Coefficients of one-loop integral functions by recursion

International Workshop:
From Twistors to Amplitudes

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Motivation

- The LHC collider at CERN is approaching
  - More perturbative calculations of amplitudes in QCD will be needed in order to interpret data from collider experiments
- Expressions for amplitudes at next-to-leading order *i.e.* at one-loop order will be needed to
  - Unravel new physics
    - Investigate the possibilities of supersymmetry extensions of the Standard Model
    - Search for the Higgs

    …

Amplitudes

- A traditional Feynman diagram approach is not very ideal for amplitude calculations
- The number of Feynman diagrams will grow very rapidly with the number of legs
- Amplitudes will contain various contractions of
  - Momentum vectors (p_i ¢ p_j)
  - Products of momentum vectors with external polarisation tensors such as (p_i ¢ ε_j) and (ε_i ¢ ε_j)

Hence amplitude expressions usually become very complicated

Amplitudes

Possible Simplifications

- Specifying the external polarisation tensors
  - Spinor-helicity formalism
  - Colour ordering of amplitudes
- Recursion techniques
- Use unitarity and a supersymmetric decomposition to evaluate loop amplitudes

Helicity states formalism

- In D=4 the SO(3,1) Lorentz group is locally isomorph to a SL(2) x SL(2) description in terms of spinors
- We can write the spinors in the SL(2) x SL(2) description as

$$\langle \lambda_1, \lambda_2 \rangle = -\epsilon^{ab} \lambda_{1a} \lambda_{2b}, \quad [\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$$

$$\lambda_a, \ (a = 1, 2) \qquad \tilde{\lambda}_{\dot{a}}, \ (\dot{a} = 1, 2)$$

Helicity states

- All momentum parts of amplitudes can be written in terms of spinors

$$2(p \cdot q) = s_{ij} = -\langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$$

- We can also define spin-1 polarisation tensors. Up to a gauge choice these are defined as (Xu,Zhang,Chang)

$$\varepsilon^{-}_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]}, \qquad \tilde{\varepsilon}^{+}_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}$$

- Also polaristion for other matter types such as gravitons, fermions, scalars etc

Scattering amplitudes in D=4

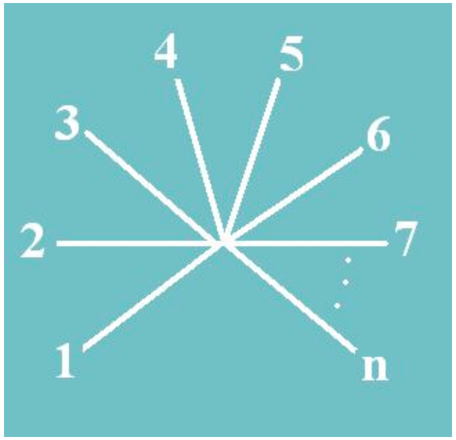
- Amplitudes can hence be expressed completely in terms of
  - The external helicies
  - The spinor variables

$$\lambda,\ \tilde{\lambda}$$

Color-ordering

- The convention is to look at the amplitude part which is leading in color
- For a gluon amplitude we concentrate on the term with coefficient.

$$\mathrm{Tr}(T_1\, T_2 \,..\, T_n)$$



Notation

- We will use the notation:

$$s_{i,i+1} \equiv K_{i,i+1}^2 = (p_i + p_{i+1})^2$$
$$t_{i,j} \equiv K_{i,j}^2 = (p_i + \ldots + p_j)^2 \,,$$

$$[k|K_{i,j}|l\rangle \equiv \langle k^+|\not{K}_{i,j}|l^+\rangle \equiv \langle l^-|\not{K}_{i,j}|k^-\rangle \equiv \langle l|K_{i,j}|k] \equiv \sum_{a=i}^{j} [k\,a]\,\langle a\,l\rangle \,,$$

$$\langle k|K_{i,j}K_{m,n}|l\rangle \equiv \langle k^-|\not{K}_{i,j}\not{K}_{m,n}|l^+\rangle \equiv \sum_{a=i}^{j}\sum_{b=m}^{n} \langle k\,a\rangle\,[a\,b]\,\langle b\,l\rangle \,,$$
$$\langle k|[q,K]|l\rangle \equiv \langle k|qK|l\rangle - \langle k|Kq|l\rangle \,.$$

New techniques

- Proposal that N=4 super Yang-Mills is dual to a string theory in twistor space? (Witten)
- Recently there has been a huge activity and progress in the calculation of tree and loop amplitudes for gauge theories and gravity
- New very efficient techniques for calculating amplitudes

Key feature to rapid progress:

New ideas + the realization that expressions for amplitudes in a helicity formalism seem to be much simpler than one would expect *a priori!*

Recursion for loops

- Recursion for a certain class of loop amplitudes which are finite have been considered by (Bern, Dixon, Kosower).
- Recursion for the rational pieces in one-loop QCD amplitudes (Bern,Dixon,Kosower; Forde, Kosower).
- Here we will consider the possibility of recursion for the pieces of amplitudes which can be obtained by unitarity cuts.

