constructible Parts of MHV one-loop Amplitudes in pure Yang-Mills (Bedford-AB-Spence-Travaglini)
    New results for QCD from MHV diagrams!
    For rational parts other techniques needed (see Bern, Forde, Dixon, Kosower)

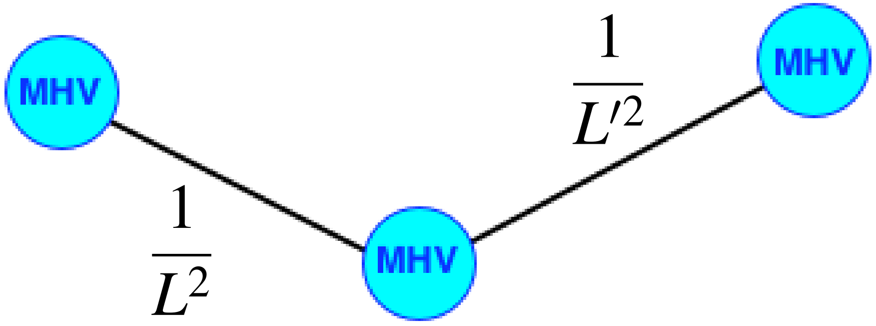
MHV Diagrams

$$A^{tree}_{MHV} = ig^{n-2}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

(Parke-Taylor, Berends-Giele)

- MHV amplitude (holomorphic) = line in twistor space = local interaction in Minkowski space (Witten, Cachazo-Svrcek-Witten)
- CSW Rules (Cachazo-Svrcek-Witten)
  - MHV amplitudes promoted to local vertices using off-shell continuation: $L_\mu = l_\mu + z\,\eta_\mu$, $\lambda_a \sim L_{a\dot{a}}\tilde{\eta}^{\dot{a}}$
  - Connect MHV vertices with scalar propagators

MHV Diagrams cont'd

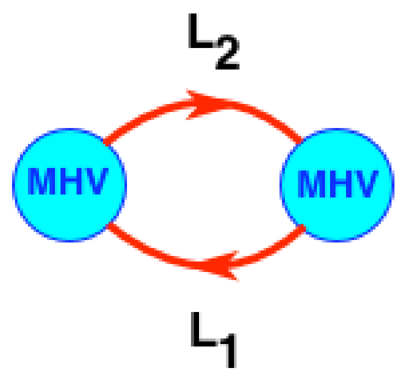


- Proof at Tree Level:
  - Covariance (= $\eta-$independence) is achieved after summing all MHV diagrams (CSW)
  - Equivalence with Feynman diagrams
    - Britto-Cachazo-Feng-Witten
    - Risager: MHV diagrams = special recursion relation (shifts invisible since MHV amplitudes are holom.)

From Trees to Loops (AB-Spence-Travaglini)

- Original prognosis from twistor string theory was negative (Berkovits-Witten), Conformal SUGRA modes spoil duality
- Try anyway:
  - Connect MHV vertices, using the same off-shell continuation as for trees
  - Chose measure, perform loop integration
- MHV 1-loop amplitudes in N=4/N=1 SYM (agrees with BDDK)

$$\int d\mathcal{M} \sum_{m_1, m_2, h}$$



From Trees to Loops, cont'd

$$d\mathcal{M} = 2\pi i\, \theta(P_{L;z}^2)\, dLIPS(l_2^{\mp}, -l_1^{\pm}; P_{L;z})\, \frac{dP_{L;z}^2}{P_{L;z}^2 - P_L^2 - i\varepsilon}$$

