CSW rules from exotic recursive shifts

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Outline

1. BCFW and CSW
2. CSW Rules for Yang–Mills
3. CSW Rules for Gravity
4. Perspectives

Outline

1. BCFW and CSW
   - Britto, Cachazo, Feng, Svrček, Witten (and others).
2. CSW Rules for Yang–Mills
3. CSW Rules for Gravity
4. Perspectives

BCFW Recursion – A

In its purest form, consists of shifting two momenta by a reference vector (defining $\widehat{A}(z)$)

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and using Cauchys theorem to write

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$$= A(0) - \sum_{i=3}^{n} A(\widehat{2}, \ldots, i, -\widehat{P}_{2i}) \frac{i}{P_{2i}^2} A(\widehat{P}_{2i}, i+1, \ldots, \widehat{1})$$

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If we can calculate C_∞ (*e.g.*, it's zero if $\widehat{A}(z) \to 0$ as $z \to \infty$), we have an on-shell recursion relation.

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- Much more complicated shifts are allowed as long as they conserve momentum.
- Three-vertices may be non-zero, however (roughly), the holomorphic MHV three-vertex is zero if the shifted momentum sitting on it has its holomorphic spinor shifted (and vice versa).
- Proving that $\lim_{z\to\infty} \widehat{A}(z) = 0$ can be difficult. Generally, the behaviour becomes safer when negative helicity gluons have their anti-holomorphic spinor shifted (and vice versa).

CSW Rules – A

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3. For insertion in the PT amplitudes, the holomorphic spinors of the internal lines are $[\eta P|$ where $[\eta|$ is a constant, but otherwise arbitrary, anti-holomorphic spinor.
4. Or, equivalently, use the spinor of

$$\widetilde{P}^\mu = P^\mu - \frac{P^2}{2P \cdot \eta}\eta^\mu$$

CSW Rules – B

Some points worth noting here:

- The rules combine projected momenta along with untouched propagators (hints a connection to BCFW).
- Centered around negative helicity gluons. Uses vertices with the minimum sensible number of negative gluons.
- Three-point MHV appears while googlies are absent.

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Be aware that the CSW rules for Yang–Mills as presented here have been rigorously proven (original CSW and BCFW papers together).

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Strong CSW construction:

- $|\widetilde{P}_i\rangle \propto |P_i\eta]$, $|\widetilde{P}_i] \propto |P_i\eta\rangle$.
- $P_i^\flat = P_i - \frac{P_i^2}{2P_i\cdot\eta}\eta$
- Yang–Mills may have a strong CSW, gravity does not.
- Strong CSW seems essential for the BST application to loops.

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   - [hep-th/0508206]
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- Presence of an antiholomorphic reference spinor $|\eta]$.
- Controlled $z \to \infty$ behaviour.

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Conclusion:

- Shift the antiholomorphic spinors of the negative helicity gluons by a reference spinor.

Decomposing NMHV Amplitudes – 1

Take the amplitude $A(1^+, \ldots, m_1^-, \ldots, m_2^-, \ldots, m_3^-, \ldots, n^+)$ and make the shifts

$$\begin{aligned} |m_1] &\rightarrow |m_1] + z\langle m_2 m_3\rangle|\eta] \\ |m_2] &\rightarrow |m_2] + z\langle m_3 m_1\rangle|\eta] \\ |m_3] &\rightarrow |m_3] + z\langle m_1 m_2\rangle|\eta] \end{aligned}$$

which conserve momentum due to the Schouten identity,

$$\langle m_2 m_3\rangle\langle m_1| + \langle m_3 m_1\rangle\langle m_2| + \langle m_1 m_2\rangle\langle m_3| = 0.$$

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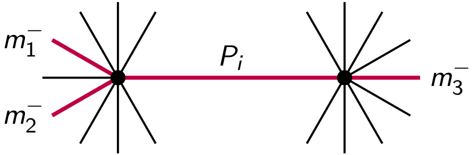
As $z \rightarrow \infty$, the shifted amplitude goes as z^{-2} or better, so there are no boundary terms.

Decomposing NMHV Amplitudes – 2

The channels where propagators depend on z are exactly those with the structure



and the internal momentum is displaced to

$$\widehat{P}_i = P_i - \frac{P_i^2}{\langle m_j P_i \eta]} |m_j\rangle[\eta|, \qquad |\widehat{P}_i\rangle \propto |P_i \eta], \qquad |\widehat{P}_i] \propto |P_i m_j\rangle$$

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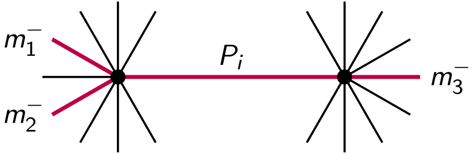

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This is exactly the CSW prescription at NMHV.

Going Beyond NMHV

Shifting only three negative helicity gluons when there are four or more would produce only a subset of the actual CSW diagrams, so we take the democratic approach

$$|m_i] \rightarrow |m_i] + z r_i |\eta]$$

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For a N^nMHV amplitude, the shifted amplitude goes as $z^{-(n+1)}$ or better as $z \to \infty$.

Proof by Induction – A

Let us now assume that the CSW rules hold for $\mathrm{N}^{(<n)}$MHV and apply the above shift to a N^nMHV amplitude. This results in:

$$\sum_j \widehat{A}^L(z_j)\frac{i}{P_j^2}\widehat{A}^R(z_j)$$

where j runs over all internal propagators allowed in the CSW rules.

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where j runs over all internal propagators allowed in the CSW rules.

