Applications of on-shell methods: Massive recursion relations and multicollinear splitting functions

Nigel Glover

IPPP, University of Durham

from twistors to amplitudes

Queen Mary, University of London, 4 November 2005

Off-shell methods: Feynman diagrams

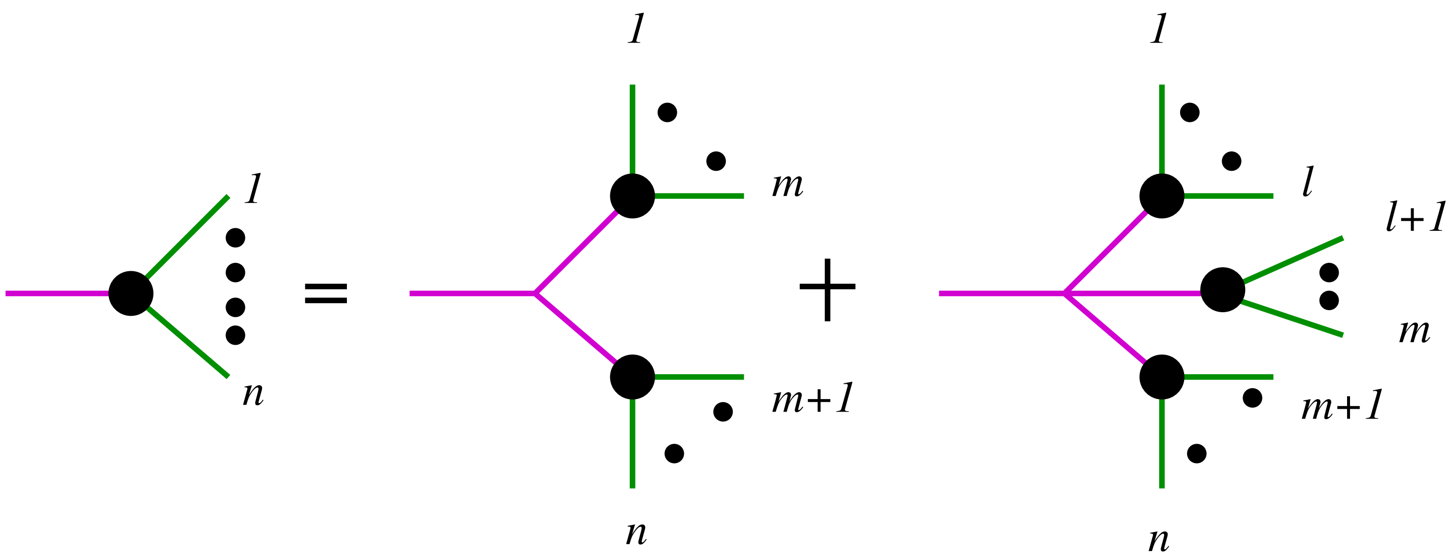
- ✓ Direct link to Lagrangian
- ✓ Easy to adapt to any model
- ✓ Easy to include massive particles with/without spin
- ✓ Easy to automate
  ⇒ tree-level packages `Madgraph/Grace/CompHep/...`
- ✗ Many Feynman diagrams
- ✗ Large cancellations between diagrams
- ✓ Off-shell Berends-Giele recursion relations
  ⇒ tree-level packages `Alpgen/HELAC/PHEGAS/...`
- ✓ Easy to include massive particles with/without spin

Rodrigo

- ✗ Loop amplitudes manpower intensive

Off-shell Recursion relations

Full amplitudes can be built up from simpler amplitudes with fewer particles



Purple gluons are off-shell, green gluons are on-shell.
This is a recursion relation built from off-shell currents.

Berends, Giele

Particularly suited to numerical solution

ALPGEN, HELAC/PHEGAS, Papadopoulos and Worek

NLO Wish List - Salam

Experiments priorities

1. $pp \to WW$+ jet
2. $pp \to H + 2$ jets
   $\to$ VBF Higgs background
3. $pp \to t\bar{t}b\bar{b}$
4. $pp \to t\bar{t}$ + 2 jets
   $\to$ $t\bar{t}H$ backgrounds
5. $pp \to WWb\bar{b}$
6. $pp \to VV + 2$ jets
   $\to$ WW scattering back-ground
7. $pp \to V$+3 jets
8. $pp \to VVV$+jet
   $\to$ SUSY background

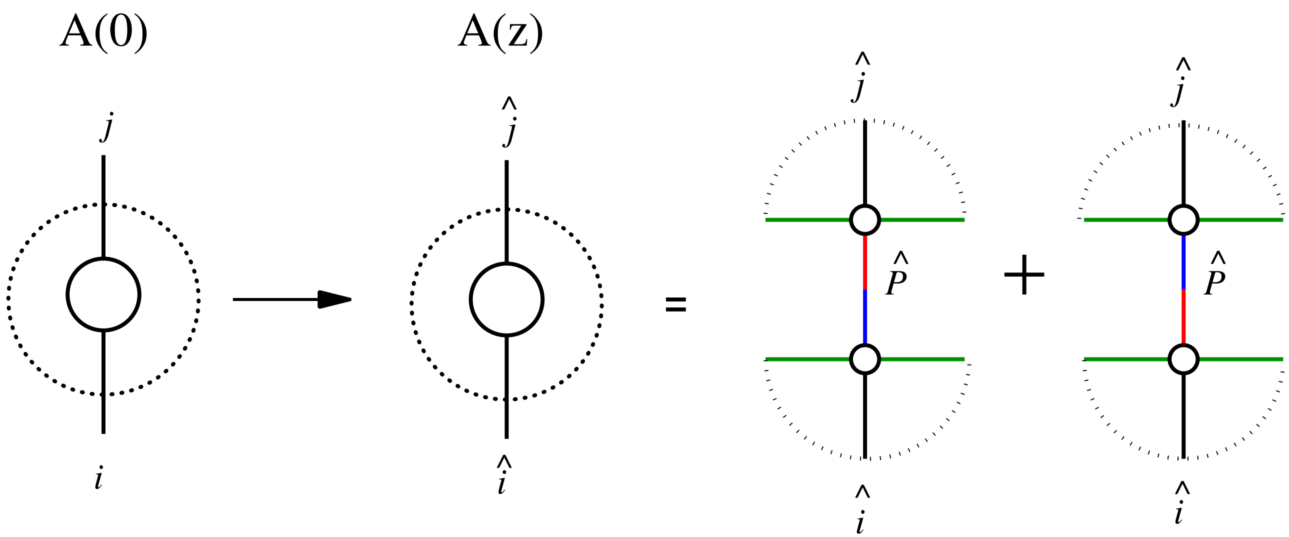
Already available
NLOJET++, MCFM, PHOX, ...
`http://www.cedar.ac.uk/hepcode`
Still to come

- $pp \to WW$+jet
- $pp \to VVV$
- $pp \to H$+2 jet
- $pp \to 4$ jets
- $pp \to t\bar{t}$+2 jets
- $pp \to t\bar{t}b\bar{b}$
- $pp \to VV$+2 jets
- $pp \to VVV$+jet
- $pp \to WWb\bar{b}$

On-shell recursion relations

Britto, Cachazo, Feng, Witten; Roiban, Spradlin, Volovich

Consider an n particle amplitude $A(0)$.



hatted momenta are shifted to put on-shell

$$\hat{i} = i + z\eta, \qquad \hat{j} = j - z\eta, \qquad \hat{P} = P + z\eta$$

$\Rightarrow$ each vertex is an on-shell amplitude

Recursion relations with massive particles

- On-shell condition:

$$\eta.i = \eta.j = \eta.\eta = 0$$

- Can in principle mark massive particles, however, in practice solution of on-shell condition simpler for massless particles.

$$\eta = \lambda_i \tilde{\lambda}_j, \qquad \text{or} \qquad \eta = \lambda_j \tilde{\lambda}_i.$$

- Vanishing of the amplitude at large z implies that

$$\eta = |j\rangle|i] \; : \qquad (h_i, h_j) \; = \; (+,-) \; , \; (+,+) \; , \; (-,-)$$

but not $(h_i, h_j) = (-,+)$.

