Supertwistors, Chern-Simons And Super Gauge Theories

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Contents

Double Fibrations

- The central objects of this talk are double fibrations of the form

$$\begin{array}{ccc} & \mathscr{Y} & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathscr{Z} & & \mathscr{X} \end{array}$$

where $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ are complex (super)manifolds:

  - $\mathscr{X}$ space-time
  - $\mathscr{Y}$ correspondence space
  - $\mathscr{Z}$ twistor space

Examples

- $\mathscr{X}$: complexified (super) space-time $\mathbb{C}^{m|n}$ with $m = 3, 4$.
- $\mathscr{Z}$: open subsets of $\mathbb{C}P^{3|n}$, $W\mathbb{C}P^{3|n}[b_1, \ldots, b_4|f_1, \ldots, f_n]$, degree 2 hypersurface in $\mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$, $W\mathbb{C}P^{2|4}[2, 1, 1|1, 1, 1, 1]$, exotic supermanifolds.
- In due course, we shall also sometimes introduce real structures on these spaces.

Twistor Correspondence

- Then we've a correspondence between $\mathscr{Z}$ and $\mathscr{X}$, i.e., between points in one space and subspaces of the other:

$$\begin{array}{ccc} z \in \mathscr{Z} & \longleftrightarrow & \pi_1(\pi_2^{-1}(z)) \subset \mathscr{X} \\ \pi_2(\pi_1^{-1}(x)) \subset \mathscr{Z} & \longleftrightarrow & x \in \mathscr{X} \end{array}$$

- Using the correspondence, we can transfer data given on $\mathscr{Z}$ to data on $\mathscr{X}$ and vice versa.
- E.g., take some analytic object $\mathrm{Ob}_{\mathscr{Z}}$ on $\mathscr{Z}$ and transform it to an object $\mathrm{Ob}_{\mathscr{X}}$ on $\mathscr{X}$ which will be constrained by some PDEs since $\pi_2^*\mathrm{Ob}_{\mathscr{Z}}$ has to be constant along the fibers of $\pi_2 : \mathscr{Y} \to \mathscr{Z}$.

Penrose-Ward Transform

- Under suitable topological conditions, the map

$$\mathrm{Ob}_{\mathscr{Z}} \;\mapsto\; \mathrm{Ob}_{\mathscr{X}}$$

  is a bijection between equivalence classes $[\mathrm{Ob}_{\mathscr{Z}}]$ and $[\mathrm{Ob}_{\mathscr{X}}]$ (i.e., a bijection between moduli spaces).
- In the following, my objects are certain holomorphic vector bundles.

Penrose-Ward Transform

In fact, suppose that the fibers of $\pi_2 : \mathscr{Y} \to \mathscr{Z}$ are simply connected and that $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{Z}$ is compact and connected for all $x \in \mathscr{X}$. Suppose further that

$$\Gamma(\mathscr{X}, T^*\mathscr{X}) \;\cong\; \pi_{1*}\{\Gamma(\mathscr{Y}, T^*\mathscr{Y})/\pi_2^*\Gamma(\mathscr{Z}, T^*\mathscr{Z})\}.$$

Then there is a 1-1 correspondence between $\mathscr{X}$-trivial holomorphic vector bundles $\mathcal{E} \to \mathscr{Z}$, holomorphic vector bundles on $\mathscr{Y}$ equipped with flat relative connection and holomorphic vector bundles $E \to \mathscr{X}$ equipped with a connection flat on each $\pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathscr{X}$ for $z \in \mathscr{Z}$.

What I'm considering.

- In the sequel, I shall be considering three examples of twistor manifolds $\mathscr{Z}$:
  - the supertwistor space $\mathcal{P}^{3|\mathcal{N}} \subset \mathbb{C}P^{3|\mathcal{N}}$.
  - the mini-supertwistor space $\mathcal{P}^{2|4} \subset W\mathbb{C}P^{2|4}[2,1,1|1,1,1,1]$.
  - the superambitwistor space $\mathcal{L}^{5|6} \subset \mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$.

