Queen Mary University of London

From Twistors to Amplitudes

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N=1 Super-Yang-Mills at One Loop

or, Constructing Cut-Constructible Amplitudes

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Here I present a systematic procedure to carry out finite unitarity cut integrals.

This technique suffices to compute gluon amplitudes in $\mathcal{N} = 1$ Super-Yang-Mills at one loop.

It applies equally well to the cut-constructible part of nonsupersymmetric Yang-Mills at one loop.

Overview

- This will be a refinement of the unitarity method.
- We need only the usual double cuts.
- By doing some initial simplification, we can split the cut integral into terms corresponding to particular coefficients.
- This method lets us compute a given coefficient directly as an integral in a single variable (Feynman parameter).

One-Loop Amplitudes in Super-Yang-Mills

SUSY multiplet decomposition:

$$
\begin{aligned}
A^{\mathrm{QCD}} &= g \\
&= (g + 4f + 3s) - 4(f + s) + s \\
&= A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1\ \mathrm{chiral}} + A^{scalar}
\end{aligned}
$$

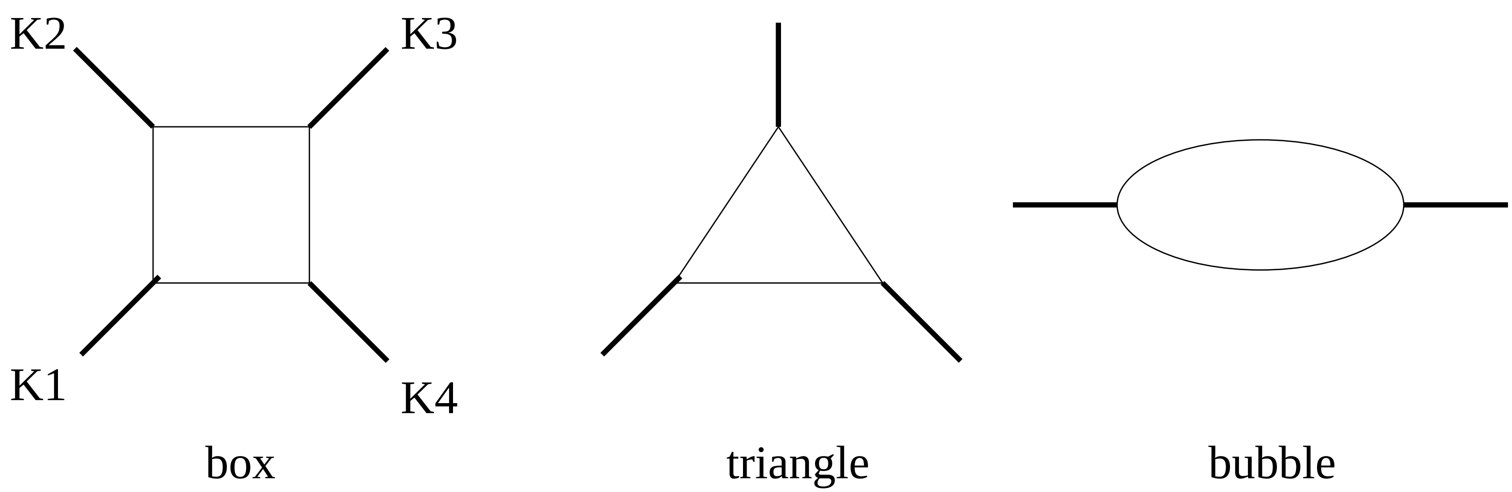
or

$$
\begin{aligned}
A^{\mathcal{N}=1\ \mathrm{vector}} &= g + f \\
&= (g + 4f + 3s) - 3(f + s) \\
&= A^{\mathcal{N}=4} - 3A^{\mathcal{N}=1\ \mathrm{chiral}}
\end{aligned}
$$

Passarino-Veltman Reduction

$$A_{n,1} = \sum_i c_{2,i}\,(\text{bubble}) + \sum_i c_{3,i}\,(\text{triangle}) + \sum_i c_{4,i}\,(\text{box}) + \text{extra}$$

where expressions for scalar bubble, scalar triangle and scalar box functions are known explicitly. (In Dim. Reg.: Bern, Dixon, Kosower 1993)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K3
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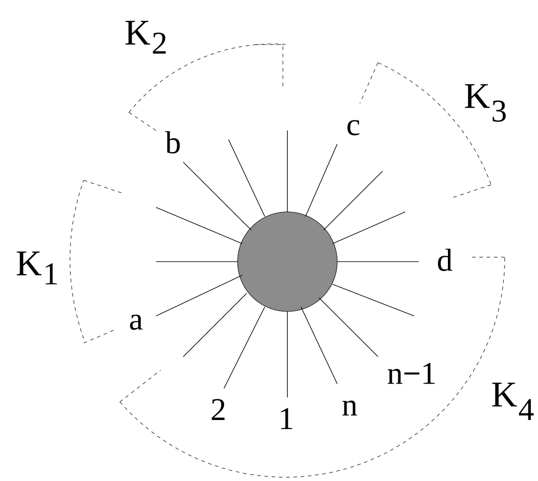
K_2
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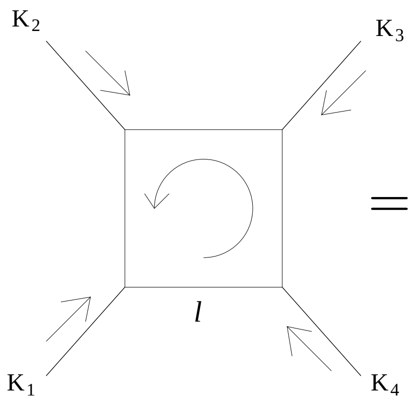