Britto, Cachazo, Feng, Witten; Badger, NG, Khoze and Svrček

Recursion relations with massive particles

The massive propagator is given by

$$\frac{1}{P(z)^2 - m^2} = \frac{1}{P^2 - m^2 + 2zP \cdot \eta}$$

with a simple pole at

$$z = -\frac{P^2 - m^2}{2P \cdot \eta} = \frac{P^2 - m^2}{\langle j|P|i]}.$$

Provided the contribution at infinity vanishes,

$$\mathcal{A} = \sum_{\alpha} \mathcal{A}_L(z_\alpha) \frac{1}{P_\alpha^2 - m_\alpha^2} \mathcal{A}_R(z_\alpha)$$

Massive recursion relations: scalars

- One-loop amplitudes with n external gluons in pure Yang-Mills can conveniently be decomposed as

$$\mathcal{A}_n^{\mathrm{gluon}} = \mathcal{A}_n^{\mathcal{N}=4} - 4\mathcal{A}_n^{\mathrm{chiral}\ \mathcal{N}=1} + \mathcal{A}_n^{\mathrm{scalar}} ,$$

✓ The SUSY amplitudes $\mathcal{A}_n^{\mathcal{N}=4}$ and $\mathcal{A}_n^{\mathrm{chiral}\ \mathcal{N}=1}$ are cut constructible in 4-dimensions.

see talks by Dixon, Bjerrum-Bohr, Britto, Kosower, Brandhuber,...

✗ $\mathcal{A}_n^{\mathrm{scalar}}$ cannot be fully reconstructed from its imaginary part evaluated in 4 dimensions and contains purely rational terms that do not have cuts in 4 dimensions.

Bern, Dixon, Dunbar, Kosower

✓ $\mathcal{A}_n^{\mathrm{scalar}}$ is cut constructible in D-dimensions.

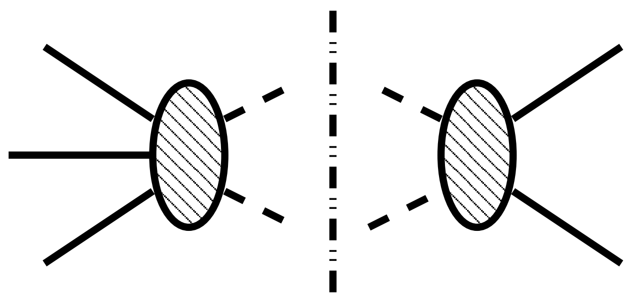
Massive recursion relations: scalars

Massless scalars ϕ with D-dimensional momenta P_D can be thought of as massive scalars with 4-dimensional momenta.

$$P_D^2 = 0 \Longrightarrow P_4^2 = \mu^2$$

These massive amplitudes can then be stitched together to obtain loops with a D-dimensional cut.

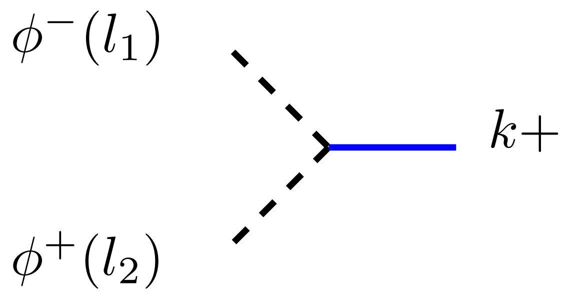
Bern, Dixon, Dunbar, Kosower



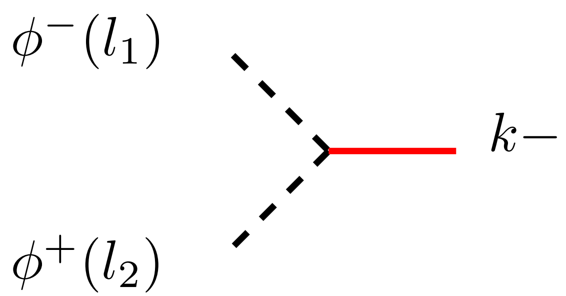
Powers of μ^2 lead to dimension changed integrals.

Massive recursion relations: scalars

Basic building block for recursion relation is 3-point vertex



$$\mathcal{A}_3(l_1^-, k^+, l_2^+) = \frac{\langle q_1|l_1|k]}{\langle q_1 k\rangle}$$

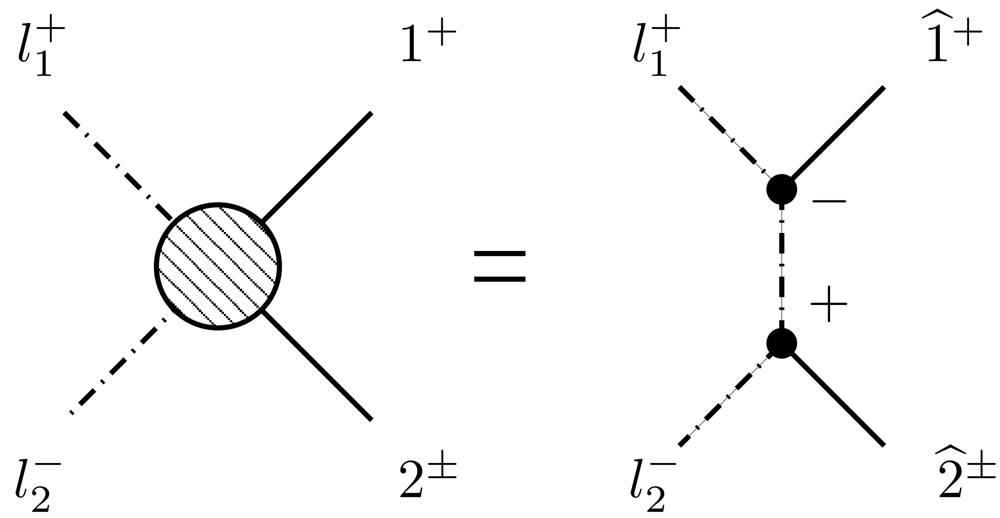


$$\mathcal{A}_3(l_1^-, k^-, l_2^+) = -\frac{\langle k|l_1|q_2]}{[q_2 k]}$$

Here q_1 and q_2 are gauge vectors such that e.g. $q_1^a k_a = 0$.
Changing $q_1^a \to q_1'^a = \alpha q_1^a + \beta k^a$ the amplitude becomes

$$\mathcal{A}'(l_1^+, k^+, l_2^-) = \frac{\alpha\langle q_1|l_1|k] + \beta\langle k|l_1|k]}{\alpha\langle q_1 k\rangle} \equiv \mathcal{A}(l_1^+, k^+, l_2^-)$$