Recursion for loops

- Simple idea:
  - Use the generality of the proof for the BCFW tree relations (Britto, Cachazo, Feng, Witten)
  - Together with factorization properties for loop amplitudes

) to generate coefficients of integral functions in loop amplitudes in a similar way as for tree amplitudes.

Recursion for loops

- Universal factorization properties for loop amplitudes has to be understood (Bern, Chalmers).
- The structure of singularities that appear in dimensionally regularized amplitudes has to be known; simple poles, spurious singularities.
- The basis of integrals for which dimensionally regularized loop amplitudes may be expanded. Scalar boxes, triangles, bubbles (Bern, Dixon, Dunbar, Kosower)
- Proof of the BCFW recursion relation (Britto, Cachazo, Feng, Witten)
- Starting expressions for loop amplitudes to build up expressions for higher point amplitudes via recursion.

Review of BCFW Recursion for trees

- Consider a generic tree amplitude

$$A(p_1,p_2,..,p_n)$$

in complex momentum space.

- We consider it behaviour under the following shift of the spinors,

$$\tilde{\lambda}_a \to \tilde{\lambda}_a + z\tilde{\lambda}_b\,,$$
$$\lambda_b \to \lambda_b - z\lambda_a$$

- This shifts momentum on legs a and b into

$$p_a(z) = \lambda_a\tilde{\lambda}_a + z\lambda_a\tilde{\lambda}_b\,,$$
$$p_b(z) = \lambda_b\tilde{\lambda}_b - z\lambda_a\tilde{\lambda}_b\,.$$

Review of BCFW Recursion for trees

- Essential point: legs a and b will remain on-shell even after shift
- The amplitude transforms as

$$A(p_1, p_2, \ldots, p_n) \;\rightarrow\; A(p_1, p_2, \ldots, p_a(z), \ldots, p_b(z), \ldots, p_n) \;\equiv\; A(z)\,,$$

- We can now evaluate the contour integral over A(z)

$$\frac{1}{2\pi i}\oint \frac{dz}{z} A(z) \;=\; C_\infty \;=\; A(0) \;+\; \sum_\alpha \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}\,,$$

Review of BCWF Recursion for trees

(Britto, Cachazo, Feng, Witten)

Given that:

- If A(z) vanish for z ! 0
- A(z) is a rational function
- A(z) has simple poles in z

$$A(0) = -\sum_{\alpha} \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z} .$$

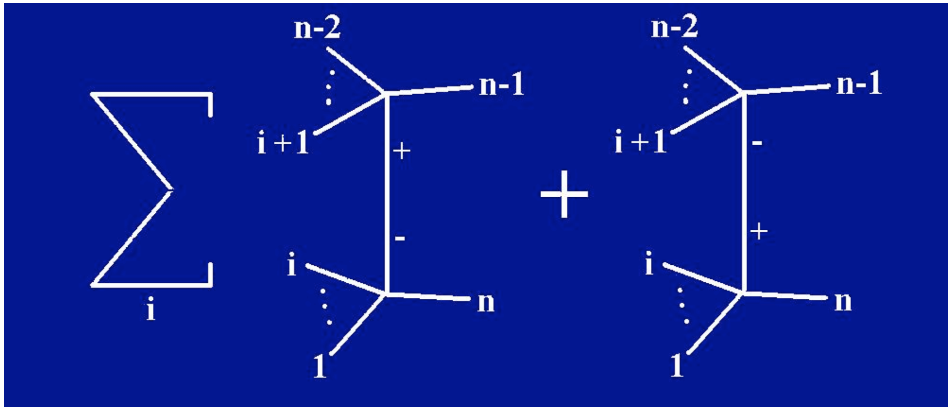
Factorization properties for trees

- The residues of the shifted amplitude is determined by the factorization properties of the amplitude.
- An on-shell tree amplitude will factories into a product of two on-shell tree amplitudes in the following way

$$A \xrightarrow{K_{i,j}^2 \to 0} A(k_i, \cdots, k_j, K_{i,j}) \times \frac{i}{K_{i,j}^2} \times A(k_{j+1}, \cdots, k_{i-1}, -K_{i,j}) \, .$$

Recursion for tree amplitudes

- At tree-level one can show that there are no other factorizations in the complex plane.
- Hence: $$A(0) = \sum_{\alpha,h} A^{h}_{n-m_\alpha+1}(z_\alpha)\frac{i}{K_\alpha^2}A^{-h}_{m_\alpha+1}(z_\alpha)\,,$$