Next step is to apply the CSW rules to the shifted amplitudes. Pick a j and the contribution of one diagram from each shifted amplitude. This is a product of

- MHV vertices (shifted, but that doesn't matter!) corresponding to a particular N^nMHV diagram,
- Propagators corresponding to that same diagram, all but one shifted.

Proof by Induction – B

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Thus, every diagram gives a contribution given by the CSW rules.

Proof Outline

The procedure used can be rephrased as

1. Make a shift and expand the result
2. Identify each term with a CSW diagram
3. Split each term into a shift independent term (vertices) and a shift dependent term (propagators)
4. Argue that the sum of shift dependent terms of each diagram come from the shifting of something which is well-defined off-shell (a product of propagators).
5. Each diagram now contributes the shift independent terms times a term which is 'reverted' to an unshifted state.

Consequences

This proves CSW, but only in the weak sense, since $|\eta\rangle$ is not a constant spinor.

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Method should be applicable more generally to field theories where many amplitudes (in this case helicity configurations) vanish.

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   - Bjerrum-Bohr, Dunbar, Ita, Perkins, KR [hep-th/0509016]
4. Perspectives

Extension to Gravity

Gravity has two main complications

1. $z \to \infty$ behaviour is far less understood. We will content ourselves with the fact that the gravity CSW rules do not introduce boundary terms themselves, and that they pass numerical tests. (z^{-n} is required for N^nMHV).

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1. $z \to \infty$ behaviour is far less understood. We will content ourselves with the fact that the gravity CSW rules do not introduce boundary terms themselves, and that they pass numerical tests. (z^{-n} is required for N^{n}MHV).
2. MHV vertices depend on antiholomorphic spinors. This means that the vertices are z dependent, and we need to be more cunning to fix this dependence to something constant. Also, proof by induction will not work, as the terms constributing to the same diagram do not obey momentum conservation.

Proof Overview – A

Begin in the same way as with Yang–Mills by choosing a set r_i such that $\sum_i r_i \langle m_i| = 0$. This allows us to write the amplitude as

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For each term in the sum, choose now a new set of r_i such that $(\widehat{P}_j)^2 = 0$, and use this shift to decompose

$$\widehat{A}^L(z_j)\widehat{A}^R(z_j) = \sum_k \widehat{\widehat{A}}^1 \widehat{\widehat{A}}^2 \widehat{\widehat{A}}^3 \frac{i}{\widehat{P}_k^2}$$

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For a given diagram (= choice of propagators) all $n+2$ negative helicity gravitons end up being shifted such that all the n propagators are on shell. This applies no matter which order the shifts were done in. Along with conservation of momentum, this fixes uniquely the a_i in

$$|m_i] \to |m_i] + a_i|\eta]$$

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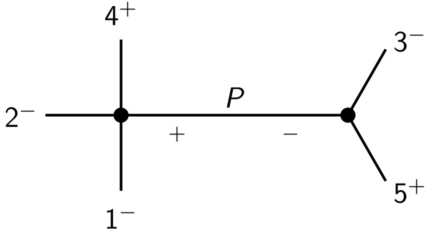
Conclusion:

- Each gravity CSW diagram contributes the unshifted propagators times vertices shifted as to put all propagators on-shell.

Example – A

The following diagram is a contribution to $A(1^-, 2^-, 3^-, 4^+, 5^+)$



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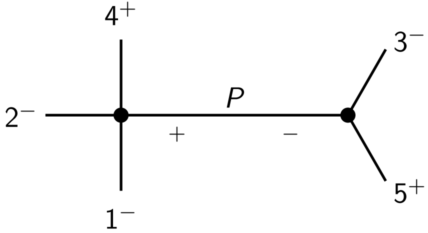

The condition that $\widehat{P}$ be on-shell is

$$[\widehat{3}5] = [35] + a_3[\eta 5] = 0$$

while conservation of momentum can be written

$$a_2\langle 12\rangle + a_3\langle 13\rangle = 0 \quad , \quad a_1\langle 21\rangle + a_3\langle 23\rangle = 0$$

Example – B

The solution is

$$a_1 = \frac{\langle 23\rangle[35]}{\langle 21\rangle[\eta 5]}, \quad a_2 = \frac{\langle 13\rangle[35]}{\langle 12\rangle[\eta 5]}, \quad a_3 = -\frac{[35]}{[\eta 5]}$$

and for plugging into the 3- and 4-point MHV gravity amplitudes we need just

$$|\widehat{P}\rangle \propto |P\eta], \quad [\widehat{\widehat{12}}] = [12] - \frac{[\eta(1+2)35]}{\langle 12\rangle[\eta 5]}$$

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When seeing the result, it does not seem terribly surprising, because it forces/shifts the vertices on-shell in a "democratic" manner, while retaining the propagators.

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In fact, $\mathrm{N}^n\mathrm{MHV} \sim z^{-n}$ comes out in both Yang–Mills and gravity, and is a general consequence one can expect in other cases.

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Generalizing to other Theories

(Weak) CSW formulations already exist for QCD...

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CSW constructions for other theories would be highly interesting (non-QCD, QCD loops, massive theories).

To Do

The following projects would be helfpful

1. Prove strong CSW for QCD.
2. Extend proof of weak CSW beyond pure QCD.
3. Understand $z \to \infty$ behaviour (important).
4. Apply to loop calculations ($\mapsto$ Andi's talk).
5. Find dramatic and unexpected applications.

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- A weak CSW construction for gravity has been given.
- A theoretical base for application and extension of CSW constructions is now in place.