Supertwistor Space

- The supertwistor space $\mathscr{Z} = \mathcal{P}^{3|\mathcal{N}}$ is a holomorphic fibration

$$\mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^{\mathcal{N}} \;\to\; \mathbb{C}P^1$$

  over the Riemann sphere $\mathbb{C}P^1$.

- On $\mathcal{P}^{3|\mathcal{N}}$ we can introduce (homogeneous) coordinates $(z^\alpha, \lambda_{\dot\alpha}, \eta_i)$. Holomorphic sections of $\mathcal{P}^{3|\mathcal{N}}$ are of the form $z^\alpha = x^{\alpha\dot\alpha}\lambda_{\dot\alpha}$, $\eta_i = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}$, where $(x, \eta) \in \mathbb{C}^{4|2\mathcal{N}}$.

Supertwistor Space

- Thus, we may write down the following double fibration:

$$\begin{array}{ccc} & \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1 \ni (x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \lambda_{\dot\alpha}) & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ \mathcal{P}^{3|\mathcal{N}} \ni (z^\alpha, \lambda_{\dot\alpha}, \eta_i) & & \mathbb{C}^{4|2\mathcal{N}} \ni (x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}) \end{array}$$

- By virtue of

$$\pi_2(x^{\alpha\dot\alpha}, \eta_i^{\dot\alpha}, \lambda_{\dot\alpha}) = (z^\alpha, \lambda_{\dot\alpha}, \eta_i) = (x^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \lambda_{\dot\alpha}, \eta_i^{\dot\alpha}\lambda_{\dot\alpha}),$$

we find:

$$\begin{array}{ccc} \text{point } p \in \mathcal{P}^{3|\mathcal{N}} & \longleftrightarrow & \text{null superplane } \mathbb{C}_p^{2|\mathcal{N}} \subset \mathbb{C}^{4|2\mathcal{N}} \\ \mathbb{C}P^1_{x,\eta} \subset \mathcal{P}^{3|\mathcal{N}} & \longleftrightarrow & \text{point } (x,\eta) \in \mathbb{C}^{4|2\mathcal{N}} \end{array}$$

Self-Dual SYM Theory

- Let's now discuss super gauge theory and consider a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ holomorphically trivial on any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ ($\mathbb{C}P^1$-trivial).
- Then the transition function f of $\pi_2^*\mathcal{E}$ is annihilated by $D_\alpha^\pm = \lambda_\pm^{\dot\alpha}\partial_{\alpha\dot\alpha}$, $D_\pm^i = \lambda_\pm^{\dot\alpha}\partial_{\dot\alpha}^i$ and $\partial_{\bar\lambda_\pm}$, where

$$\lambda_+ := \frac{\lambda_{\dot 1}}{\lambda_{\dot 2}}, \qquad \lambda_- := \frac{\lambda_{\dot 2}}{\lambda_{\dot 1}}.$$

Furthermore,

$$f_{+-} = \psi_+^{-1}\psi_-, \qquad \partial_{\bar\lambda_\pm}\psi_\pm = 0.$$

Self-Dual SYM Theory

- Therefore, we learn that

$$(D^{\pm}_{\alpha} + \mathcal{A}^{\pm}_{\alpha})\psi_{\pm} \;=\; 0, \quad \partial_{\bar{\lambda}_{\pm}}\psi_{\pm} \;=\; 0, \quad (D^{i}_{\pm} + \mathcal{A}^{i}_{\pm})\psi_{\pm} \;=\; 0,$$

where one can show that

$$\mathcal{A}^{\pm}_{\alpha} \;\equiv\; \lambda^{\dot{\alpha}}_{\pm}\mathcal{A}_{\alpha\dot{\alpha}}, \qquad \mathcal{A}^{i}_{\pm} \;\equiv\; \lambda^{\dot{\alpha}}_{\pm}\mathcal{A}^{i}_{\dot{\alpha}}.$$

- Clearly, we have certain compatibility conditions which turn out to be the constraint equations for the components $\mathcal{A}_{\alpha\dot{\alpha}}$ and $\mathcal{A}^{i}_{\dot{\alpha}}$ of the gauge potential of $\mathcal{N}$-extended self-dual SYM theory on $\mathbb{C}^{4|2\mathcal{N}}$ ($\mathbb{R}^{4|2\mathcal{N}}$) which are equivalent to the e.o.m. of this theory on $\mathbb{C}^4$ ($\mathbb{R}^4$).