Supersymmetric decomposition

- The cut parts of amplitudes in QCD for gluon scattering can be constructed from supersymmetric loop amplitudes.
- The three types of multiplets are:

$$A_n^{\mathcal{N}=4} \equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]} ,$$
$$A_n^{\mathcal{N}=1 \text{ vector}} \equiv A_n^{[1]} + A_n^{[1/2]} ,$$
$$A_n^{\mathcal{N}=1 \text{ chiral}} \equiv A_n^{[1/2]} + A_n^{[0]} .$$

- They are linked by the identity

$$A_n^{\mathcal{N}=1 \text{ vector}} \equiv A_n^{\mathcal{N}=4} - 3A_n^{\mathcal{N}=1 \text{ chiral}} .$$

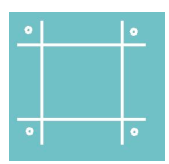
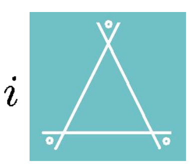
Supersymmetric decomposition

- For N=4 we sum over the states in the vector multiplet.
- For N=1 we sum over the states in the chiral multiplet.

- Results for the multiplets for various supersymmetric theories can be combined in to amplitudes for gluons including an additional scalar contribution.
- For full QCD amplitudes there is an additional rational piece due to the scalar loop.

Supersymmetric decomposition

- Super-symmetry imposes a simplicity of the expressions for loop amplitudes.
  - For N=4 only scalar boxes appear.
  - For N=1 scalar boxes, triangles and bubbles appear.
- Hence one-loop amplitudes are built up from a linear combination of terms (Bern, Dixon, Dunbar, Kosower).

$$A = \sum_i C_i F_i = \sum c_i$$

$$+ \sum t_i$$

$$+ \sum b_i$$

$$+ \mathrm{R}$$

Unitarity cuts

- Unitarity methods are building on the cut equation

$$C_{i,i+1,\ldots,j} = \mathrm{Im}_{(p_i+p_{i+1}+\ldots+p_j)^2>0} A^{1-\mathrm{loop}}$$

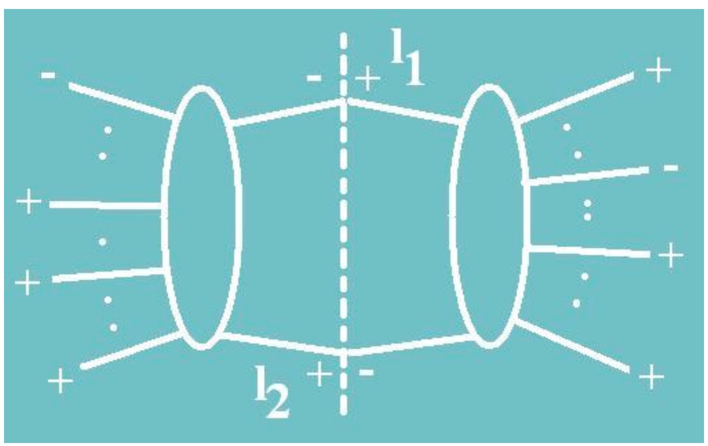
- Where the cut is

$$C_{i,\ldots,j} \equiv \frac{i}{2}\int dLIPS\Big[A^{\mathrm{tree}}(\ell_1, i, \ldots, j, \ell_2) \times A^{\mathrm{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)\Big].$$

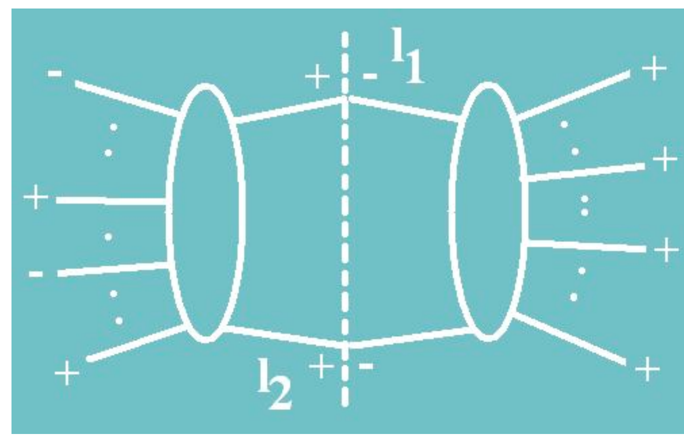
Unitarity cuts

- Loop amplitudes do inherit properties from tree amplitudes

Non-singlet



Singlet



Factorization properties for loop amplitudes

- Avoid the presence of logarithmic branch cuts.
  - Construct recursion relations not for amplitudes but for coefficients of the integral functions in one-loop amplitudes.
- Factorization properties of loop amplitudes has to be understood.
- Problems:
  - Complete set of poles in amplitudes not present in coefficients.
  - Spurious non-physical singularities can interfere in recursion.