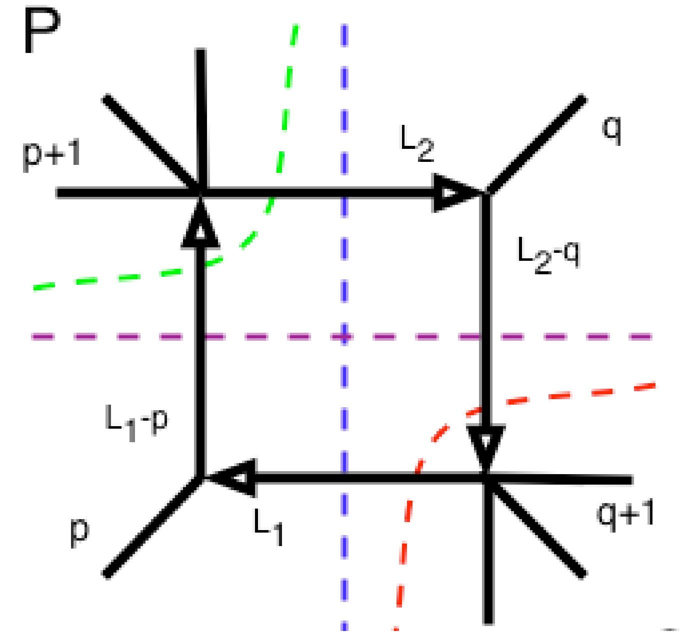
= (2-particle LIPS measure) x (dispersive measure)

➡ The Return of the Analytic S-Matrix

- $\eta-$dependence cancels after summing diagrams
- MHV one-loop amplitudes in N=4 SYM are a linear combination of "2-mass easy" box functions
- with a slight generalisation of BST, we can find the all order in ε box functions ➡

From Trees to Loops, cont'd

$A^{\text{tree}}_{\text{MHV}} \times$



the all order in ε
2-mass easy box function:

$$F^{2me}(s,t,P^2,Q^2) = -\frac{c_\Gamma}{\varepsilon^2}\Bigg[\left(\frac{-s}{\mu^2}\right)^{-\varepsilon}{}_2F_1\left(1,-\varepsilon,1-\varepsilon,as\right) + \left(\frac{-t}{\mu^2}\right)^{-\varepsilon}{}_2F_1\left(1,-\varepsilon,1-\varepsilon,at\right)$$

$$-\left(\frac{-P^2}{\mu^2}\right)^{-\varepsilon}{}_2F_1\left(1,-\varepsilon,1-\varepsilon,aP^2\right) - \left(\frac{-Q^2}{\mu^2}\right)^{-\varepsilon}{}_2F_1\left(1,-\varepsilon,1-\varepsilon,aQ^2\right)\Bigg]$$

with $\quad a := \dfrac{2(pq)}{P^2Q^2 - st}$

From Loops Back to Trees

via the Feynman Tree Theorem (FTT)

- We want to show that MHV diagrams are equivalent to Feynman diagrams for generic one-loop amplitudes in SYM
- Step1: proof of covariance using FTT
- The FTT is based on the decomposition of the conventional Feynman propagator:

$$\Delta_F(P) = \Delta_R(P) + 2\pi\delta^{(-)}(P^2 - m^2)$$

$$\delta^{(-)}(P^2 - m^2) \equiv \delta(P^2 - m^2)\theta(-P_0)$$

FTT cont'd

- Assume we use Feynman rules with $\Delta_F(P)$ instead of $\Delta_R(P)$
- Since $\Delta_R(P)$ is a causal propagator (contrary to $\Delta_F(P)$) any loop integral

$$I_R = \int \prod_i d^4x_i \, \Delta_R(x_1 - x_2) V(x_2) \Delta_R(x_2 - x_3) V(x_3) \cdots \Delta_R(x_n - x_1) V(x_1) = 0$$

with local vertices has support for: $t_1 > t_2 > \cdots > t_n > t_1$

Since there are no closed time-like curves in Minkowski space this integral vanishes!

FTT cont'd

- Now use the decomposition of $\Delta_R(P)$ into $\Delta_F(P)$ and an on-shell delta-function

$$I_R \;:=\; \int \frac{d^4L}{(2\pi)^4} f(L,\{K_i\}) \prod_i \left[\Delta_F(L+K_i) - 2\pi\delta^{(-)}\left((L+K_i)^2\right)\right] \;=\; 0$$

to find the FTT

$$I_F \;=\; -\int \frac{d^4L}{(2\pi)^4} f(L,\{K_i\}) \prod_i{}' \left[\Delta_F(L+K_i) - 2\pi\delta^{(-)}\left((L+K_i)^2\right)\right]$$