Self-Dual SYM Theory

Moreover, we have

$$\mathcal{A}_{\alpha\dot{\alpha}} = \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+} \frac{\mathcal{A}^+_\alpha}{\lambda^{\dot{\alpha}}_+}, \quad \mathcal{A}^i_{\dot{\alpha}} = \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+} \frac{\mathcal{A}^i_+}{\lambda^{\dot{\alpha}}_+}.$$

Penrose-Ward Transform

Mini-Supertwistor Space

- Let's now discuss *3D* super gauge theory. In particular, we're interested in the diagram

$$\begin{array}{ccccc}
\mathcal{P}^{3|4} & \cong & \mathbb{R}^{4|8} \times S^2 & \longrightarrow & \mathbb{R}^{4|8} \\
\downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{R}^{3|8} \times S^2 & & \\
\downarrow & \swarrow^{\pi_2} & & \searrow^{\pi_1} & \downarrow \\
\mathcal{P}^{2|4} & & & & \mathbb{R}^{3|8}
\end{array}$$

in the real setting. [A.D. Popov, C. Sämann, M. Wolf, '05]

Mini-Supertwistor Space

- In fact, one can show that $\mathcal{P}^{2|4}$ is the holomorphic fibration

$$\mathcal{O}(2) \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^4 \ \rightarrow\ \mathbb{C}P^1.$$

- Holomorphic sections are of the form $w = y^{\dot\alpha\dot\beta}\lambda_{\dot\alpha}\lambda_{\dot\beta}$ and $\eta_i = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}$.
- By virtue of these equations, we therefore have:

$$\begin{array}{ccc} \text{point } p \in \mathcal{P}^{2|4} & \longleftrightarrow & \text{oriented lines } \mathbb{R}^{1|0}_p \subset \mathbb{R}^{3|8} \\ \mathbb{C}P^1_{y,\eta} \subset \mathcal{P}^{2|4} & \longleftrightarrow & \text{point } (y,\eta) \in \mathbb{R}^{3|8} \end{array}$$

- Note that $c_1(\mathcal{P}^{2|4}) = 0$, i.e., it's CY. So, one can take it as target for twistor string theory.

[D.W. Chiou, O.J. Ganor, Y.P. Hong, B.S. Kim, I. Mitra, '05]

Cauchy-Riemann Supertwistors

- Clearly, $\mathbb{R}^{3|8} \times S^2$ cannot be a complex manifold, but ... it's a so-called Cauchy-Riemann supermanifold.
- Let $\mathscr{X}$ be a smooth (super)manifold with $\dim_{\mathbb{R}} \mathscr{X} = d_1|d_2$. A CR structure is a complex involutive subbundle $\overline{\mathscr{D}} \subset T_{\mathbb{C}}\mathscr{X}$ of rank $m_1|m_2$ with $\overline{\mathscr{D}} \cap \mathscr{D} = \{0\}$; if $d_i = 2m_i$ we talk about a complex (super)manifold.
- On $\mathbb{R}^{3|8} \times S^2$ one can introduce several CR structures, e.g.,
  - $\mathcal{F}_0^{5|8} \equiv (\mathbb{R}^{3|8} \times S^2, \overline{\mathscr{D}}_0) \cong \mathbb{R}^{1|0} \times \mathbb{C}^{1|4} \times \mathbb{C}P^1$,
  - $\mathcal{F}^{5|8} \equiv (\mathbb{R}^{3|8} \times S^2, \overline{\mathscr{D}} = \pi_2^* T^{0,1}\mathcal{P}^{2|4})$, which we call CR supertwistor space.