Factorization properties for loop amplitudes

- Factorization of loop amplitudes

$$A_n^{\text{1-loop}} \xrightarrow{K^2_{i,i+m-1} \longrightarrow 0} \sum_{h=\pm} \Bigg[A_{m+1}^{\text{1-loop}}(\ldots, K^h_{i,i+m-1}, \ldots) \frac{i}{K^2_{i,i+m-1}} A^{\text{tree}}_{n-m+1}(\ldots, (-K_{i,i+m-1})^{-h}, \ldots)$$

$$+ A^{\text{tree}}_{m+1}(\ldots, K^h_{i,i+m-1}, \ldots) \frac{i}{K^2_{i,i+m-1}} A^{\text{1-loop}}_{n-m+1}(\ldots, (-K_{i,i+m-1})^{-h}, \ldots)$$

$$+ A^{\text{tree}}_{m+1}(\ldots, K^h_{i,i+m-1}, \ldots) \frac{i}{K^2_{i,i+m-1}} A^{\text{tree}}_{n-m+1}(\ldots, (-K_{i,i+m-1})^{-h}, \ldots) \mathcal{F}_n(K^2_{i,i+m-1}; p_1, \ldots, p_n) \Bigg],$$

- The function $\mathcal{F}_n(K_{i,i+m-1}{}^2; p_1, .., p_n)$ represents a non-factorization.

Factorization properties for loop amplitudes

- Collinear singularity

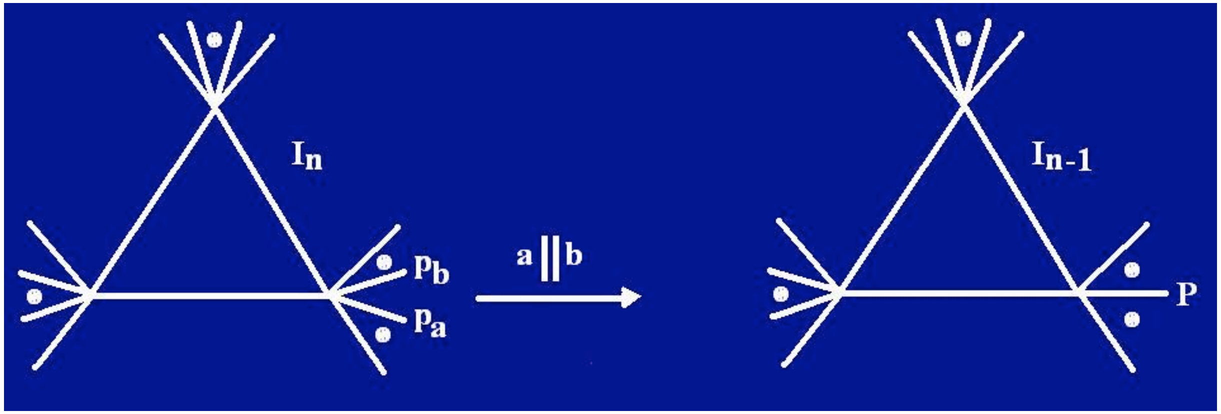
$$A_n^{\text{1-loop}}(\ldots, p_a, p_b, \ldots) \xrightarrow{a\|b} \sum_h \text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b}) A^{\text{1-loop}}_{n-1}(\ldots, (P)^h, \ldots) + \sum_h \text{Split}^{\text{1-loop}}_{-h}(a^{h_a}, b^{h_b}) A^{\text{tree}}_{n-1}(\ldots, (P)^h, \ldots),$$

- Or expanding out in terms of integral functions

$$\sum_i c_{i,n} I_{i,n} \xrightarrow{a\|b} \sum_h \text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b}) \sum_i c^h_{i,n-1} I_{i,n-1} + \sum_h \text{Split}^{\text{one-loop}}_{-h}(a^{h_a}, b^{h_b}) A^{h\,\text{tree}}_{n-1}.$$

Recursion for simple case

- Now we consider the behaviour of a single term in this expansion



$$c_{i,n} \xrightarrow{a||b} \sum_h \text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b})\; c^h_{i,n-1}\,,$$

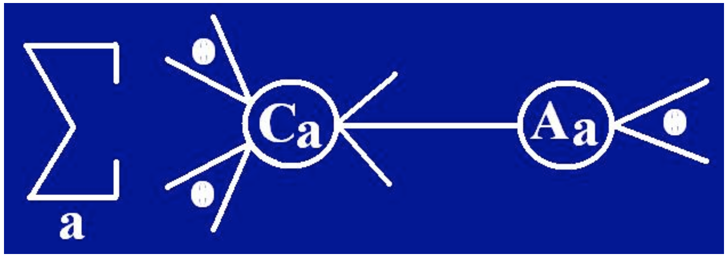
Recursion for simple case

- Applying the same logic to multiparticle factorizations we conclude that the coefficients behave as if they were tree amplitudes

$$c_{i,n} \xrightarrow{K^2\to 0} \sum_h A^h_{n-m+1}\,\frac{i}{K^2}\,c^{-h}_{i,m+1}\,.$$