In a nutshell: the FTT reduces Loops to Trees! Or more precisely to the sum of all possible cuts.

$$I_F \;=\; I_{1-cut} + I_{2-cut} + I_{3-cut} + I_{4-cut}$$

FTT and MHV Diagrams

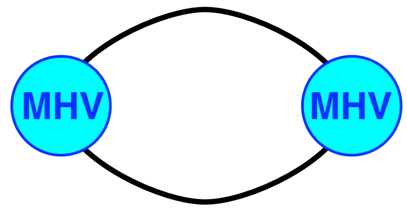
- Because of the local character in Minkowski space of MHV vertices we can apply the FTT directly to MHV diagrams.
- This will allow us to find a simple proof of covariance for the sum of MHV diagrams contributing to generic (one-)loop amplitudes
- An MHV diagram contributing to a one-loop amplitude with Q negative helicity gluons contains V=Q MHV vertices connected by scalar propagators
- The amplitude is given by a sum of terms in which at least one loop leg is cut

$$\mathcal{A} = \mathcal{A}_{1-cut} + \mathcal{A}_{2-cut} + \mathcal{A}_{3-cut} + \mathcal{A}_{4-cut}$$

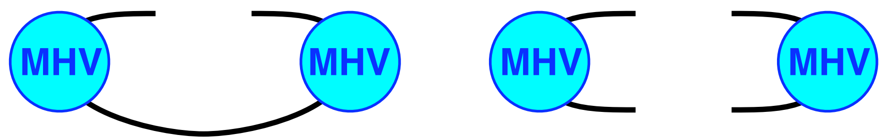
The key point is that each set of p-particle cut diagrams sums to a covariant expression!

FTT and MHV Diagrams cont'd

- MHV one-loop amplitudes

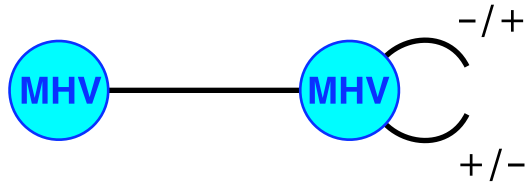


The MHV diagrams have the following 1-particle and 2-particle cuts



The 2-particle cuts give a phase space integral of a product of on-shell tree amplitudes and hence are covariant

FTT and MHV Diagrams cont'd



"missing diagrams"

- More care needed for 1-particle cuts: sum is only over diagrams with cut legs on different MHV vertices.

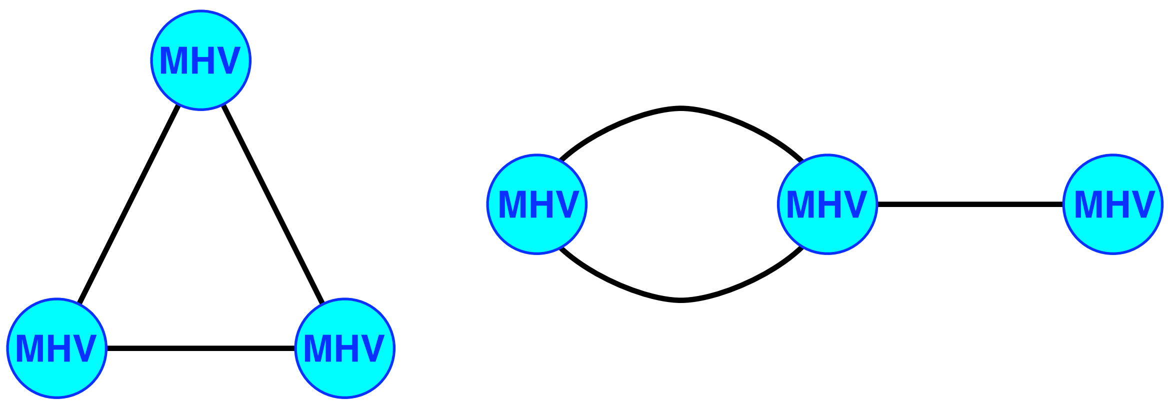
- Two alternative justifications to exclude these diagrams

  1. In supersymmetric theories the missing diagrams give a vanishing integrand, after summing over internal particle species.