$$= \int d^{4-2\epsilon}\ell \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2}$$

- Depending on how many massless legs, we have I_4^{4m}, I_4^{3m}, $I_4^{2m\ e}$, $I_4^{2m\ h}$, I_4^{1m}; I_3^{3m}, I_3^{2m}, I_3^{1m}; and I_2^{2m}.

  e.g. I_3^{2m} has 3 legs (“triangle integral”) of which 2 are massive.

- So the problem is reduced to finding the coefficients $c_{2,i}, c_{3,i}, c_{4,i}$, which are rational functions of the spinor products $\langle i\ j\rangle$ and $[i\ j]$.

With the decomposition into $\mathcal{N} = 4$, $\mathcal{N} = 1$, and $\mathcal{N} = 0$, the nonzero coefficients are the following:

$\mathcal{N}$	Box	Triangle	Bubble	Extra
4	$\surd$			
1	$\surd$	$\surd$	$\surd$	
0	$\surd$	$\surd$	$\surd$	$\surd$

Unitarity method

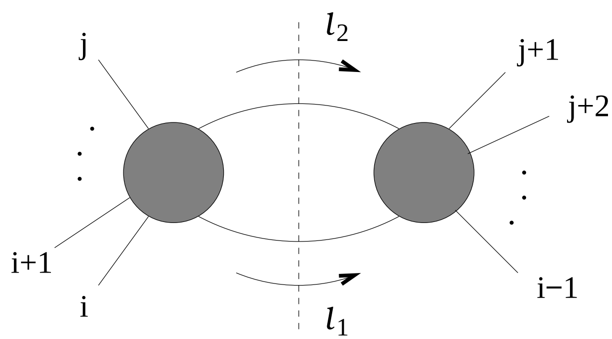
There are two ways to compute a unitarity cut.

One is from the cut integral,

$$C_{i,\ldots,j} = \int d\mu\, A^{\rm tree}(-\ell_1, i, \ldots, j, -\ell_2) A^{\rm tree}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

where

$$d\mu = d^4\ell_1\, d^4\ell_2\, \delta^{(4)}(\ell_1 + \ell_2 - P_L)\, \delta^{(+)}(\ell_1^2)\, \delta^{(+)}(\ell_2^2)$$



For two propagators, we drop the principal part, leaving the delta function that places their momenta on-shell.

The second way is by finding the discontinuity Δ of the amplitude across the branch cut of interest. (By unitarity, this is the imaginary part of the amplitude in the kinematic regime where $P_L^2 > 0$ and all other momentum invariants are negative.)

$$C_{i,\ldots,j} = \Delta A_n^{1-\mathrm{loop}}$$

If a given box, triangle, or bubble has a nonzero discontinuity across the cut in the P_L channel, then the cut integral has information about its coefficient.

Unfortunately, we always have several coefficients show up at once in a given cut. This problem has been variously addressed by reduction techniques, twistor geometry methods, generalized unitarity...

(Bern, Dixon, Dunbar, Kosower (x2); Bern, Del Duca, Dixon, Kosower; Cachazo; RB, Cachazo, Feng (x2); Bidder, Bjerrum-Bohr, Dixon, Dunbar; Brandhuber, Spence, Travaglini; Bedford, Brandhuber, Spence, Travaglini (x2); Quigley, Rozali; Bern, Dixon, Kosower; Bidder, Bjerrum-Bohr, Dunbar, Perkins; Brandhuber, McNamara, Spence, Travaglini; . . .)

Box integrals have a unique leading singularity

This underlies the "quadruple cut" method. (RB, Cachazo, Feng)

Because a quadruple cut has 4 delta functions, the integral is fully localized.

(In the method discussed here, this is an optional first step, or an easy cross check.)

So this takes care of all boxes and hence all of $\mathcal{N} = 4$. But we still have to learn to distinguish triangles and bubbles.

Triple cuts might seem like a good idea at this point, but let's see how to save ourselves some work...