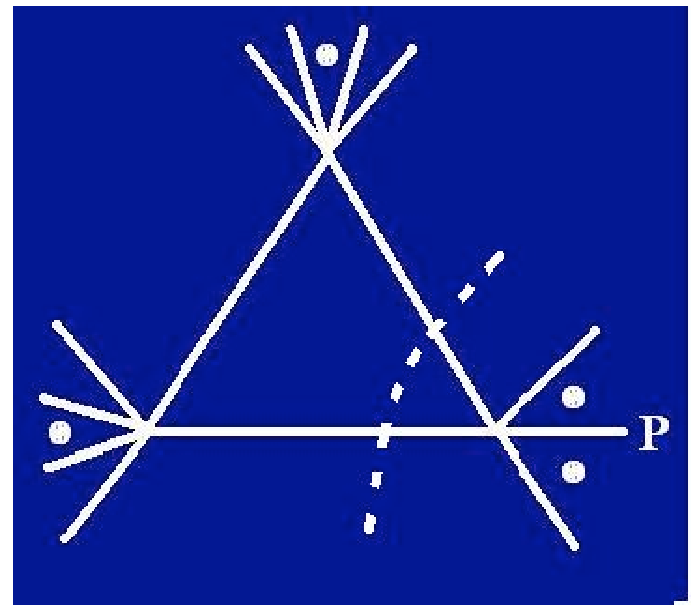
- Assuming well behaved denominators

$$c_n(0) = \sum_{\alpha,h} A^h_{n-m_\alpha+1}(z_\alpha)\,\frac{i}{K_\alpha^2}\,c^{-h}_{m_\alpha+1}(z_\alpha)\,,$$



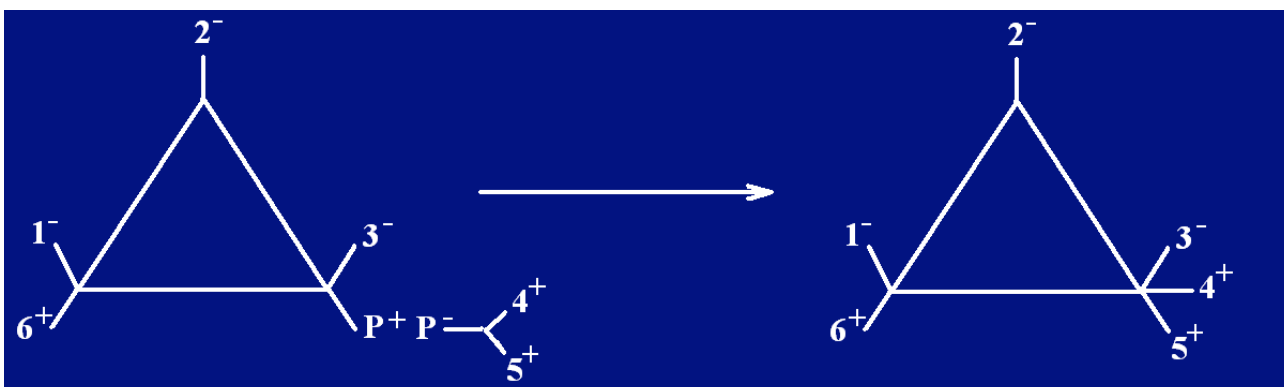
Recursion for simple case

- Sufficient conditions:
  - Shifted tree cluster well behaved as z ! 1
  - All loop momenta dependent kinematic poles are unmodified by the shift.



Example 5pt ! 6pt

- As a simple example of a recursion that satisfies the simple criteria
- We can consider recursion of a 5pt integral coefficient into a 6pt integral coefficient



Example 5pt ! 6pt

- Integral functions:

$$\mathrm{K}_0(r) = \frac{1}{\epsilon\,(1-2\,\epsilon)}(-r)^{-\epsilon} = \left(-\log(-r) + 2 + \frac{1}{\epsilon}\right) + \mathcal{O}(\epsilon)\,,$$

$$\mathrm{L}_0(r) = \frac{\log(r)}{1-r}\,, \qquad \mathrm{L}_2(r) = \frac{\log(r) - (r - 1/r)/2}{(1-r)^3}\,.$$