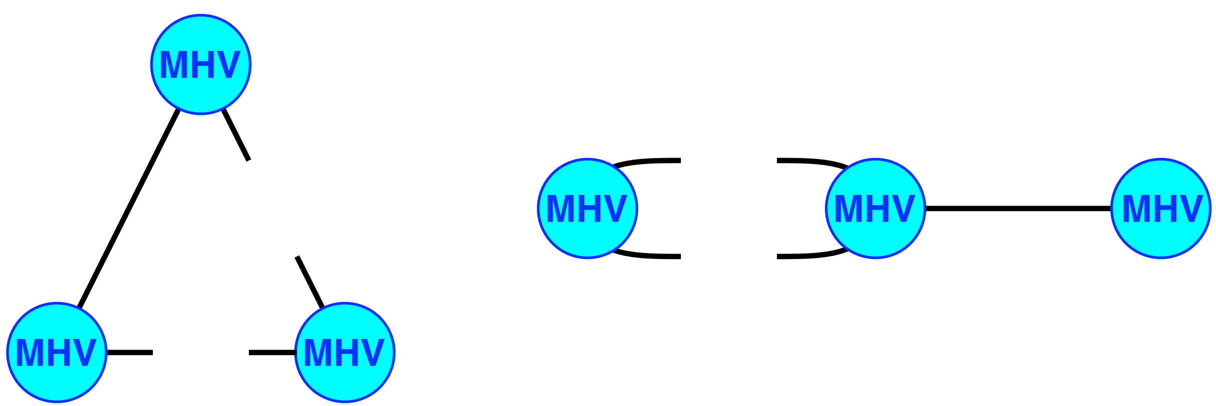
  2. cut legs are (anti-)collinear ⇒
     "missing diagram" = (splitting function) x (tree diagram)
     These tree diagrams sum to an amplitude.