Massive recursion relations: scalars



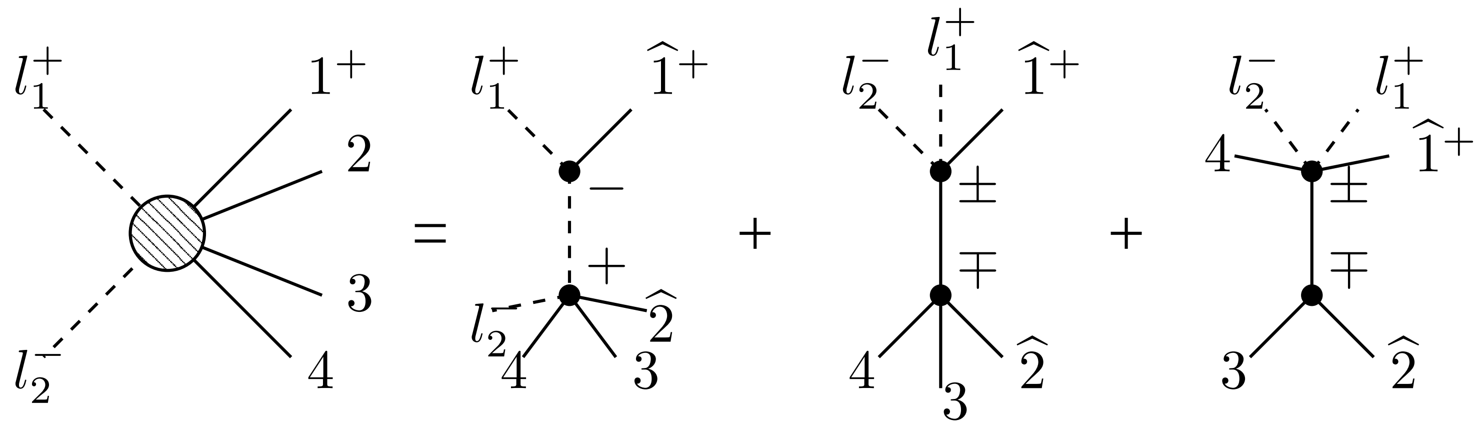
Simplfying choice $q_2 = \widehat{k}_1 = |\widehat{k}_1\rangle|\widehat{k}_1]\ , \qquad q_1 = \widehat{k}_2 = |\widehat{k}_2\rangle|\widehat{k}_2]$

$$\mathcal{A}_4(l_1^+, 1^+, 2^+, l_2^-) = \frac{\mu^2[12]}{\langle 12\rangle((l_1+k_1)^2-\mu^2)}$$

Bern, Dixon, Dunbar, Kosower

$$\mathcal{A}_4(l_1^+, 1^+, 2^-, l_2^-) = -\frac{\langle 2|l_1|1]^2}{\langle 12\rangle[12]((l_1+k_1)^2-\mu^2)}$$

Massive recursion relations: scalars



Explicit results for up to 4 gluons. e.g.

$$\mathcal{A}_6(l_1^+, 1^+, 2^+, 3^+, 4^+, l_2^-) = -\frac{\mu^2[4|l_2(3+4)(1+2)l_1|1]}{Q_1 Q_2 Q_3 \langle 12\rangle\langle 23\rangle\langle 34\rangle}$$

where $Q_1 = ((l_1+k_1)^2 - \mu^2)$, $Q_2 = ((l_1+k_1+k_2)^2 - \mu^2)$ and $Q_3 = ((l_2+k_4)^2 - \mu^2)$.

Badger, NG, Khoze and Svrček

All orders results for all-plus and one-minus

Forde, Kosower

Massive recursion relations: vector bosons

- Consider a generic theory with a non-Abelian gauge group being a product $G_1 \times G_2$, where G_1 is unbroken, and G_2 is broken by the Higgs mechanism.
- The two gauge groups are coupled to each other via fermions which are charged under both groups.
- Work with the G_1- and G_2-colour-stripped purely kinematic multi-currents

$$S_{\mu_1 \ldots \mu_m}(1_q, 2, 3, \ldots, n-1, n_{\bar{q}}).$$

- Lorentz indices $\mu_1 \ldots \mu_m$ are associated with massive (off-shell) vectors $V_{\mu_1}, \ldots V_{\mu_m}$
- Single vector currents will be used in calculations of multi currents

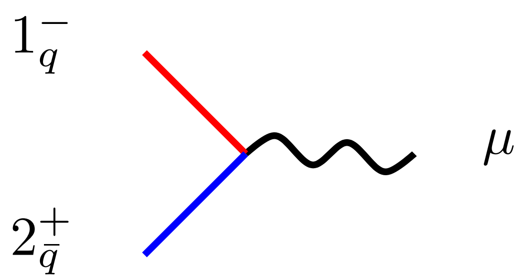
Berends, Giele, Kuijf, Tausk; Bern, Forde, Kosower, Mastrolia; Badger, NG, Khoze

Single vector boson current

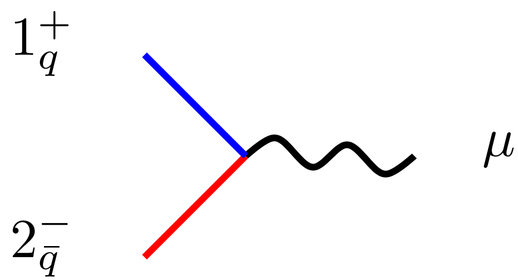
- First construct the single vector boson currents,

$$S_\mu(1_q^\lambda, 2^{h_2}, \ldots, (n-1)^{h_{n-1}}, n_{\bar{q}}^{-\lambda}).$$

- Need to introduce two new primitive vertices derived from Feynman rules



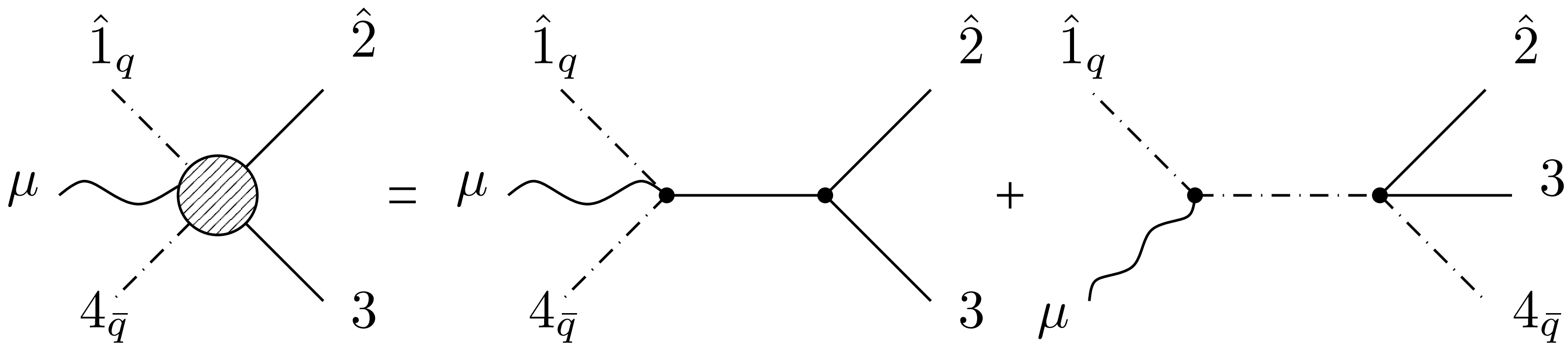
$$S_\mu(1_q^-, 2_{\bar{q}}^+) = \langle 1|\sigma_\mu|2] \equiv [2|\bar{\sigma}_\mu|1\rangle$$



$$S_\mu(1_q^+, 2_{\bar{q}}^-) = [1|\bar{\sigma}_\mu|2\rangle \equiv \langle 2|\sigma_\mu|1]$$

✓ Parity and Line reversal

e.g. Five parton current



- Note that the vector boson V directly couples to fermion-antifermion pair and does not propagate
- Usual rules for marking massless particles
- For example

$$S_\mu(1_q^-, 2^+, 3^-, 4_{\bar{q}}^+) =$$

$$\frac{[24]^3 \langle 1|\sigma_\mu P_V|1\rangle}{s_{234}[23][34]\langle 1|2+3|4]} + \frac{\langle 13\rangle^3 [4|P_V \sigma_\mu|4]}{s_{123}\langle 12\rangle\langle 23\rangle\langle 1|2+3|4]}$$

