Twistor actions for non-self-dual fields

a new foundation for twistor-string theory

L.J.Mason

The Mathematical Institute, Oxford
lmason@maths.ox.ac.uk

From twistors to amplitudes, Queen Mary College

Part 1: JHEP10(2005)009, hep-th/0507269,
Part 2: hep-th/0510262 with D.Skinner

Outline

1. Twistor action for CSW twistor strings
   - Introduction to Witten's conjecture
   - Twistor Actions
   - Derivation of CSW twistor-strings
   - Summary

2. Ambitwistor Lagrangian
   - Ambitwistor space
   - Ambitwistor Lagrangians
   - Perturbation theory

Asymptotically, gauge fields $\rightsquigarrow$ linearised fields
these decompose into left (ASD) and right handed (SD).
Representations on $\mathbb{M}$ for

- left handed are

$$A \in \Omega^1 \otimes \mathfrak{g} / \mathrm{d}\left(\Omega^0 \otimes \mathfrak{g}\right) \qquad \text{such that} \qquad \mathrm{d}A \in \Omega^{2-} \otimes \mathfrak{g}$$

- right handed are

$$G = \mathrm{d}A \in \Omega^{2+} \otimes \mathfrak{g}\,, \qquad \mathrm{d}G = 0\,.$$

where $\Omega^{2\pm}$ are the $\pm$-dual 2-forms under Hodge $*$.

The generating function (wave function) $\mathcal{A}[A, G]$ determine the n-particle scattering amplitudes by

$$\mathcal{A}[G_1, \ldots, G_q, A_q+1, \ldots A_n] = \delta_{G_1} \ldots \delta_{G_q} \delta_{A_{q+1}} \ldots \delta_{A_n} \mathcal{A}[A, G]|_{A=0=G}$$

It is an axiom of QFT that

$$\mathcal{A}[A, G] = \int_{\mathrm{d}A' \sim \mathrm{d}A + G \text{ at } \infty} \mathrm{D}A' \exp \frac{i}{\hbar} S[A']$$

Classically $\mathcal{A}[A, G] = \exp \frac{i}{\hbar} S[A']$

here A' = classical solution with $\mathrm{d}A' \sim \mathrm{d}A + G$ at ∞.

Euclidean signature twistor correspondence

- $\mathbb{PT} = \mathbb{CP}^3 = \{Z^\alpha \in \mathbb{C}^4\}/\{Z^\alpha \sim \lambda Z^\alpha\}$, $\lambda \in \mathbb{C}^*$, $\alpha = 0, \ldots 3$.
- $\mathbb{PT}' := \mathbb{CP}^3 - \mathbb{CP}^1$,
- $Z^\alpha = (\omega^A, \pi_{A'})$, incidence relation with $x^{AA'} \in \mathbb{M}$

$$\omega^A = x^{AA'}\pi_{A'}.$$

- In Euclidean signature have $p : \mathbb{PT}' \to \mathbb{M}$, fibre $\mathbb{CP}^1$ with homogeneous coords $\pi_{A'}$.
- $\mathcal{O}(n)$ = sheaf of holomorphic functions on $\mathbb{PT}$ of homogeneity n, $f(\lambda Z) = \lambda^n f(Z)$.
- $D^3 Z = \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha \mathrm{d}Z^\beta \mathrm{d}Z^\gamma \mathrm{d}Z^\delta \in \Omega^3(4)$.

Use p^* and p_* with $p : \mathbb{PT} \to \mathbb{E}$:

Define $E \to \mathbb{PT}'$ by $E = p^* E$ (abuse of notation!).

- Left handed
  - Let $a =$ projection of $p^* A$ into $\Omega^{0,1}(\mathrm{End}(E))$.
  - $\mathrm{d}A \in \Omega^{2-} \Rightarrow \bar{\partial} a = 0$
  - Gauge freedom $a \to a + \bar{\partial} f$ so $[a] \in H^1(\mathbb{PT}', \mathrm{End}(E))$.
- Right handed $G \leftrightarrow g \in H^1(\mathbb{PT}', \mathcal{O}(-4))$ by

$$G(x) = \int_{p^{-1}(x)} g \wedge D^3 Z \,.$$

Consider the generating functional $\mathcal{A}[a, g]$ instead of $\mathcal{A}[A, G]$.