FTT and MHV Diagrams cont'd

- more complicated examples can be treated in complete analogy
- NMHV Amplitudes

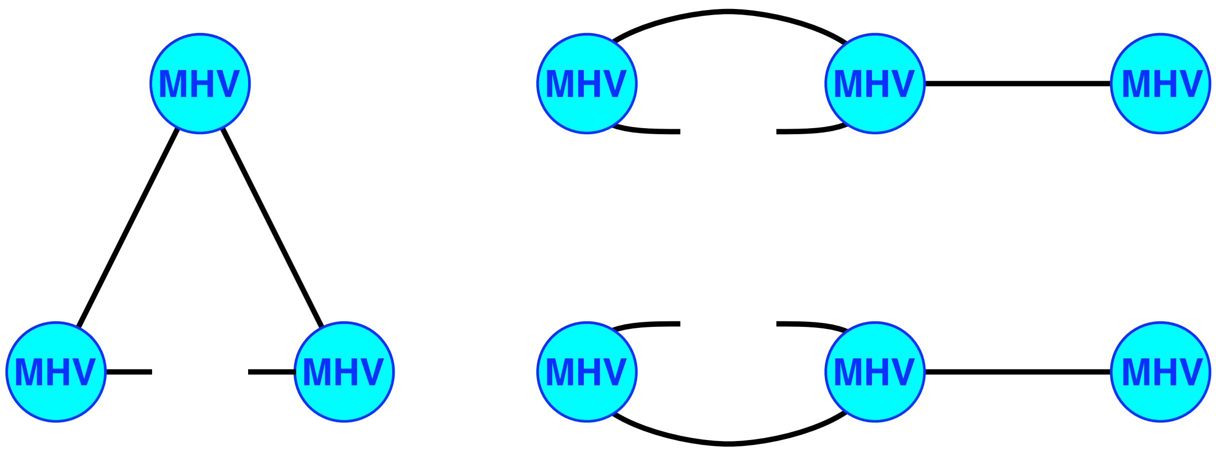


MHV diagrams

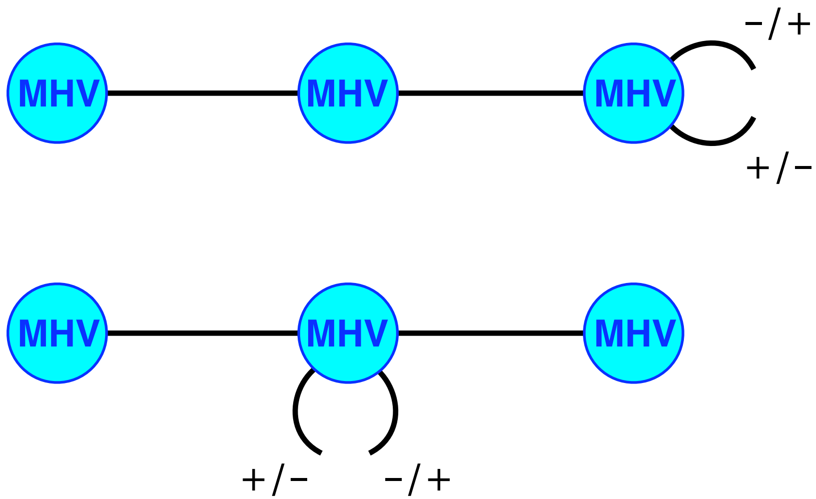


2-particle cut diagrams

FTT and MHV Diagrams cont'd



one-particle cut
diagrams



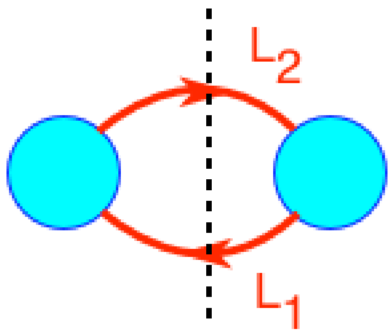
"missing" diagrams

Discontinuities

- One-loop MHV diagrams give covariant expressions
- Step 2: check that these expressions have the correct discontinuities or unitarity cuts in all channels.
- Straighforward; the diagrammatics is the same for Feynman 2-particle cuts in the FTT and a unitarity 2-particle cuts.
- In a particular channel one fixes two propagators and replaces them by two on-shell delta functions. Summing all MHV diagrams sharing the same 2-particle cut, one obtains the full tree amplitudes on both sides of the cut. LIPS integration produces then the expected discontinuity.

Discontinuities cont'd

- This argument applies also to generalised unitarity cuts.
- Note: although the diagrammatics look the same, a Feynman 2-particle cut is different from a unitarity 2-particle cut. In particular a Feynman/Unitarity 2-particle cut vanishes above/below the 2-particle threshold!



In a Feynman cut:

$L_{10} < 0$ and $L_{20} < 0$

In a Unitarity cut:

$L_{10} < 0$ and $L_{20} > 0$

Factorisation

- MHV one-loop diagrams give covariant formulas with all the correct (generalised) cuts
- Step 3: Check that all physical poles in possible kinematic limits are correct. In particular we will check the universal collinear and some of the soft limits.
- Unphysical, $\eta-$dependent singularities (and cuts)can be excluded by our proof of covariance
- The remaining ambiguity must be a polynomial term, which can be ruled out on dimensional grounds (as in BCFW)

Universal collinear factorisation

- Consider a one-loop amplitude $\mathcal{A}_n^{1-loop}$ in the limit when legs a and b become collinear

$$\mathcal{A}_n^{1-loop}(1,\ldots,a^{\lambda_a},b^{\lambda_b},\ldots,n) \xrightarrow{a\|b}$$
$$\sum_{\sigma}\Big[Split_{-\sigma}^{tree}(a^{\lambda_a},b^{\lambda_b})\ \mathcal{A}_{n-1}^{1-loop}(1,\ldots,(a+b)^{\sigma},\ldots,n)$$
$$+\ Split_{-\sigma}^{1-loop}(a^{\lambda_a},b^{\lambda_b})\ \mathcal{A}_{n-1}^{tree}(1,\ldots,(a+b)^{\sigma},\ldots,n)\Big]$$