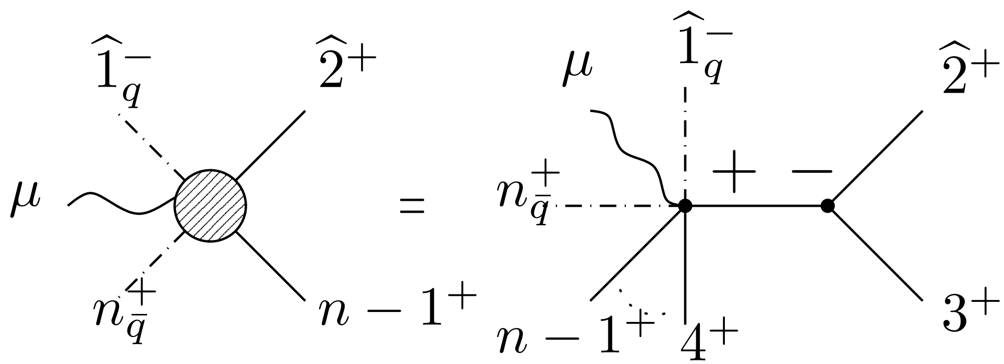
MHV n-parton currents

- The current for vector boson decaying to a quark pair and any number of positive helicity gluons has been known for some time

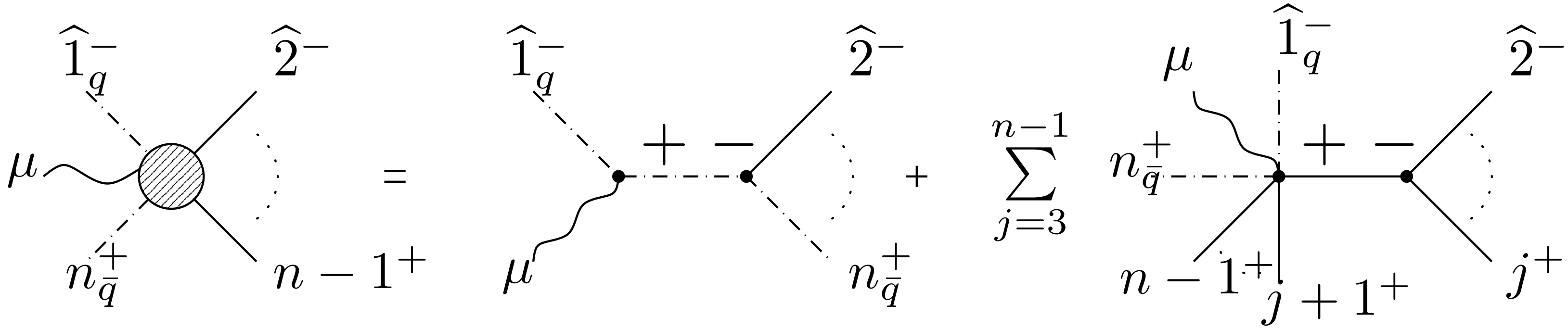
Berends,Giele

$$S_\mu(1_q^-, 2^+, \ldots, (n-1)^+, n_{\bar{q}}^+) = (-1)^n \frac{\langle 1|\sigma_\mu P_V|1\rangle}{\prod_{\alpha=1}^{n-1}\langle \alpha\, \alpha+1\rangle}$$

- Straightforward to prove recursively



NMHV n-parton currents

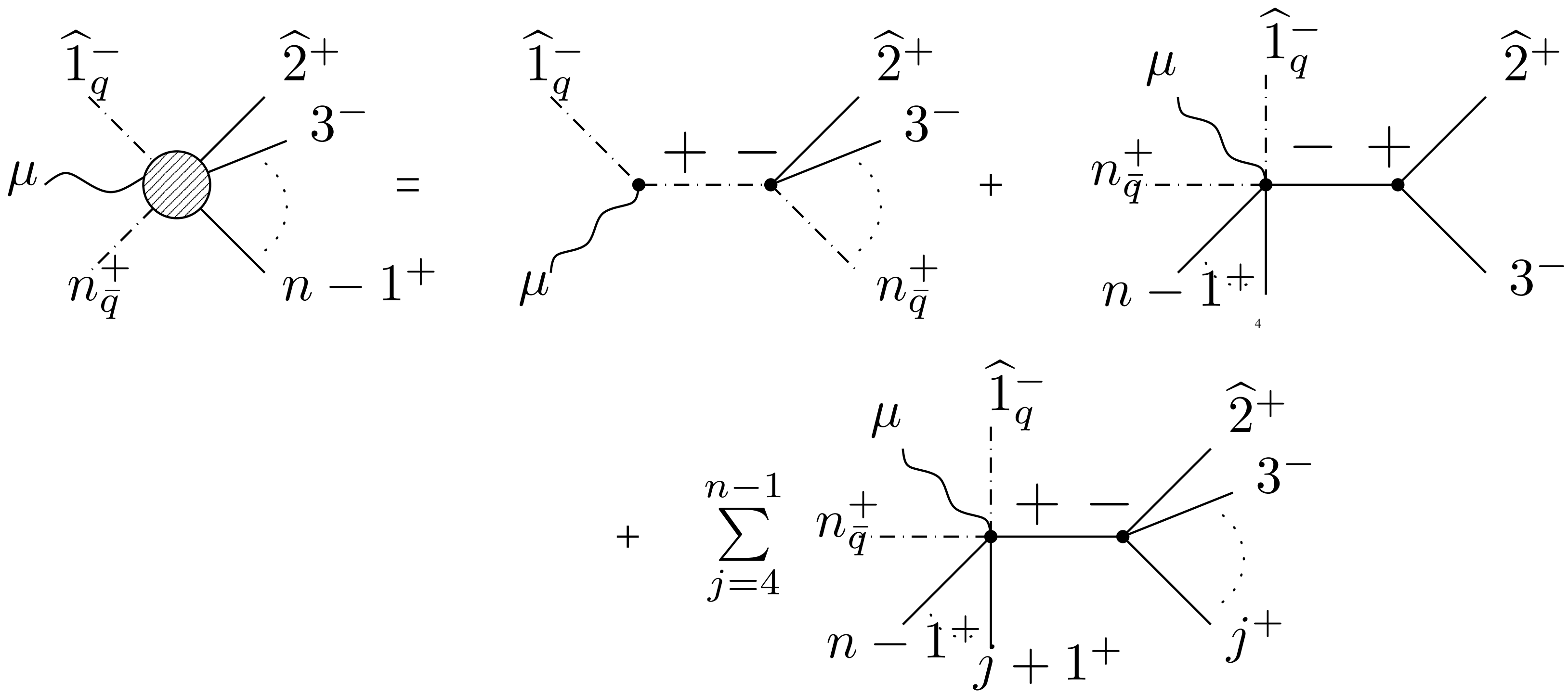


$$S_\mu(1_q^-, 2^-, 3^+, \ldots, (n-1)^+, n_{\bar{q}}^+) = \frac{-(-1)^n}{\prod_{\alpha=2}^{n-1}\langle \alpha\, \alpha+1\rangle}\left(\frac{\langle 2|K_{3,n}P_V\sigma_\mu K_{3,n}|2\rangle\langle 2n\rangle}{s_{2,n}\langle n|K_{2,n-1}|1]}\right.$$

$$\left.+\sum_{j=3}^{n-1}\frac{\langle 2|K_{3,j}K_{1,j}P_V\bar{\sigma}_\mu K_{1,j}K_{3,j}|2\rangle\langle 2|K_{3,j}|1]\langle j\, j+1\rangle}{s_{2,j}s_{1,j}\langle j|K_{2,j-1}|1]\langle j+1|K_{2,j}|1]}\right)$$