Super-twistor space $\mathbb{PT}'_s = \mathbb{CP}^{3|4} - \mathbb{CP}^{1|4}$

$$\mathbb{CP}^{3|4} = \{(Z^\alpha, \eta^i) \in \mathbb{C}^{4|4}\}/\{(Z, \eta) \sim (\lambda Z, \lambda\eta)\}$$

where η^i anticommute, $i = 1, \ldots, 4$.

- $\mathbb{CP}^{3|4}$ is Calabi-Yau as

$$\Omega_s = D^3 Z \wedge \mathrm{d}\eta^1 \wedge \mathrm{d}\eta^2 \wedge \mathrm{d}\eta^3 \wedge \mathrm{d}\eta^4$$

  has weight zero, $\mathrm{d}(\lambda\eta^i) = (\mathrm{d}\eta^i)/\lambda$ for η^i anticommuting.
- Generic $a_s \in H^1(\mathbb{PT}'_s, \mathrm{End}(E)) \leftrightarrow N = 4$ SYM multiplet

$$a_s = a + \eta^i b_i + \ldots + \eta_1\eta_2\eta_3\eta_4 g.$$

Conjecture: [Witten] Let $\bar{\partial}_{a_s} = \bar{\partial} + a_s$, then

$$\mathcal{A}[a, g] = \sum_{d=1}^{\infty} \int_{\mathscr{M}_s^d} \mathrm{Det}(\bar{\partial}_{a_s}|_C) \mathrm{d}\mu$$

$\mathscr{M}_s^d =$ contour in moduli space of connected algebraic curves degree d in $\mathbb{PT}_s$, $C \in \mathscr{M}_s^d$, $\mathrm{d}\mu$ some measure.

- l :=Number of loops, $g :=$ genus, then $g \leq l$.
- $\mathscr{M}_s^d$ contribution $\leftrightarrow$ processes with $d + 1 - l$ SD gluons.

CSW formulation; maximally disconnected curves

If $C = d$ lines then $\mathscr{M}^d_s = \mathbb{M}^d_s/\mathrm{Sym_d}$, *but* must include propagators from SUSY Chern-Simons action

$$S_{\mathrm{asd}}[a_s] = \int_{\mathbb{PT}_s} \mathrm{CS}(a_s) \wedge \Omega_s$$

$\rightsquigarrow$ diagrammatic formalism with MHV diagrams glued together with Chern-Simons propagators (also a 3-vertex).
Gluing can be done on-shell.

Gukov, Motl & Nietzke argue $\Leftrightarrow$ connected version.

Chalmers & Siegel action is

$$S_{\mathrm{asd}}[A,G] = \int_{\mathbb{M}} \mathrm{tr} G \wedge F\,, \qquad G \in \Omega^{2+}(\mathrm{End}(E))\,.$$

$\rightsquigarrow D_A G = 0$, & $F \in \Omega^{2-}$: $A =$ASD, $G =$SD but not fully coupled.
For full Yang Mills include $I[G] = \int_{\mathbb{M}} \mathrm{tr} G \wedge G$

$$S[A,G] = \int_{\mathbb{M}} \mathrm{tr} G \wedge F - \frac{\epsilon}{2} \int_{\mathbb{M}} \mathrm{tr} G \wedge G\,.$$

$\rightsquigarrow F^+ = \epsilon G,\; D_A G = D_A^* F = 0.$

Theorem (Ward)

Given connection $D = \mathrm{d} + A$ on $E \to \mathbb{M}$, let $\bar{\partial}_a$ be the d-bar operator induced on p^*E by p^*A, then $f_a := \bar{\partial}_a^2 = 0 \Leftrightarrow F^+ = 0$. E and A can be reconstructed from $(p^*E, \bar{\partial}_a)$.

Witten gave the action

$$S_{\mathrm{asd}}[a, g] = \int_{\mathbb{PT}'} \mathrm{tr}\, g \wedge \bar{\partial}_a^2 \wedge D^3 Z$$

bosonic part of the holomorphic Chern-Simons action for $\bar{\partial}_{a_s}$.