$\mathscr{T}$-Flat Vector Bundles

- To introduce super gauge theories, we first need some preliminaries.
- Let $\mathscr{X}$ be as before and $\mathscr{T} \subset T_{\mathbb{C}}\mathscr{X}$ be an involutive subbundle with $\text{rank}(\mathscr{T} \cap \overline{\mathscr{T}}) = \text{const}$.
- The exterior derivative $\mathrm{d}_{\mathscr{T}}$ is given by

$$\Gamma(\mathscr{X}, \Lambda^q T^*\mathscr{X}) \xrightarrow{\mathrm{d}} \Gamma(\mathscr{X}, \Lambda^{q+1} T^*\mathscr{X}) \rightarrow \Gamma(\mathscr{X}, \Lambda^{q+1}\mathscr{T}^*).$$

- Let $\mathcal{E} \to \mathscr{X}$ be a complex vector bundle. A $\mathscr{T}$-connection is a map $D_{\mathscr{T}} : \Gamma(\mathscr{X}, \Lambda^q\mathscr{T}^* \otimes \mathcal{E}) \to \Gamma(\mathscr{X}, \Lambda^{q+1}\mathscr{T}^* \otimes \mathcal{E})$ locally given by $D_{\mathscr{T}} = \mathrm{d}_{\mathscr{T}} + \mathcal{A}_{\mathscr{T}}$ with $\mathcal{A}_{\mathscr{T}} \in \Gamma(\mathscr{X}, \mathscr{T}^* \otimes End\ \mathcal{E})$.
- $D^2_{\mathscr{T}}$ induces $\mathcal{F}_{\mathscr{T}} \in \Gamma(\mathscr{X}, \Lambda^2\mathscr{T}^* \otimes End\ \mathcal{E})$; if $\mathcal{F}_{\mathscr{T}} = 0$ then $\mathcal{E}$ is said to be $\mathscr{T}$-flat.

Partially Holomorphic Chern-Simons Theory

- Consider the following short exact sequence (on $\mathcal{F}^{5|8}$):

$$0 \rightarrow \pi_2^* T^{0,1}\mathcal{P}^{2|4} \rightarrow \mathscr{T} \rightarrow \pi_2^*(T\mathcal{F}^{5|8}/T\mathcal{P}^{2|4}) \rightarrow 0$$

- Since, $\mathcal{P}^{2|4}$ is CY $\rightarrow \exists$ a holomorphic volume form Ω; $\pi_2^*\Omega$ is globally well-defined on $\mathcal{F}^{5|8}$ and allows to integrate over objects holomorphic in the fermionic coordinates.
- Consider a complex vector bundle $\mathcal{E} \rightarrow \mathcal{F}^{5|8}$ which is $\mathbb{C}P^1$-trivial and equipped with $\mathcal{A}_{\mathscr{T}} \in \Gamma(\mathcal{F}^{5|8}, \mathscr{T}_b^* \otimes End\ \mathcal{E})$ holomorphic in the fermionic coordinates.

Partially Holomorphic Chern-Simons Theory

- Then

$$S_{phCS} = \int \pi_2^*\Omega \wedge \mathrm{tr}\left(\mathcal{A}_{\mathscr{T}} \wedge \mathrm{d}_{\mathscr{T}} \mathcal{A}_{\mathscr{T}} + \tfrac{2}{3}\mathcal{A}_{\mathscr{T}} \wedge \mathcal{A}_{\mathscr{T}} \wedge \mathcal{A}_{\mathscr{T}}\right)$$

- The resulting e.o.m. are

$$\mathrm{d}_{\mathscr{T}} \mathcal{A}_{\mathscr{T}} + \mathcal{A}_{\mathscr{T}} \wedge \mathcal{A}_{\mathscr{T}} = 0,$$

that is, the bundle $\mathcal{E} \to \mathcal{F}^{5|8}$ is $\mathscr{T}$-flat.

- The geometry of $\mathcal{F}^{5|8}$ fixes the superfield expansions of $\mathcal{A}_{\mathscr{T}}(A_{\dot\alpha\dot\beta}, \Phi, \chi^i_{\dot\alpha}, \tilde\chi_{i\dot\alpha}, \phi^{ij}, G_{\dot\alpha\dot\beta})$ (up to super gauge transformations).