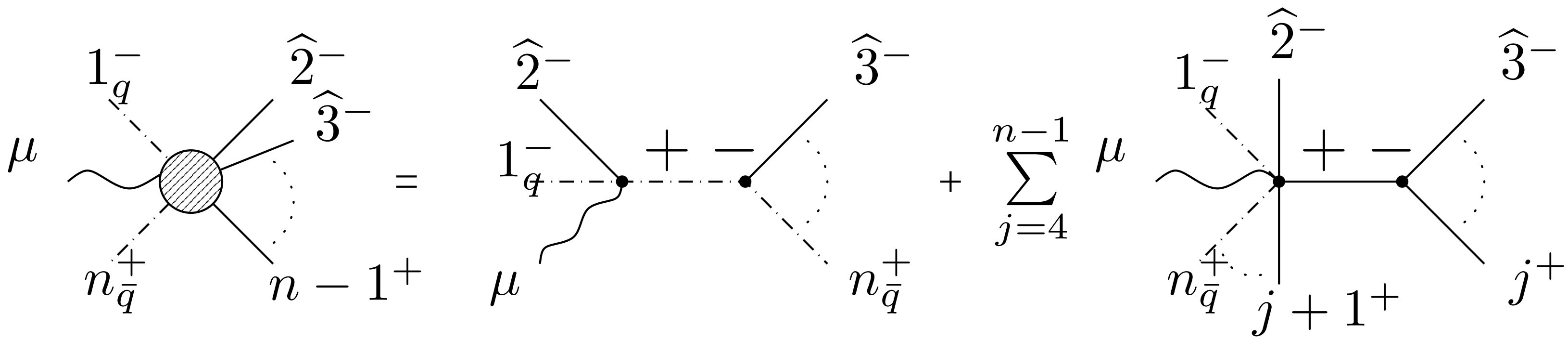
Similar result for $S_\mu(1_q^-, 2^+, \ldots, (n-2)^+, (n-1)^-, n_{\bar{q}}^+)$

NMHV n-parton currents



- For NMHV currents with non-adjacent negative helicities we can re-use the above result to find the amplitude where the negative helicities are separated by one positive helicity.
- .. and continue adding more positive helicities

NMHV n-parton currents

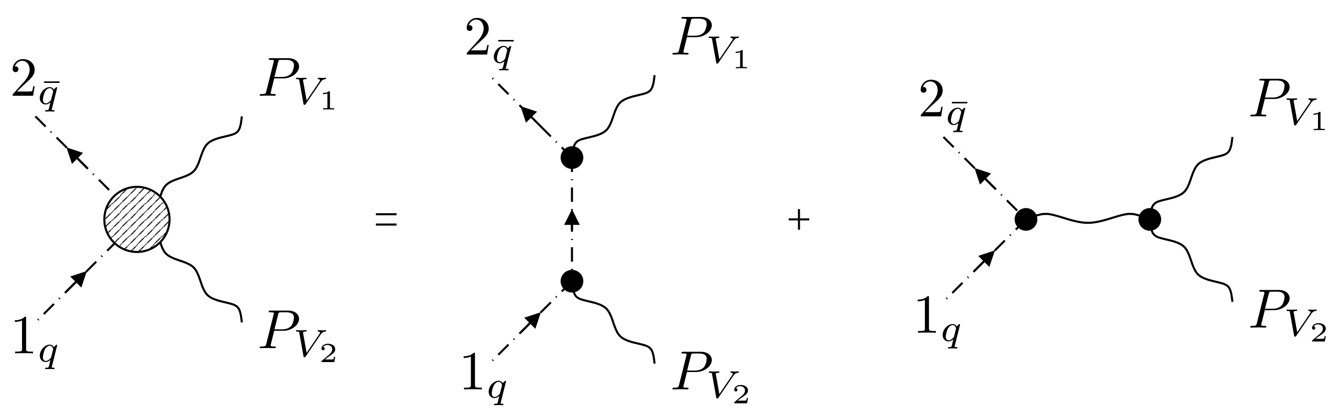


- By marking particles $i = 2$ and $j = 3$ only the adjacent minus NMHV current is needed.
- Numerically checked against explicit results up to 6 partons

  Berends, Giele, Kuijf, Tausk

- Other general formulae can be obtained by repeated use of recursion relations

Double vector boson currents



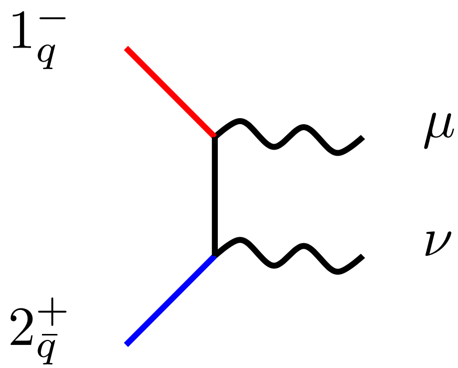
- Cannot mark the two adjacent massless fermions and use on-shell recursion relations can be used to derive $S_{\mu\nu}(q,\bar{q})$ from two single vector boson amplitudes $S_\mu(q,\bar{q})$.
- Split current into Abelian and non-Abelian parts

Dixon, Kunszt, Signer

$$S_{\mu\nu}(1_q,\ldots,n_{\bar q}) = T^{(3)}_{\mu\nu\rho}(P_{V_1},P_{V_2},-P)\,\frac{1}{(P^2-M_P^2)}\,S^\rho(1_q,\ldots,n_{\bar q}) + S^{Abelian}_{\mu\nu}(1_q,\ldots,n_{\bar q}). \qquad (1)$$

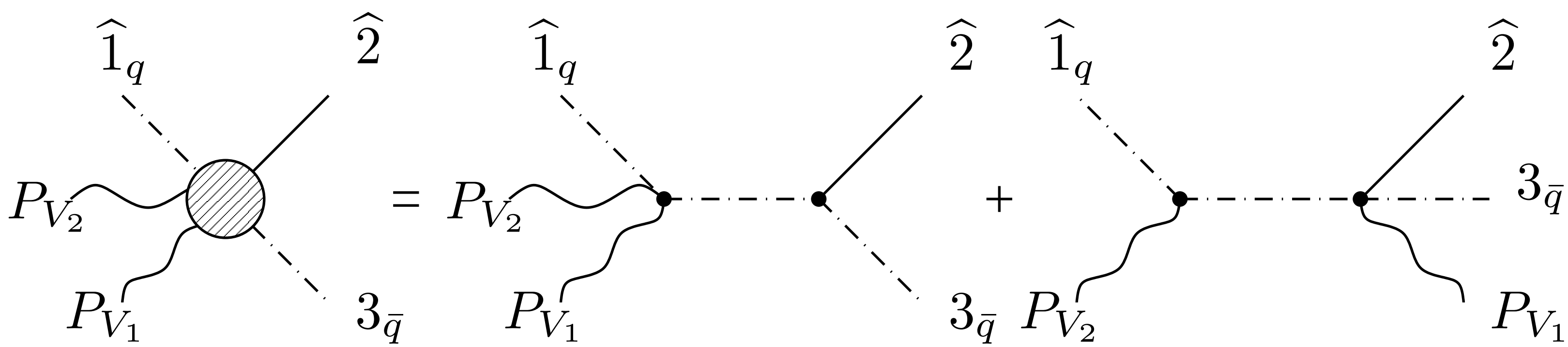
Double vector boson currents

There is a new primitive 4-point vertex for the Abelian contribution



$$S^{Abelian}_{\mu\nu}(1^-_q, 2^+_{\bar{q}}) = \frac{1}{s_{1P_{V_2}}} \langle 1|\sigma_\nu(1 + P_{V_2})\sigma_\mu|2]$$