- Five gluon (N=1) amplitude is

$$A^{\mathcal{N}=1\ \mathrm{chiral}}(1^-,2^-,3^-,4^+,5^+) = \frac{1}{2}A^{\mathrm{tree}}(1^-,2^-,3^-,4^+,5^+)\left(\mathrm{K}_0(t_{5,1}) + \mathrm{K}_0(t_{3,4})\right)$$
$$+\frac{1}{2}c^{\mathcal{N}=1\ \mathrm{chiral}}(5^+,1^-;2^-;3^-,4^+)\frac{\mathrm{L}_0\left(-t_{5,1}/(-t_{5,2})\right)}{t_{5,2}}\,,$$

Example 5pt ! 6pt

- The 5pt point coefficient is

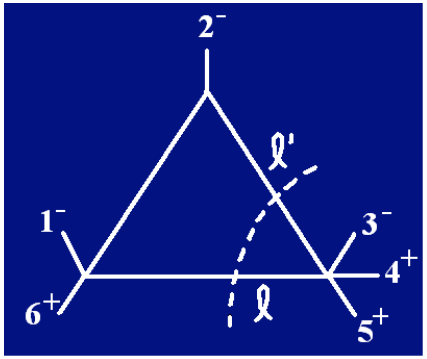
$$c_5 = c(5^+, 1^-; 2^-; 3^-, 4^+) \; = \; -i \frac{[4|P|5]\,[4|\tilde{P}|5]\;\,[4|[k_2, K_{5,1}]|5]}{[5\,1][1\,2][2\,3][3\,4]}\,.$$

- We will do the shift

$$\begin{aligned} \tilde{\lambda}_3 \;&\rightarrow\; \tilde{\lambda}_3 \;-\; z\tilde{\lambda}_4\,, \\ \lambda_4 \;&\rightarrow\; \lambda_4 \;+\; z\lambda_3\,. \end{aligned}$$

Example 5pt ! 6pt

- We can first check that the two criteria for recursion is satisfied
- Looking at the cutted diagram
- The shifted tree amplitude is



$$A^{\text{tree}}(3^-,4^+,5^+,\ell_s^+,{\ell'_s}^-;z) \;=\; i\frac{\langle 3\,\ell_s\rangle^2\,\langle 3\,\ell'_s\rangle^2}{\langle 3\,4\rangle\,(\langle 4\,5\rangle + z\,\langle 3\,5\rangle)\,\langle 5\,\ell_s\rangle\,\langle \ell_s\,\ell'_s\rangle\,\langle \ell'_s\,3\rangle},$$

Example 5pt ! 6pt

- Both criteria immediately satisfied

  - z-dependence factors out of the integrant
  - The shifted coefficient times the integral vanishes as |z|!1

$$c_6(6^+,1^-;2^-;3^-,4^+,5^+) \xrightarrow{4\|5}$$
$$\mathrm{Split}^{\mathrm{tree}}_{-}(4^+,5^+)\, c_5(6^+,1^-;2^-;3^-,(4+5)^+)\,.$$

Example 5pt ! 6pt

- Hence we can do recursion

$$z_1 = -\langle 4\,5\rangle/\langle 3\,5\rangle\,, \qquad \omega\bar\omega = \langle 3|K_{4,5}|4]\,,$$

$$[\hat 4\,\hat K_{4,5}] = [4|K_{4,5}|3\rangle/\bar\omega\,, \qquad [5\,\hat K_{4,5}] = [5|K_{4,5}|3\rangle/\bar\omega\,,$$

$$[2\,\hat 3] = [2\,3] - z[2\,4] = [2|K_{3,4}|5\rangle/\langle 3\,5\rangle\,, \qquad [\hat 3\,\hat K_{4,5}] = t_{3,5}/\bar\omega\,,$$

$$c_6 = c(6^+,1^-;2^-;\hat 3^-,\hat K_{45}^+)\,\frac{i}{s_{45}}\,A(\hat 4^+,5^+,(-\hat K_{45})^-)\,,$$

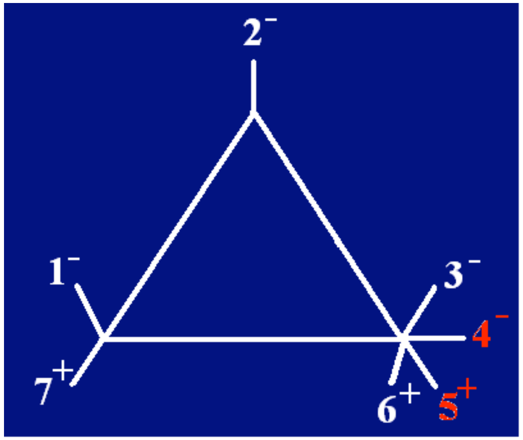
$$= -i\,\frac{[\hat K_{45}|P|6]\,[\hat K_{45}|\tilde P|6]\,\left[\hat K_{45}|[k_2,K_{6,2}]|6\right]}{[6\,1][1\,2][2\,\hat 3][\hat 3\,\hat K_{45}]}\;\frac{i}{s_{45}}\;\frac{(-i)\,[\hat 4\,5]^3}{[(-\hat K_{45})\,\hat 4][5\,(-\hat K_{45})]}\,,$$

$$= i\,\frac{\langle 3|K_{3,5}P|6]\,\langle 3|K_{3,5}\tilde P|6]\,\langle 3|K_{3,5}[k_2,K_{6,2}]|6]}{[2|K_{3,5}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,t_{3,5}}\,.$$