Contains tree and one-loop splitting functions:

$$Split_{-}^{tree}(a^{+},b^{+}) = \frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle a\,b\rangle},\ \ Split_{+}^{tree}(a^{-},b^{-}) = -\frac{1}{\sqrt{z(1-z)}}\frac{1}{[a\,b]}$$

With $k_a := zk_P,\ k_b := (1-z)k_P,\ k_P^2 \to 0$

Universal Collinear Factorisation

The one-loop splitting function is

$$Split^{1-loop}_{-\sigma}(a^{\lambda_a}, b^{\lambda_b}) \;=\; Split^{tree}_{-\sigma}(a^{\lambda_a}, b^{\lambda_b})\, r(z)$$

where, to all orders in ε

$$r(z) \;:=\; \frac{c_\Gamma}{\varepsilon^2}\left(\frac{-s_{ab}}{\mu^2}\right)^{-\varepsilon}\left[1 - {}_2F_1\left(1, -\varepsilon, 1-\varepsilon, \frac{z-1}{z}\right) - {}_2F_1\left(1, -\varepsilon, 1-\varepsilon, \frac{z}{z-1}\right)\right]$$

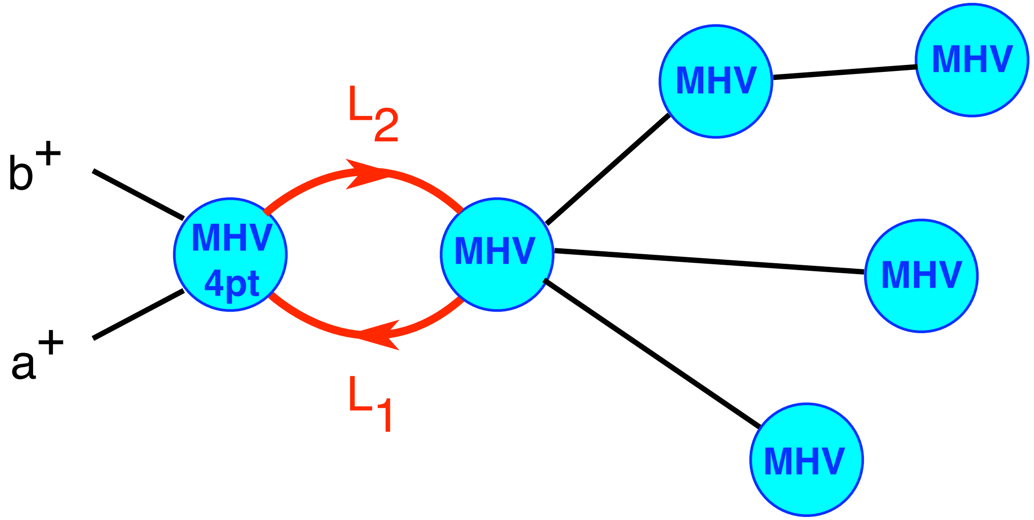
- All order expression for $r(z)$ was obtained by (Kosower-Uwer, Bern-Del Duca-Kilgore-Schmidt)
- This all order formula can be reproduced from MHV diagrams: work to all orders in ε in the calculation of LIPS integrals.

Collinear Limits from MHV Diagrams

- At tree level, collinear limits come out as expected (CSW) as well as soft limits
- Two types of collinear limits in MHV diagram method
  - A-type: $++ \Rightarrow +$ and $+- \Rightarrow -$
  - B-type: $+- \Rightarrow +$ and $-- \Rightarrow -$
- Both legs belong to the same MHV vertex, for B-type this must be a 3-point vertex

Collinear Limits from MHV Diagrams

- Also at one level we have to distinguish A/B-type
- "singular channel" (Kosower) and "non-singular channel" MHV diagrams
  - "non-singular channel": Tree splitting function
  - "singular channel": One-loop splitting function



"singular channel" MHV diagram

One-loop splitting functions from MHV diagrams

- If legs a and b end on different MHV vertices
  ➡ no contribution to collinear limit