Super Bogomolny Model

- We get

$$S_{sB} = \int \mathrm{d}^3x \,\mathrm{tr}\Big\{\epsilon^{\dot\alpha\dot\delta}\epsilon^{\dot\beta\dot\gamma}G_{\dot\gamma\dot\delta}\left(f_{\dot\alpha\dot\beta} + \tfrac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi\right) + \\ + \mathrm{i}\,\epsilon^{\dot\alpha\dot\delta}\epsilon^{\dot\beta\dot\gamma}\chi^i_{\dot\alpha}D_{\dot\delta\dot\beta}\tilde\chi_{i\dot\gamma} + \tfrac{1}{2}\phi_{ij}\triangle\phi^{ij} - \\ - \tfrac{1}{2}\epsilon^{\dot\alpha\dot\delta}\chi^i_{\dot\alpha}[\tilde\chi_{i\dot\delta},\Phi] - \epsilon^{\dot\alpha\dot\gamma}\phi_{ij}\{\chi^i_{\dot\alpha},\chi^j_{\dot\gamma}\} + \tfrac{1}{8}[\phi_{ij},\Phi][\phi^{ij},\Phi]\Big\}.$$

- The e.o.m. are

$$f_{\dot\alpha\dot\beta} = -\tfrac{\mathrm{i}}{2}D_{\dot\alpha\dot\beta}\Phi, \qquad \ldots$$

Dolbeault vs. Čech

Here, I've chosen the Dolbeault picture of CR vector bundles for the derivation of all equations. However, you can also work in Čech language as

$$(\mathcal{E}, f = \mathbb{1}, \mathrm{d}_{\mathscr{T}} + \mathcal{A}_{\mathscr{T}}) \sim (\tilde{\mathcal{E}}, \tilde{f}, \mathrm{d}_{\mathscr{T}})$$

[A.D. Popov, C. Sämann, M. Wolf, '05]

Missing Box

Question: What's the missing box?

phCS theory on $\mathcal{F}^{5|8}$

??? on $\mathcal{P}^{2|4}$ ↔ phCS theory on $\mathcal{F}^{5|8}$ ↔ supersymmetric Bogomolny model on $\mathbb{R}^3$

??? on $\mathcal{P}^{2|4}$ ↔ supersymmetric Bogomolny model on $\mathbb{R}^3$

Answer: It's a holomorphic BF theory. [A.D. Popov, '99]

Holomorphic BF Theory

- In fact, consider a trivial complex vector bundle $\mathcal{E} \to \mathcal{P}^{2|4}$ together with the action functional

$$S_{hBF} = \int \Omega \wedge \mathrm{tr}\left\{B(\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1})\right\},$$

where $\mathcal{A}^{0,1}$ and B are *End* $\mathcal{E}$-valued and holomorphic in the fermionic coordinates; $\mathcal{A}^{0,1}$ doesn't contain components along antiholomorphic fermionic directions.

- The resulting e.o.m. are

$$\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0, \qquad \bar{\partial}_{\mathcal{A}}B = 0,$$

i.e., the bundle is holomorphic ($\delta_g \mathcal{A}^{0,1} = \bar{\partial}_{\mathcal{A}}\omega$).

- Next, one can do the superfield expansions to eventually get the super Bogomolny model on $\mathbb{R}^3$.

Deformations

Now one may also study deformations of the CR structure on $\mathcal{F}^{5|8}$ (respectively, of the complex structure on $\mathcal{P}^{2|4}$). Doing this in a very particular fashion, one obtains a similar correspondence between theories but this time with massive spinors and scalars.

[D.W. Chiou, O.J. Ganor, Y.P. Hong, B.S. Kim, I. Mitra, '05]

[A.D. Popov, C. Sämann, M. Wolf, '05]

Signs Of Integrability In Quantum $\mathcal{N} = 4$ SYM Theory

- First signs of quantum integrability in $SU(N)$ $\mathcal{N} = 4$ SYM theory have been discovered by Minahan and Zarembo in the large N-limit.