(Bidder, NEJBB, Dixon, Dunbar)

General case and 7 and 8

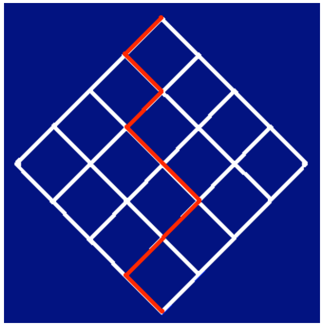
- The recursion can easily be extended to more general cases.
- 7pt: Two orders for recursion:
  - First add plus leg then minus
  - First add minus leg then plus

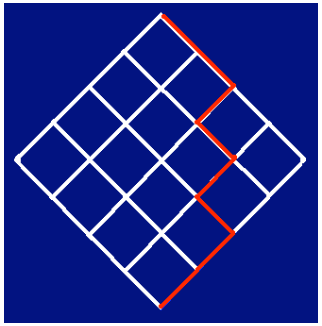


- The different orders of recursions can be displayed as paths in helicity diagrams

Helicity diagrams

- The various contributions to the coefficients can be organized in terms of helicity diagrams





- A path going to the left means a minus path
- A path going to the right means a plus path

All paths (T_R, T_L)

$$
\begin{aligned}
T_{P_L,P_R} \;=\; & i\,(-1)^{l+\kappa+\kappa'} \frac{\langle l|Q_R\,P\,Q_L|1\rangle\,\langle l|Q_R\,\tilde{P}\,Q_L|1\rangle\,\langle l|Q_R[k_2,K_L]Q_L|1\rangle\,\langle r,r+1\rangle}{[1\,2][2\,3]...[l-1,l]\langle l,l+1\rangle...\langle n-1,n\rangle\langle n,1\rangle} \\
& \times \frac{\prod_{i=1}^{N}[\alpha_i-1,\alpha_i]\;\prod_{i=2}^{N}\langle\rho_i,\rho_i+1\rangle}{\langle\rho_N|K_N|\alpha_N-1]\;\prod_{i=1}^{N-1}\langle\rho_i|K_i|\alpha_i-1]\;[\alpha_i|\bar{K}_i|\rho_{i+1}+1\rangle} \\
& \times \frac{\prod_{j=1}^{N'}[\beta_j,\beta_j+1]\;\prod_{j=2}^{N'}\langle\sigma_j-1,\sigma_j\rangle}{\langle\sigma_{N'}|K'_{N'}|\beta_{N'}+1]\;\prod_{j=1}^{N'-1}\langle\sigma_j|K'_j|\beta_j+1][\beta_j|\bar{K}'_j|\sigma_{i+1}-1\rangle} \\
& \times \frac{1}{K_N^2\,K_{N'}^2\;\prod_{i=1}^{N-1}\bar{K}_i^2\,K_i^2\;\;\prod_{j=1}^{N'-1}K'^2_j\bar{K}'^2_j}\,,
\end{aligned}
$$

All paths (T_R,T_L)

- Where

$$
\begin{aligned}
K_i &= K_{\alpha_i,\rho_i}\,, & K'_j &= K_{\beta_j,\sigma_j}\,,\\
\bar{K}_i &= K_{\alpha_i,\rho_{i+1}}\,, & \bar{K}'_j &= K_{\beta_j,\sigma_{j+1}}\,,\\
K_R &= K_{\alpha_\kappa,\rho_1}\,, & K'_L &= K_{\beta_{\kappa'},\sigma_1}\,,\\
Q_R &= K_N\bar{K}_{N-1}K_{N-1}\ldots\bar{K}_1K_1\,, & Q_L &= K'_1\bar{K}'_1\ldots K'_{N'-1}\bar{K}'_{N'-1}K'_{N'}\,,\\
P &= k_mK_R\,,1\,, & \tilde{P} &= K_Rk_m\,,1\,.
\end{aligned}
$$