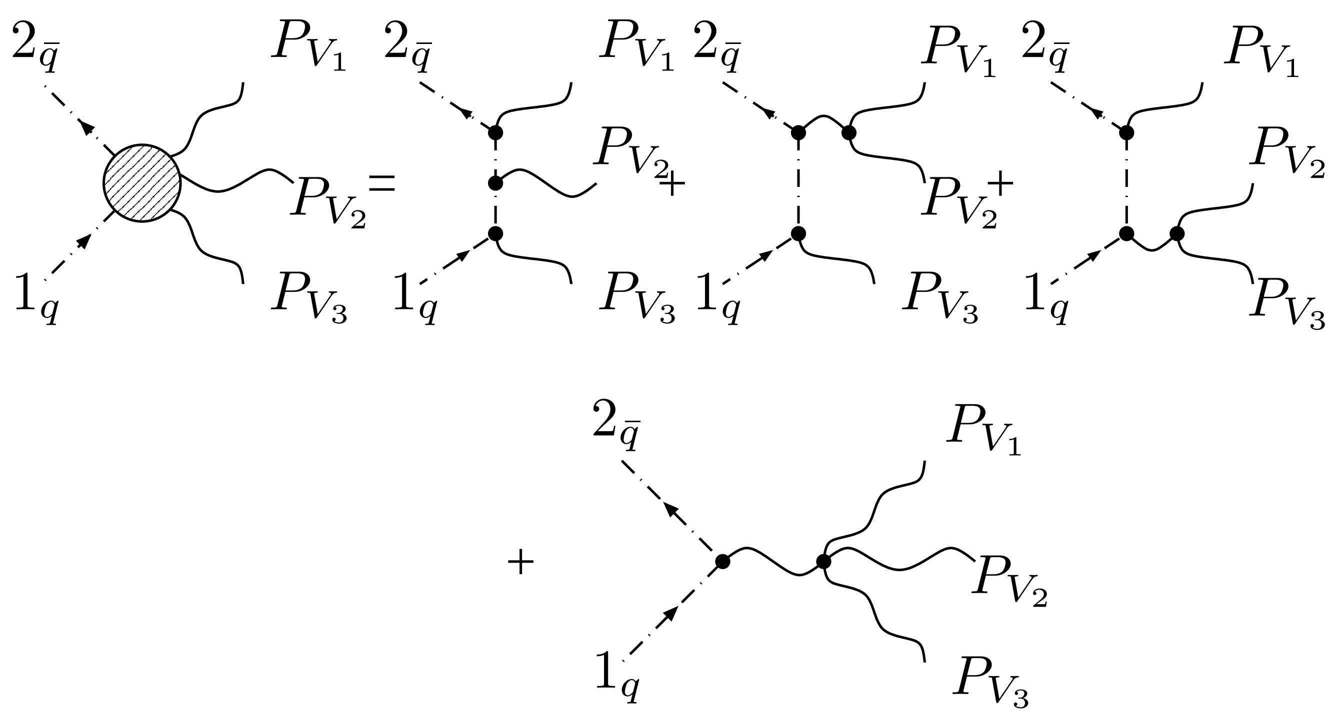
Use usual recursion to generate higher point Abelian contributions e.g.



Multi vector boson currents

- The generalisation to more vector bosons always reuses the known currents and introduces a new Abelian primitive vertex.

$$S^{Abelian}_{\mu_1 \ldots \mu_n}(1_q, 2_{\bar{q}})$$



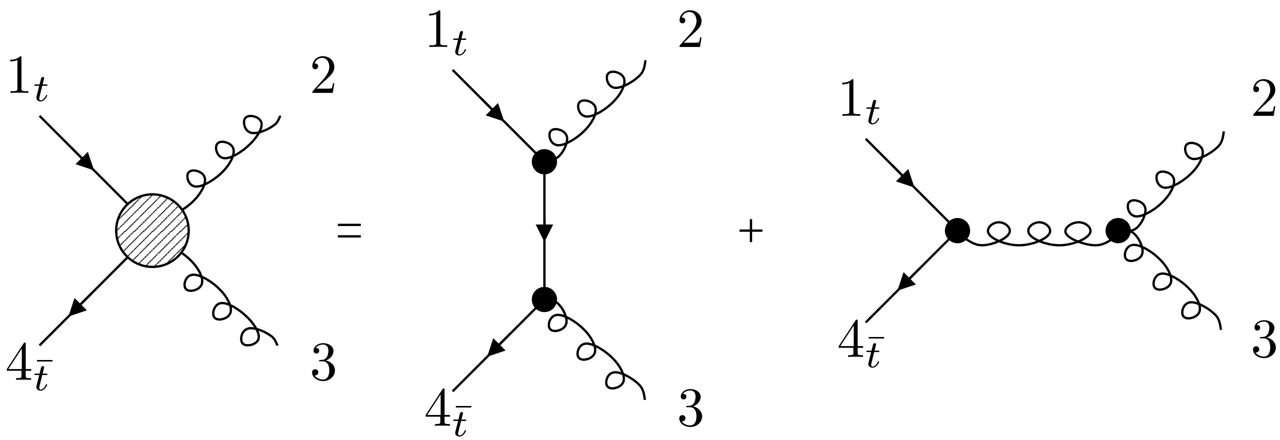
Massive recursion relations: fermions

- In the standard recursion relation all particles are assumed to be in a state with fixed helicity, and there is a summation over all these states.
- Try to avoid using helicity states for internal massive particles.

$$\begin{aligned}&\sum_s \mathcal{A}_L(p_{r;\bar{q}},\ldots,\widehat{p}_i,\ldots,p_s,-\widehat{P}_q^s)\frac{1}{P^2-m_P^2}\mathcal{A}_R(\widehat{P}_{\bar{q}}^{-s},p_{s+1},\ldots,\widehat{p}_j,\ldots,p_{r-1;q})\\ =&\mathcal{A}_L(p_{r;\bar{q}},\ldots,\widehat{p}_i,\ldots,p_s,-\widehat{P}^*)\frac{\widehat{\not{P}}+m_P}{P^2-m_P^2}\mathcal{A}_R(\widehat{P}^*,p_{s+1},\ldots,\widehat{p}_j,\ldots,p_{r-1;q})\end{aligned} \tag{2}$$

where P^* indicates the external spinor wave-function has been stripped off this amplitude.

Calculation of $\mathcal{A}_4(1_t, 2, 3, 4_{\bar{t}})$



The recursion relation result is,

$$\mathcal{A}(1_t, 2, 3, 4_{\bar{t}}) = \frac{1}{(P^2 - m_t^2)} \bar{u}(p_1) \not{\epsilon}(p_2, \xi_2) \left(\sum_s u_s(\widehat{P}) \bar{u}_s(\widehat{P}) \right) \not{\epsilon}(p_3, \xi_3) v(p_4).$$

The shifts must generate the contributions from both Feynman diagrams.

Calculation of $\mathcal{A}_4(1_t, 2^-, 3^+, 4_{\bar{t}})$

- First consider the case where the gluons have opposite helicity, $\mathcal{A}(1_t, 2^-, 3^+, 4_{\bar{t}})$. It is convenient to choose $\xi_2 = p_3$ and $\xi_3 = p_2$ so that,

$$\mathcal{A}(1_t, 2^-, 3^+, 4_{\bar{t}}) =$$

$$\frac{1}{(P^2 - m_t^2)s_{23}} \bar{u}(p_1) \begin{pmatrix} 0 & |\hat{2}\rangle[3| \\ |3]\langle\hat{2}| & 0 \end{pmatrix} \begin{pmatrix} m_t & \hat{\not{P}} \\ \hat{\not{P}} & m_t \end{pmatrix} \begin{pmatrix} 0 & |2\rangle[\hat{3}| \\ |\hat{3}]\langle 2| & 0 \end{pmatrix} v(p_4).$$

✓ The marking prescription $i = 3$ and $j = 2$ ensures that the shifts on the polarisation vectors and $\widehat{P}$ disappear.

✓ The recursion relation therefore exactly reproduces the first Feynman diagram.