Twistor action for full Yang-Mills

The transform $g \to G$ is

$$G(x) = \int_{p^{-1}(x)} H(x,\pi)^{-1} g H(x,\pi) \wedge D^3 Z$$

H is a holomorphic frame of E over $p^{-1}(x)$, $\bar{\partial}_a H|_{p^{-1}(x)} = 0$.

$$\text{Defining} \qquad I[a,g] = \int_{\mathbb{M}} \mathrm{tr} G \wedge G \qquad \rightsquigarrow$$

$$I[a,g] = \int_{\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT}} \mathrm{tr} H_1^{-1} g_1 H_1 \wedge H_2^{-1} g_2 H_2 \wedge D^3 Z_1 \wedge D^3 Z_2$$

where $\mathbb{PT} \times_{\mathbb{M}} \mathbb{PT} = \{(Z_1, Z_2) \in \mathbb{PT} \times \mathbb{PT} | p(Z_1) = p(Z_2)\}$ and subscript 1 or 2 denotes dependence on Z_1 or Z_2.

Theorem

For small a, $S[a,g] := S_{\rm asd}[a,g] - \frac{\epsilon}{2} I[a,g]$ *is equivalent to the full space-time Yang-Mills action.*

- gauge freedom $a \to h^{-1}ah + h^{-1}\bar\partial h$, $g \to g + \bar\partial_a \xi$ on $\mathbb{PT}$.
- Solutions mod gauge on $\mathbb{PT}$ = solutions mod gauge on $\mathbb{M}$.
- $S[a,g] = S[A,G]$ when field equs are satisfied.
- Classical equivalence $\not\Rightarrow$ quantum equivalence.

Ideas in reformulation of $I[a, g]$

- The $\bar{\partial}_a$ Green's function on E-valued spinors on $p^{-1}(x)$ is

$$K_{12} := K(x, \pi_1, \pi_2) = \frac{H_1 H_2^{-1}}{\pi_1 \cdot \pi_2}$$

  with $\pi_1 \cdot \pi_2 = \pi_1^{A'} \pi_2^{B'} \varepsilon_{A'B'}$.
- In the QFT of E-valued spinors with action

$$S[\alpha, \beta] = \int_{p^{-1}(x)} \beta \bar{\partial}_a \alpha$$

  we have $K_{12} = \langle \alpha_1 \beta_2 \rangle$.
- Let $\psi = \eta^1 \eta^2 \eta^3 \eta^4$ then with $\eta_1^i = \theta^{iA'} \pi_{1A'}$ we have

$$\int \mathrm{d}^8 \theta \psi_1 \psi_2 = (\pi_1 \cdot \pi_2)^4$$

$I[a, g]$ as a determinant on curves

With $\mathrm{d}\pi = \pi^{A'}\mathrm{d}\pi_{A'}$, $\frac{\epsilon^2}{2} I[a, g] =$

$$
\begin{aligned}
&= \frac{\epsilon^2}{2}\int_{\mathbb{M}}\int_{p^{-1}(x)\times p^{-1}(x)} \mathrm{tr}(g_1 K_{12} \wedge g_2 K_{21})(\pi_1 \cdot \pi_2)^4 \mathrm{d}\pi_1 \mathrm{d}\pi_2 \wedge \mathrm{d}^4 x \\
&= \frac{\epsilon^2}{2}\int_{\mathbb{M}_s}\int_{p^{-1}(x)\times p^{-1}(x)} \langle(\beta_1 g_1 \alpha_1) \wedge (\beta_2 g_2 \alpha_2)\rangle \psi_1 \psi_2 \mathrm{d}\pi_1 \mathrm{d}\pi_2 \mathrm{d}^8\theta \mathrm{d}^4 x \\
&= \frac{1}{2}\int_{\mathbb{M}_s} \left\langle \left(\int_{p^{-1}(x)} (\epsilon\psi\beta g\alpha)\mathrm{d}\pi \right)^2 \right\rangle \mathrm{d}^8\theta \mathrm{d}^4 x \\
&= \int_{\mathbb{M}_s} \left\langle \exp\left(\int_{p^{-1}(x)} (\psi\beta g\alpha)\mathrm{d}\pi \right) \right\rangle \mathrm{d}^8\theta \mathrm{d}^4 x \\
&= \int_{\mathbb{M}_s}\int \mathrm{D}\alpha \mathrm{D}\beta \exp\left(\int_{p^{-1}(x)} (\beta\bar{\partial}_a\alpha + \beta\epsilon\psi g\alpha)\mathrm{d}\pi \right) \mathrm{d}^8\theta \mathrm{d}^4 x
\end{aligned}
$$

From twistor action to CSW twistor string

Thus

$$\frac{\epsilon^2}{2} I[a,g] = \int_{\mathbb{M}_s} \mathrm{Det}(\bar\partial_{a_s}|_{p^{-1}(x,\theta)}) \mathrm{d}^8\theta \mathrm{d}^4 x \,.$$