  [J.A. Minahan, K. Zarembo, '02]

- It has then been shown that it's possible to interpret the full one-loop dilatation operator as Hamiltonian of an integrable quantum spin chain.

  [N. Beisert, M. Staudacher, '03]

Signs Of Integrability In Quantum $\mathcal{N} = 4$ SYM Theory

- Another development which has pointed towards integrable structures was triggered by the AdS/CFT conjecture and initiated by Bena, Polchinski and Roiban. They showed that the classical Green-Schwarz superstring on $AdS_5 \times S^5$ possesses an infinite number of conserved nonlocal charges.

  [I. Bena, J. Polchinski, R. Roiban, '03]

- Within the spin chain approach, Dolan, Nappi and Witten related these nonlocal charges for the superstring to a corresponding set of nonlocal charges in the gauge theory; they found also field theoretic expressions of these charges at zero 't Hooft coupling.

  [L. Dolan, C. Nappi, E. Witten, '03]

Charges From First Principles?

- It's certainly reasonable to ask whether such charges in the gauge theory can be understood from first principles, i.e., is there some non-perturbative approach for their construction?
- As a modest step towards an answer, let's first study a simplification – namely its self-dual truncation.
- In the following, I'll explain how one can (at least classically) construct hidden symmetry algebras (and hence an infinite number of conserved nonlocal charges) in (self-dual) SYM theory via the supertwistor correspondence.

[M. Wolf, '04]

Starting Point

- Above, we've seen that via the PW transform we got a bijection:

$$\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}) \ \ni \ [f] \ \longleftrightarrow \ [\mathcal{A}] \ \in \ \mathcal{M}^{\mathcal{N}}_{\rm SDYM}$$

- However, we can associate with any open subset $\mathcal{U} \subset \mathcal{U}_+ \cap \mathcal{U}_-$, where $\mathcal{P}^{3|\mathcal{N}} = \mathcal{U}_+ \cup \mathcal{U}_-$, an infinite number of such $[f]$.
- Each class $[f]$ corresponds to a class $[\mathcal{A}]$ and vice versa.

Starting Point

Question: How can this observation help us to learn something about symmetry algebras of the e.o.m. of our super gauge theory on space-time?

Answer: The key ingredient is to study infinitesimal deformations of our holomorphic vector bundles on supertwistor space!

Infinitesimal Deformations

- Consider $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$. Then Kodaira tells us that any infinitesimal deformation is allowed, as small enough perturbations of $\mathcal{E}$ will preserve its trivializability properties on the curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$.
- A generic deformation looks like

$$\delta \ : \ f_{+-} \ \mapsto \ \delta f_{+-} \ = \ \sum \epsilon_a \delta_a f_{+-}$$

and thus,

$$f_{+-} + \delta f_{+-} \ = \ (\psi_+ + \delta\psi_+)^{-1}(\psi_- + \delta\psi_-).$$

Infinitesimal Deformations

- As the trivializability properties of the bundle are preserved, δf_{+-} leads to solving the infinitesimal Riemann-Hilbert Problem (on $\mathbb{C}P^1$)

$$\varphi_{+-} \equiv \psi_+(\delta f_{+-})\psi_-^{-1} = \phi_+ - \phi_-, \quad \delta\psi_\pm = -\phi_\pm\psi_\pm.$$

- Thus, upon linearization of the linear system and by virtue of the PW transform we find $PW : [\delta f] \longleftrightarrow [\delta\mathcal{A}]$:

$$\delta\mathcal{A}_{\alpha\dot\alpha} = \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+} \frac{\nabla^+_\alpha\phi_+}{\lambda^{\dot\alpha}_+}, \quad \delta\mathcal{A}^i_{\dot\alpha} = \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+} \frac{\nabla^i_+\phi_+}{\lambda^{\dot\alpha}_+}.$$

Infinitesimal Deformations

For $\delta f_{+-} = \sum \epsilon_a \delta_a f_{+-}$, let $\{\delta_a\}$ be a set of infinitesimal deformations satisfying $[\delta_a, \delta_b\} = C_{ab}{}^c \delta_c$.