- A-type collinear limits

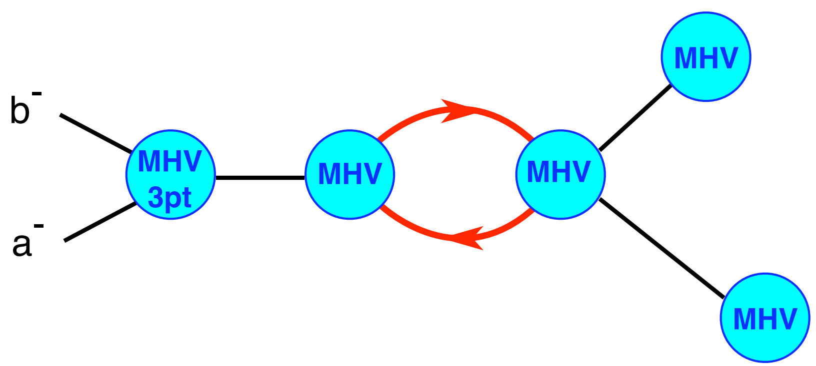
  - all order one-loop splitting function from generic "singular channel" diagram shown before

  - requires all order in ε form of the 2-mass easy box function (slight generalisation of calculation in (AB-Spence-Travaglini))

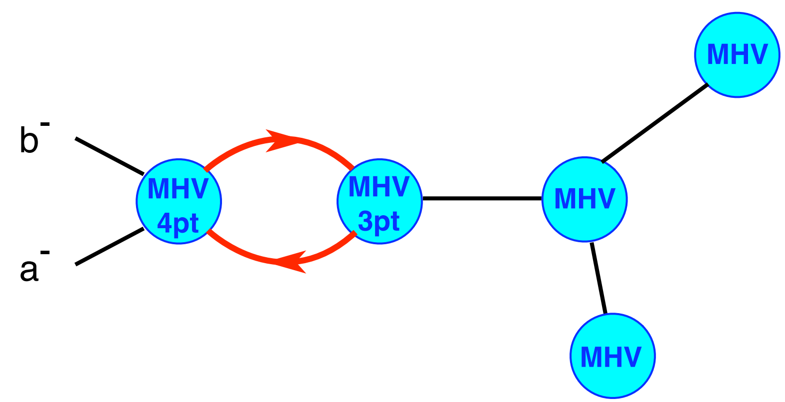
  - tree-level splitting function from "non-singular channel" diagrams, where legs a and b are a proper subset of legs attached to MHV vertex

One-loop splitting functions from MHV diagrams, cont'd

- B-type collinear limits need special attention
  - all order one-loop splitting function from "singular channel" diagram
  - agrees with known results



"non-singular channel"
MHV diagram, contributes to
tree level splitting function



"singular channel" MHV diagram

Soft Limits

- Behaviour of one-loop amplitudes when one leg s becomes soft is given by:

$$\mathcal{A}_n^{1-loop}(1,\ldots,a,s,b,\ldots,n) \xrightarrow{k_s \to 0}$$

$$Soft^{tree}(a,s,b)\ \mathcal{A}_{n-1}^{1-loop}(1,\ldots,a,b,\ldots,n)$$

$$+\ Soft^{1-loop}(a,s,b)\ \mathcal{A}_{n-1}^{tree}(1,\ldots,a,b\ldots,n)$$

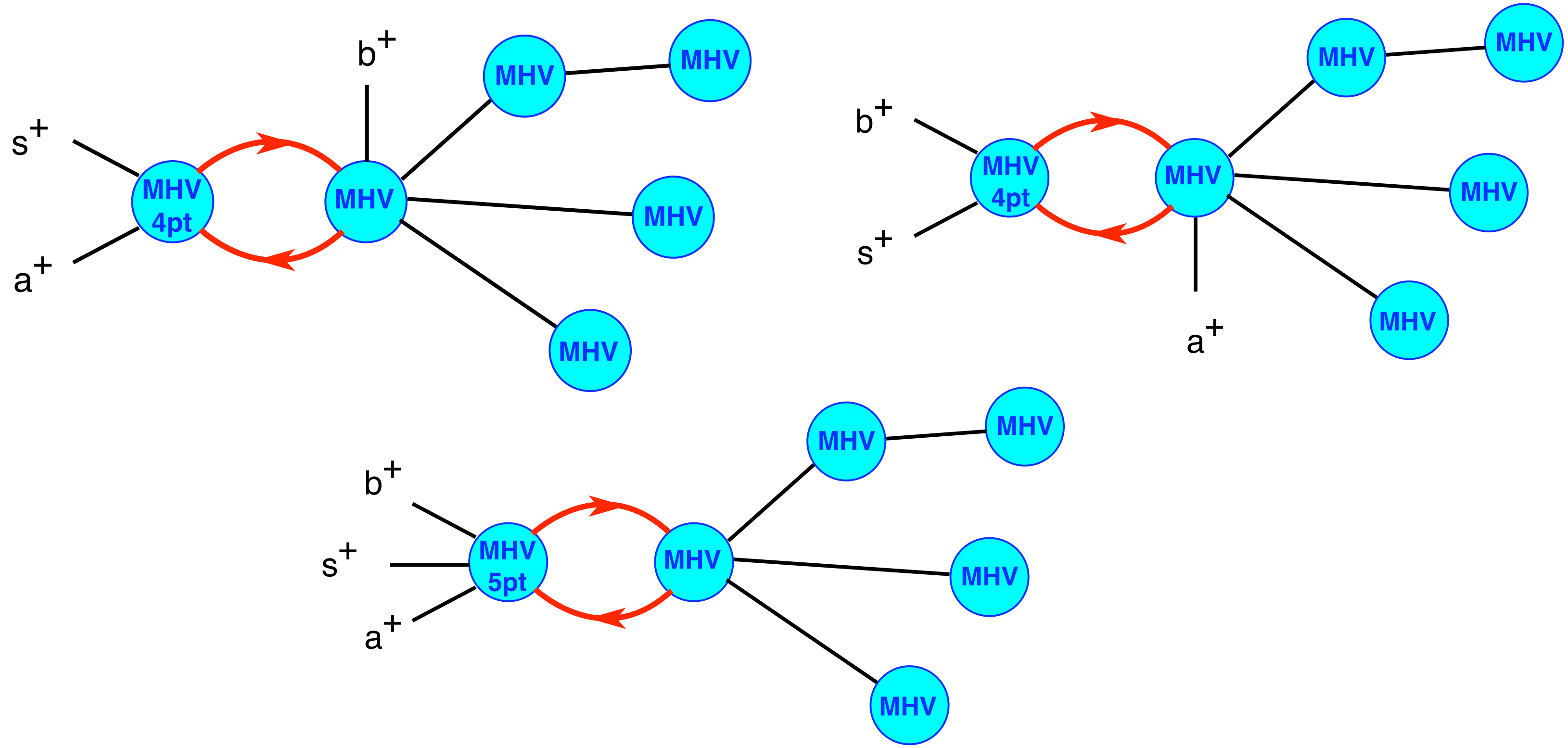
with

$$Soft^{tree}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\,\langle s\,b\rangle}$$

$$Soft^{1-loop}(a,s,b) = Soft^{tree}(a,s,b)\left(-\frac{c_\Gamma}{\varepsilon^2}\frac{\pi\varepsilon}{\sin(\pi\varepsilon)}\right)\left(-\frac{s_{ab}}{s_{as}s_{sb}}\mu^2\right)^{\varepsilon}$$

Soft Limits from MHV Diagrams

- For concreteness we discuss the soft limit: $a^+ s^+ b^+ \longrightarrow a^+ b^+$
- Three MHV diagrams contribute in this case, the first two being familiar from the collinear limits
- Again the MHV diagrams reproduce the all order, one-loop soft function



Conclusions

- MHV diagrams at one-loop in SYM
  - covariance (from FTT)
  - correct cuts (by construction)
  - correct soft and collinear limits (to all orders in ε)
- Further applications of FTT
  - rederivation of MHV 1-loop measure $d\mathcal{M}$
  - 1-loop measure for non-MHV amplitudes?
  - FTT applies also to massive/non-susy theories