Where $a_s = a + \epsilon \eta^1 \eta^2 \eta^3 \eta^4 g$.
In the classical limit

$$\begin{aligned} \mathcal{A}[a,g] &= \exp\left(\frac{i}{\hbar} S_{\mathrm{asd}} - \frac{1}{2}\epsilon^2 I[a,g]\right) \\ &= \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{asd}}} \sum_d \frac{(\epsilon^2 I[a,g])^d}{2^d d!} \\ &= \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{asd}}} \sum_d \left(\frac{\epsilon^2}{2}\right)^d \int_{\mathbb{M}_s^d} \mathrm{d}\mu_d \mathrm{Det}(\bar\partial_{a_s}|_C) \end{aligned}$$

where $C = \cup_{r=1}^d p^{-1}(x_r, \theta_r) \in \mathbb{M}_s^d$, the CSW formula.

CSW perturbation theory from action

$$S[a] = \int_{\mathbb{PT}} \mathrm{tr}\,(\frac{1}{2} g \wedge \bar{\partial} a + \frac{1}{3} g \wedge a \wedge a) - \int_{\mathbb{M}_s} \mathrm{Det}(\bar{\partial}_{a_s}|_{p^{-1}(x,\theta)}) \mathrm{d}^8\theta \mathrm{d}^4 x \,.$$

- Propagator $= \bar{\partial}^{-1}$ = Serre class (Atiyah) $\leftrightarrow 1/p^2$.
- Cubic $+--$ vertex, *gaa*. On momentum space:

$$V(p_1, p_2, p_3) = \delta^4(p_1 + p_2 + p_3) \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$$

- $\int \mathrm{Det}(\bar{\partial}_{a_s}|_{p^{-1}(x,\theta)})$ = universal MHV vertex:
  - quadratic in g but nonpolynomial in a:
  - Generates all MHV $++--\ldots--$ vertices.
  - Holomorphic analogue of Wilson loop: generates holomorphic linking in Chern-Simons theory?

- Twistor action for $N = 4$ super YM:

$$S[a_s] = \int_{\mathbb{PT}_s} \mathrm{CS}(a_s) \wedge \Omega_s - \int_{\mathbb{M}_s} \mathrm{Det}(\bar{\partial}_{a_s}|_{p^{-1}(x,\theta)}) \mathrm{d}^8\theta \mathrm{d}^4 x$$

- Conformal gravity, $S[[g]] = \int \mathrm{Weyl}^2$, works similarly.
- Generates CSW version of twistor-string theory *but* disentangles Yang-Mills from conf. gravity (& susy).

- Outlook
  - Regularization of infrared divergences & determinants, diagrammatic treatment, full QFT.
  - Other couplings and theories.
  - Berkowitz version—Cech formulation?

Ambitwistor space $\mathbb{A} = \{Z \cdot W = 0 \subset \mathbb{PT} \times \mathbb{PT}^*\}$;
the space of complex light rays in complex space-time.

Theorem (Isenberg, Yasskin, Green 1978)

There is a $1:1$ *correspondence:*

$$\left\{\begin{matrix}\text{connections on bundle} \\ E' \to U \subset \mathbb{M}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{topologically trivial holomor-} \\ \text{phic bundles } E \text{ on } \mathbb{A}_U.\end{matrix}\right\}$$

Yang-Mills equations $\Leftrightarrow$ *E extends to 3rd order in* $\mathbb{PT} \times \mathbb{PT}^*$.

Super ambitwistor space
$\mathbb{A}_{[3]} = \{Z \cdot W + \xi \cdot \eta = 0 \subset \mathbb{PT}_{[3]} \times \mathbb{PT}^*_{[3]}\} =$ super light-rays.

Theorem (Witten 1978)

For $N = 3$ super Yang-Mills fields on $U \subset \mathbb{M}_{[3]}$:
Constraint equations
$\Leftrightarrow$
integrability on superlight rays
$\Leftrightarrow$
Solution corresponds to bundle $E \to \mathbb{A}_{[3]U}$.