Question: What is the corresponding symmetry algebra of the supergauge theory on $\mathbb{R}^4$?

Kac-Moody And Superconformal Symmetries

- Let $\mathfrak{g}$ be some (finite-dimensional) Lie superalgebra with $[X_a, X_b\} = f_{ab}{}^c X_c$ and define the following perturbation:

$$\delta_a^m f_{+-} \equiv \lambda_+^m [X_a, f_{+-}], \qquad m \in \mathbb{Z}$$

- Then, it's easy to see that

$$[\delta_a^m, \delta_b^n\} = f_{ab}{}^c \delta_c^{m+n},$$

i.e., we get a centerless Kac-Moody algebra. Furthermore, by virtue of the PW transform we eventually obtain the same algebra on the (self-dual) SYM side.

Kac-Moody And Superconformal Symmetries

- Another example is affine extensions of superconformal symmetries: The generators of the superconformal algebra can be realized as vector fields on superspace-time. Their pull-back to supertwistor space yields an action of the superconformal group on the transition functions of our vector bundles. In a similar manner as above, one then obtains Kac-Moody-Virasoro type algebras on the (self-dual) SYM side.

[M. Wolf, '04]

Conclusions

What we've got:

- We saw how twistor correspondences can help to understand super gauge theories in three and four space-time dimensions.
- In particular, I discussed massless/massive super Bogomolny theories in $3D$ and introduced hBF and phCS theories – all being equivalent.
- I've shown how to construct hidden symmetry algebras of the self-dual SYM equations by using supertwistors.
- I exemplified the discussion by studying affine extensions of global (gauge type) and space-time symmetries.

Outlook

What's next?

- One can generalize the above construction to the full $SU(N)$ $\mathcal{N} = 4$ SYM theory. But how?
- Let's choose $\mathscr{Z} = \mathcal{L}^{5|6}$, which is the superambitwistor space. It's given by the locus

$$z^\alpha \mu_\alpha - w^{\dot\alpha} \lambda_{\dot\alpha} + 2\theta^i \eta_i \;=\; 0$$

in $\mathcal{P}^{3|3} \times \mathcal{P}_*^{3|3}$. Geometrically, it's the space of complex super light-rays in $\mathbb{C}^{4|12}$, i.e.,

$$\begin{array}{ccc} \text{point } p \in \mathcal{L}^{5|6} & \longleftrightarrow & \text{super light-ray } \mathbb{C}_p^{1|6} \subset \mathbb{C}^{4|12} \\ (\mathbb{C}P^1 \times \mathbb{C}P_*^1)_{x,\eta,\theta} \subset \mathcal{L}^{5|6} & \longleftrightarrow & \text{point } (x,\eta,\theta) \in \mathbb{C}^{4|12} \end{array}$$

Outlook

What's next?

- $\mathbb{C}P^1 \times \mathbb{C}P^1_*$-trivial holomorphic vector bundles $\mathcal{E} \to \mathcal{L}^{5|6}$ give rise to the e.o.m. of $\mathcal{N} = 3$ SYM theory on Minkowski space, which is fully equivalent to the $\mathcal{N} = 4$ theory.

  [E. Witten, '78]

- Now one may proceed and study infinitesimal deformations of $\mathcal{E}$ to eventually obtain hidden symmetries.
- One again finds affine extensions of global (gauge type) and superconformal symmetries.
- Furthermore, one obtains an infinite number of conserved non-local currents and hence charges, expressed by certain Wilson lines.

  [A.D. Popov, S. Uhlmann, R. Wimmer, M. Wolf, '??]

Outlook

What's next?

- Next, one should quantize these charges to see whether they are still conserved after including quantum corrections.
- Certainly, one might only hope these charge to be conserved in the large N-limit of the gauge theory.
- Hopefully, it will also be possible to relate these charges to those obtained by the spin-chain approach and in particular to the quantum Yangian.

  [L. Dolan, C. Nappi, E. Witten, '03]
- ...