✓ With this choice of reference momenta $\epsilon^-(2,3) \cdot \epsilon^+(3,2) = 0$ and the second Feynman diagram gives a vanishing contribution

Calculation of $\mathcal{A}_4(1_t, 2^+, 3^+, 4_{\bar{t}})$

$$\mathcal{A}(1_t, 2^+, 3^+, 4_{\bar{t}}) =$$

$$\frac{1}{(P^2 - m_t^2)\langle 23\rangle^2}\bar{u}(p_1)\begin{pmatrix} 0 & |3\rangle[\widehat{2}| \\ |\widehat{2}]\langle 3| & 0 \end{pmatrix}\begin{pmatrix} m_t & \widehat{\not{P}} \\ \widehat{\not{P}} & m_t \end{pmatrix}\begin{pmatrix} 0 & |2\rangle[\widehat{3}| \\ |\widehat{3}]\langle 2| & 0 \end{pmatrix} v(p_4).$$

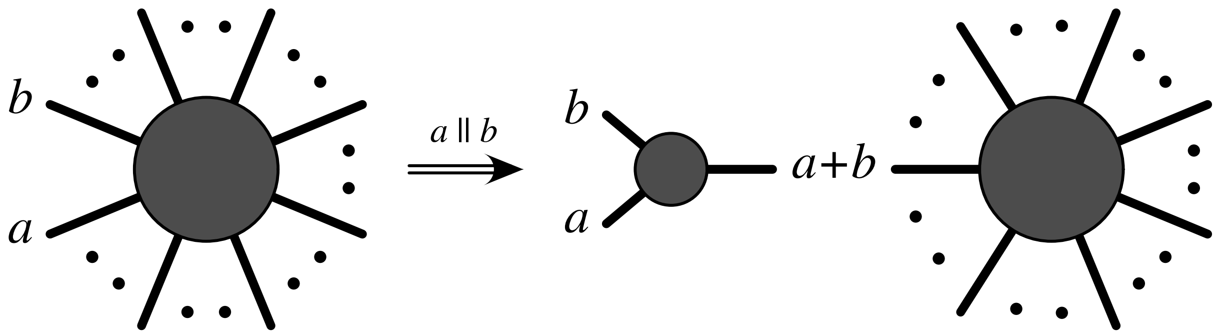
Choosing $i = 3$ and $j = 2$ removes the shifts on the propagator and the polarisation vector of gluon p_3. However, $|\widehat{2}] = |2] - z|3]$ produces an extra term

$$-\frac{1}{\langle 23\rangle^2}\bar{u}(p_1)\begin{pmatrix} 0 & \not{p}_3 \\ \not{p}_3 & 0 \end{pmatrix} v(p_4).$$

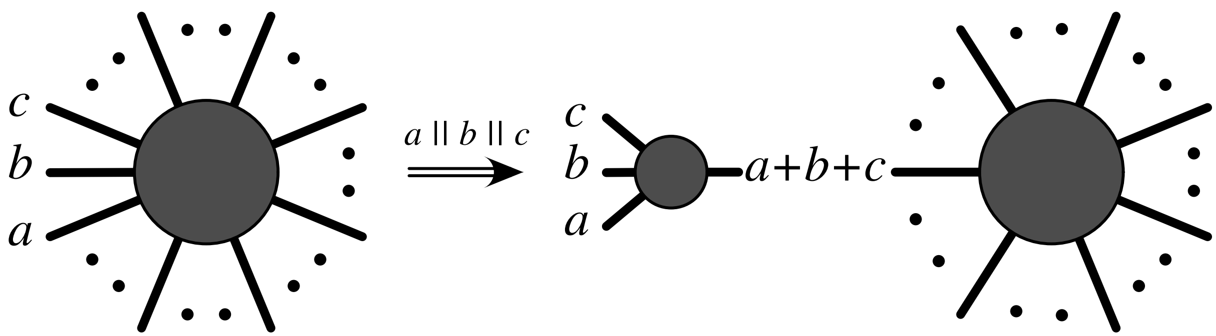
which is the same as that obtained from the second Feynman diagram

Multicollinear limits

- Gauge theory amplitudes undergo a universal factorisation when two or more particles become collinear.



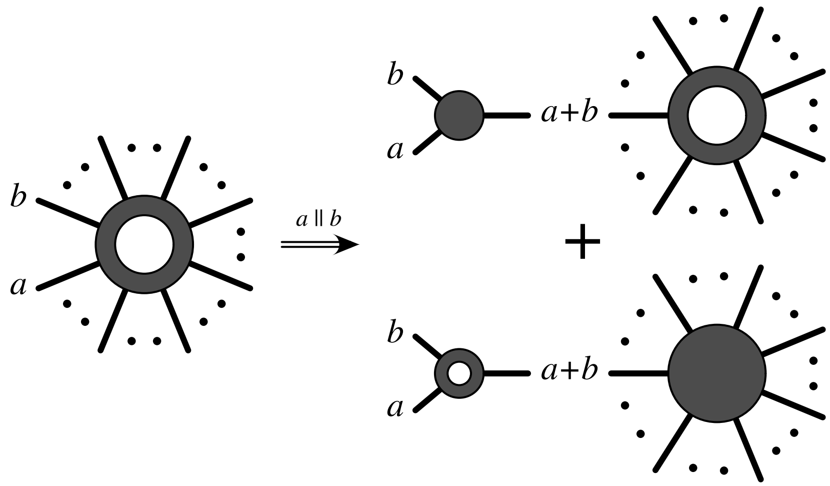
- Triple splitting functions known for some time



Campbell, NG; Catani, Grazzini; Del Duca, Frizzo, Maltoni

Multicollinear limits

- Factorisation takes place at the loop level



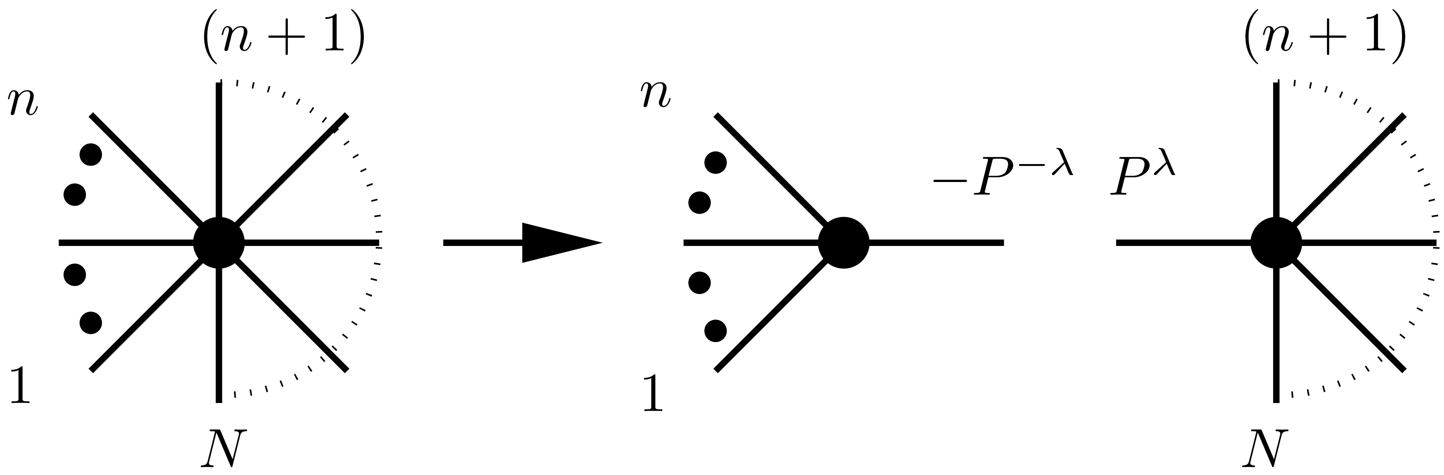
Bern, Dixon, Dunbar, Kosower

- Similar structure occurs at two-loops

Bern, Dixon, Kosower; Badger, NG

- Collinear properties play an important role in
  ✓ testing conjectures for loop amplitudes
  ✓ constructing infrared subtraction terms

Multicollinear limits



$$A_N(1^{\lambda_1},\ldots,N^{\lambda_N}) \to \text{split}(1^{\lambda_1},\ldots,n^{\lambda_n} \to P^{\lambda})\times A_{N-n+1}((n+1)^{\lambda_{n+1}},\ldots,N^{\lambda_N},P^{\lambda})$$

- ✓ In the limit, $p_1,\ldots,p_n$ become parallel and all $s_{ij}=(p_i+p_j)^2$, with $i,j=1,\ldots,n$ are small.
- ✓ The longitudinal-momentum fractions z_i given by $z_i=\xi\cdot p_i/\xi\cdot P$ where the light-like vector P^{μ} lies along the collinear direction.