NMHV result

- Reproduction of result for NMHV case split-helicity amplitudes N=1 (Bidder, NEJBB, Dunbar, Perkins)

$$A_n^{N=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+)$$

$$= \frac{A^{\text{tree}}}{2}\left(\mathrm{K}_0(s_{n1})+\mathrm{K}_0(s_{34})\right) - \frac{i}{2}\sum_{r=4}^{n-1}\hat{d}_{n,r}\,\frac{\mathrm{L}_0[t_{3,r}/t_{2,r}]}{t_{2,r}}$$

$$-\frac{i}{2}\sum_{r=4}^{n-2}\hat{g}_{n,r}\,\frac{\mathrm{L}_0[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}} - \frac{i}{2}\sum_{r=4}^{n-2}\hat{h}_{n,r}\,\frac{\mathrm{L}_0[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}},$$

NMHV result

- where

$$\hat{d}_{n,r} = \frac{\langle 3|K_{3,r}K_{2,r}|1\rangle^2 \langle 3|K_{3,r}\left[k_2, K_{2,r}\right]K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle[2|K_{2,r}|r+1\rangle \; \langle 3\,4\rangle \ldots \langle r-1\, r\rangle \; \langle r+1\, r+2\rangle \ldots \langle n\,1\rangle \; t_{2,r}\, t_{3,r}},$$

$$\hat{g}_{n,r} = \sum_{j=1}^{r-3} \frac{\langle 3|K_{3,j+3}K_{2,j+3}|1\rangle^2 \langle 3|K_{3,j+3}K_{2,j+3}\left[k_{r+1}, K_{2,r}\right]|1\rangle \langle j+3\; j+4\rangle}{[2|K_{2,j+3}|j+3\rangle[2|K_{2,j+3}|j+4\rangle \; \langle 3\,4\rangle \langle 4\,5\rangle \ldots \langle n\,1\rangle \; t_{3,j+3}\, t_{2,j+3}},$$

$$\hat{h}_{n,r} = (-1)^n \hat{g}_{n,n-r+2}\Big|_{(123..n)\to(321n..4)}.$$

NMHV result

- New result for NMHV case split-helicity amplitudes $A^{[0]}$

$$A_n^{[0]}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) =$$

$$\frac{1}{3}A_n^{N=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) - \frac{i}{3}\sum_{r=4}^{n-1}\hat{d}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{2,r}]}{t_{2,r}^3}$$

$$-\frac{i}{3}\sum_{r=4}^{n-2}\hat{g}_{n,r}\frac{\mathrm{L}_2[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}^3} - \frac{i}{3}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}^3}$$

$$+\ \text{rational}\,,$$

NMHV result

- where

$$\hat{d}_{n,r} = \frac{\langle 3|K_{3,r}k_2|1\rangle \langle 3|k_2 K_{2,r}|1\rangle \langle 3|K_{3,r}\left[k_2, K_{2,r}\right] K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle [2|K_{2,r}|r+1\rangle \langle 3\,4\rangle \ldots \langle r-1\,r\rangle \langle r+1\,r+2\rangle \ldots \langle n\,1\rangle},$$

$$\hat{g}_{n,r} = \sum_{j=1}^{r-3} \frac{\langle 3|K_{3,j+3}K_{2,j+3}P|1\rangle \langle 3|K_{3,j+3}K_{2,j+3}\tilde{P}|1\rangle \langle 3|K_{3,j+3}K_{2,j+3}\left[k_{r+1}, K_{2,r}\right]|1\rangle \langle j+3\,j+4\rangle}{[2|K_{2,j+3}|j+3\rangle [2|K_{2,j+3}|j+4\rangle \langle 3\,4\rangle \langle 4\,5\rangle \ldots \langle n\,1\rangle\; t_{3,j+3}\, t_{2,j+3}},$$

$$\hat{h}_{n,r} = (-1)^n \hat{g}_{n,n-r+2}\big|_{(123..n)\to(321n..4)},$$

and $P = k_{r+1}K_{r+1,1}$ and $\tilde{P} = K_{r+1,1}k_{r+1}$.

New results

- With the supersymmetric decomposition the split helicity amplitude contributions that we have considered are pieces of QCD amplitudes.
- N=4 amplitudes are already calculated.
- Split-helicity trees (Britto,Feng,Roiban,Spradlin,Volovich)
- Rational pieces only missing ingredient..

Conclusion

- In order to calculate amplitudes in QCD, methods that build on previous calculations are preferable.
  - Unitarity methods,
  - Recursive techniques are examples of this.
- Calculations of amplitudes via recursion are usually the most efficient option possible. For loop amplitudes integrations are avoided via recursion.
  - In this talk we have discussed that in some cases recursion for integral coefficients is possible.
- Set of criteria for valid recursions.

Conclusion

- Apply to large classes of coefficients
- Illustrated our recursion with an explicit result for *n* gluon scattering NMHV amplitudes with split helicity.
- Perhaps possible to extend by looking at more generic shifts although it is not clear what such a shift should be..
- All coefficients for loops in this way?

Additional results will be needed for use in colliders. Still long way to go.