$\mathbb{A}_{[3]}$ is super-Calabi-Yau:

$$\Omega = \int_{S^1} \frac{D^3 Z \mathrm{d}^3\xi \wedge D^3 W \mathrm{d}^3\eta}{Z \cdot W + \xi \cdot \eta} = \theta \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}^3\eta \wedge \mathrm{d}^3\xi\,, \quad \theta = Z \cdot \mathrm{d}W\,.$$

Aganagic, Neitzke, Policastro, Prem-Kumar, Vafa:

$$\mathbb{A}_{[3]} \text{ is mirror to } \mathbb{CP}^{3|4}\,.$$

Suggests holomorphic Chern-Simons
$\bar\partial$-operator $\bar\partial_a = \bar\partial + a : E \to E \otimes \Omega^{(0,1)}$

$$\mathrm{CS}(a) = \mathrm{tr}\,(\frac{1}{2} a \wedge \bar\partial a + \frac{1}{3} a^3)\,.$$

But $\mathrm{CS}(a) \wedge \Omega$ is an $8|6$-form on a $10|6$-manifold.

Super ambitwistor Lagrangian

Define $\mathbb{A}_{\mathbb{E}} = \{Z \cdot \hat{W} = 0 \subset \mathbb{A}\}$, similarly $\mathbb{A}_{[3]\mathbb{E}}$; complex light rays that intersect Euclidean real space-time $\rightsquigarrow$

$$S[a] = \int_{\mathbb{A}_{[3]\mathbb{E}}} \mathrm{CS}(a) \wedge \Omega\,,$$

$\bar{\partial}\mathrm{CS}(a) = \mathrm{tr}(F^{(0,2)})^2$ and $\bar{\partial}\frac{1}{Z \cdot W + \xi \cdot \eta} = \delta(Z \cdot W + \xi \cdot \eta)$ so

$$S[a] = \int_{Z \cdot \hat{W} = 0 \subset \mathbb{PT}_{[3]} \times \mathbb{PT}^*_{[3]}} \frac{\mathrm{tr}(F^{(0,2)})^2 \wedge \Omega}{Z \cdot W + \xi \cdot \eta}$$

Note $\delta(z) = \delta(\Re z)\delta(\Im z)\mathrm{d}\bar{z})$

Non supersymmetric ambitwistor Lagrangian

Expand to get non supersymmetric version

$$S[a] = \int_{Z\cdot\hat{W}=0\subset\mathbb{PT}\times\mathbb{PT}^*} \frac{\mathrm{tr}(F^{(0,2)})^2 \wedge \mathrm{D}^3 Z \wedge D^3 W}{(Z\cdot W)^4}$$
$$= \int_{Z\cdot\hat{W}=0\subset\mathbb{PT}\times\mathbb{PT}^*} \mathrm{CS}(a)\wedge \delta(Z\cdot W)''' \wedge \mathrm{D}^3 Z \wedge D^3 W$$

Note field equation $F^{(0,2)} \wedge \delta(Z\cdot W)''' \wedge \mathrm{D}^3 Z \wedge D^3 W = 0$
$\Leftrightarrow$ extension to 3rd order.

Theorem

Action is equivalent to the standard Yang-Mills action at the classical level.

Let $a \leftrightarrow A$, YM connection on $E \rightarrow \mathbb{E}$, then

$$S[a] = \int_{\mathbb{E}} \mathrm{tr}\,(H \wedge F - \frac{g^2}{2} H \wedge^* H)$$

where $F = (d + A)^2$ and $H \in \Omega^2 \otimes \mathrm{End}(E)$
(cf. Chalmers & Siegel).

- In linear theory $a = a^-(Z) + a^+(W)$.
- Vertices are just the cubic $++-$ and $--+$.
- Can evaluate vertices on space-time to get

$$V(a_1^-, a_2^-, a_3^+) = \int_{\mathbb{E}} \mathrm{tr}\,(A_1^- \wedge A_2^- \wedge G_3^+)$$

- Momentum space, on shell $--+$ vertex is

$$V(p_1, p_2, p_3) = \delta^4(p_1 + p_2 + p_3)\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

- Propagators are $+-$ or $-+$
  usual formulae on space-time & momentum space.

Conclusions

- Gives Yang-Mills Feynman diagram expansion based on cubic (virtual) MHV vertices.
- Suggests rules underly BCF(W) recursion (although with integration).
- Classical equivalence $\Rightarrow$ tree level works.
- Outlook
  - $\rightsquigarrow$ Behrends-Giele type recurrence with just 3-vertices?
  - Quantum equivalence?
  - Underlying string theory?
  - representation on $\mathbb{PT} \times \mathbb{PT}^* \rightsquigarrow$ twistor diagrams.
  - Hodges encoding of on shell recursion relations.