Multicollinear limits

- ΔM (ΔP) is the difference in negative (positive) helicities before/after taking the limit

$$\Delta M = 0 \quad \Rightarrow \quad \begin{array}{l} 1^+, 2^+, 3^+, \ldots, n^+ \to P^+ \\ 1^-, 2^+, 3^+, \ldots, n^+ \to P^- \end{array} \quad A_N = \mathrm{MHV}$$

$$\Delta M = 1 \quad \Rightarrow \quad \begin{array}{l} 1^-, 2^+, 3^+, \ldots, n^+ \to P^+ \\ 1^-, 2^-, 3^+, \ldots, n^+ \to P^- \end{array} \quad A_N = \mathrm{NMHV}$$

- A simple power counting argument shows

$$\mathrm{split} \propto \frac{1}{[\ \ ]^{\Delta M} \langle\ \ \rangle^{\Delta P}}\ .$$

There are only two potential sources of the anti-holomorphic spinor products : scalar propagators and the off-shell continuation. The anti-holomorphic factors of the form $[i\eta]$ all cancel.

Multicollinear limits

- ✓ Only a subset of the MHV diagrams contribute - those that have all internal propagators on-shell in the multi-collinear limit.
- ✓ More precisely,

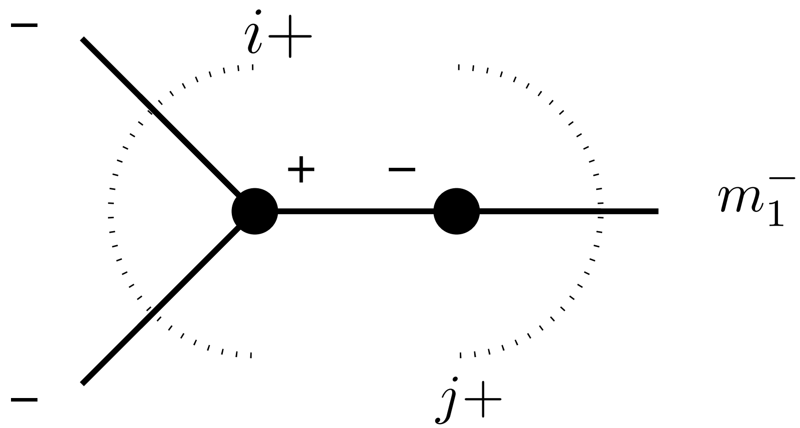
$$\text{split} = \sum \frac{1}{\prod_{i,j=1}^{\Delta M} s_{i\ldots j}} f(\langle\,\rangle)$$

where $f(\langle\,\rangle) \propto 1/\langle\ \ \rangle^{\Delta P - \Delta M}$ has a holomorphic nature

- ✓ Could work entirely with $\overline{MHV}$ rules. Role of ΔP and ΔM exchanged, $f(\langle\,\rangle) \to \bar{f}([\,])$
- ✗ Identification of which diagrams contribute *a priori* is lost in recursion rule approach - shifts change kinematic poles. In general find mixed result with both $\langle\,\rangle$ and $[\,]$

Multicollinear limits: $\Delta M = 1$

For example;



Hard partons can only be emitted from the LH vertex, collinear partons can be emitted from both vertices

$$\text{split}_+(1^+, \ldots, m_1^-, \ldots, n^+) = \sum_{i=0}^{m_1-1} \sum_{j=m_1}^{n} \text{spinor products}$$

Spinor products involving collinear particles simplify in the limit

$$\langle b\, q\rangle \longrightarrow [P\,\eta]\,\langle b\, P\rangle \sum_{l=i+1}^{j} z_l, \qquad \langle b\, a\rangle \longrightarrow \langle b\, P\rangle \sqrt{z_a}$$

Multicollinear limits: $\Delta M = 1$

When $\Delta M = 1$, there are three amplitudes in the triple limit

$$\text{split}(1^-, 2^+, 3^+ \to P^+) = \frac{\langle 1\,2\rangle\, {z_2}^2}{\sqrt{z_1 z_2 z_3}\, s_{12}\, (z_1 + z_2) \left(\langle 1\,3\rangle \sqrt{z_1} + \langle 2\,3\rangle \sqrt{z_2}\right)} + \frac{\left(\langle 1\,2\rangle \sqrt{z_2} + \langle 1\,3\rangle \sqrt{z_3}\right)^3}{s_{123}\, \langle 1\,2\rangle \langle 2\,3\rangle \left(\langle 1\,3\rangle \sqrt{z_1} + \langle 2\,3\rangle \sqrt{z_2}\right)},$$

$$\begin{aligned}\text{split}(1^+, 2^-, 3^+ \to P^+) &= \frac{\langle 1\,2\rangle\, {z_1}^2}{\sqrt{z_1 z_2 z_3}\, s_{12}\, (z_1 + z_2) \left(\langle 1\,3\rangle \sqrt{z_1} + \langle 2\,3\rangle \sqrt{z_2}\right)} \\ &+ \frac{\left(\langle 2\,1\rangle \sqrt{z_1} + \langle 2\,3\rangle \sqrt{z_3}\right)^4}{s_{123}\, \langle 1\,2\rangle \langle 2\,3\rangle \left(\langle 1\,3\rangle \sqrt{z_1} + \langle 2\,3\rangle \sqrt{z_2}\right) \left(\langle 1\,2\rangle \sqrt{z_2} + \langle 1\,3\rangle \sqrt{z_3}\right)} \\ &+ \frac{\langle 2\,3\rangle\, {z_3}^2}{\sqrt{z_1 z_2 z_3}\, s_{23}\, (z_2 + z_3) \left(\langle 1\,2\rangle \sqrt{z_2} + \langle 1\,3\rangle \sqrt{z_3}\right)},\end{aligned}$$

$$\text{split}(1^+, 2^+, 3^- \to P^+) = \text{split}(3^-, 2^+, 1^+ \to P^+)\,.$$

Multicollinear limits: Summary

- Find agreement with known results for $g \longrightarrow 4g$ and $q \longrightarrow qxx$

  Del Duca, Frizzo, Maltoni

- Specific and general results for specific helicity configurations for both gluons and quarks

  Birthwright, NG, Khoze, Marquard

✓ MHV rules very efficient way of generating collinear limits. Instantly evaluate gauge independent splitting function by selecting specfic MHV rule diagrams

✓ Coefficients are holomorphic

✓ Possible use in testing conjectures for amplitudes and in constructing infrared subtraction terms

Massive recursion relations: Summary

- On-shell recursion relations have been applied to tree processes involving
  - ✓ Massive scalars

    Badger, NG, Khoze, Svrček; Forde, Kosower
  - ✓ Vector bosons

    Badger, NG, Khoze
  - ✓ Heavy quarks

    Badger, NG, Khoze; Schwinn, Weinzierl
- Some new (slightly more compact) explicit formulae, and some new all order results for specific helicity configurations
- Long way to go if want to compete with existing tree-level solutions

  ALPGEN, HELAC/PHEGAS, MADEVENT,...