Eliminate 1m- and 2m-triangles

- We work in dimensional regularization with $D = 4 - 2\epsilon$.
- The IR + UV divergent behavior of the $\mathcal{N} = 1$ amplitude is

$$A_{n,1}\big|_{\text{singular}} = \frac{1}{\epsilon} A_{tree}.$$

(Kunszt, Soper; Kunszt, Signer, Trocsanyi; Giele, Glover; Giele, Glover, Kosower; Bern, Chalmers)

The scalar integrals diverge as follows:

function	I_4^{4m}	$I_4^{1m/2m/3m}$	I_3^{3m}	$I_3^{1m/2m}$	I_2
divergence	none	$\frac{(-K^2)^{-\epsilon}}{\epsilon^2}$	none	$\frac{(-K^2)^{-\epsilon}}{\epsilon^2}$	$\frac{1}{\epsilon}$

- So the $\frac{(-K^2)^{-\epsilon}}{\epsilon^2}$ divergences must cancel.
- However, I_3^{1m} and I_3^{2m} contain only singular pieces like $\frac{(-K^2)^{-\epsilon}}{\epsilon^2}$ without other finite parts. The only role of these two functions is to cancel the singular pieces of the boxes. Thus we need not calculate their coefficients explicitly.

Modified basis

$$A_n^{\mathcal{N}=1\text{ chiral}} = \sum \left(c_2 I_2 + c_3^{3m} I_3^{3m} + c_4^{1m} I_{4F}^{1m} + c_4^{2m\ e} I_{4F}^{2m\ e} \right.$$
$$\left. + c_4^{2m\ h} I_{4F}^{2m\ h} + c_4^{3m} I_{4F}^{3m} + c_4^{4m} I_4^{4m} \right).$$

The $1/\epsilon$ divergent piece comes entirely from bubble integrals.

Specifically, the relation between coefficients and tree amplitudes is in this case

$$\sum c_2 = A_n^{\text{tree}}.$$

It is easy to distinguish the contributions from three-mass triangles and bubbles even when they participate in the same unitarity cut. The three-mass triangles have a “signature” square root function, while the bubbles are entirely rational.

So there is no need for triple cuts.

It all comes down to the question of how to carry out the integration of double cuts.

We will see that our approach also separates the box contributions in a natural way.

Double Cut Integrals

A closer look at the phase space integral:

$$C_{i,\ldots,j} = \int d\mu\, A^{\mathrm{tree}}(-\ell_1, i, \ldots, j, -\ell_2) A^{\mathrm{tree}}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

$$d\mu = d^4\ell_1\, d^4\ell_2\, \delta^{(4)}(\ell_1 + \ell_2 - P_L)\, \delta^{(+)}(\ell_1^2)\, \delta^{(+)}(\ell_2^2)$$

Cachazo, Svrček, Witten (2004): Perform the $\delta^{(4)}$ integral to reduce to a single loop momentum variable ℓ.

$$\ell_{a\dot{a}} = t\lambda_a\tilde{\lambda}_{\dot{a}}.$$

t is real and the spinors λ and $\tilde{\lambda}$ are independent homogeneous coordinates on two copies of CP^1. The integral is over the diagonal CP^1 defined by $\tilde{\lambda} = \bar{\lambda}$. (See also Cachazo (2004).)

$$\int d^4\ell\delta^{(+)}(\ell^2)\,(\bullet) = \int_0^\infty dt\, t \int \langle\lambda, d\lambda\rangle[\tilde{\lambda}, d\tilde{\lambda}](\bullet)$$

Cut contribution of bubble

$$\Delta I_2 \quad = \quad \int d^4\ell \delta^+(\ell^2)\delta^+((\ell-K)^2)$$

- Since $\delta^+(\ell^2)$, we can write $\ell = t\lambda\widetilde{\lambda}$ so that

$$\int d^4\ell \, \delta^+(\ell^2) = \int_0^\infty t \, dt \int_{\widetilde{\lambda}=\lambda} \langle \lambda \, d\lambda \rangle \, [\widetilde{\lambda} \, d\widetilde{\lambda}]$$

(Cachazo, Svrček, Witten 2004)

In Minkowski space, we can have a choice of sign in $\widetilde{\lambda} = \pm\lambda$. The plus sign in $\delta^+(\ell^2)$ corresponds to one choice. Here we will just write $\delta(\ell^2)$.

- Now

$$\begin{aligned} \delta((\ell-K)^2) \quad &= \quad \delta(K^2 - tK_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}}) \\ &= \quad \frac{1}{(K_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})}\delta\left(t - \frac{K^2}{(K_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})}\right) \end{aligned}$$

- Putting it back and doing the t-integration we get

$$\Delta I_2 \quad = \quad \int \langle \lambda \, d\lambda \rangle \, [\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{K^2}{\left\langle \lambda | K | \widetilde{\lambda} \right]^2}$$

- The key to the calculation is another observation of Cachazo, Svrček, Witten (2004)

$$[\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | K | \widetilde{\lambda} \right]^2} = [d\widetilde{\lambda} \, \partial_{\widetilde{\lambda}}] \left(\frac{[\eta \, \widetilde{\lambda}]}{\langle \lambda | K | \eta] \left\langle \lambda | K | \tilde{\lambda} \right]} \right)$$

The integral is naively zero.

However, the contour of integration is where $\widetilde{\lambda}$ is the complex conjugate of λ, so there is a delta-function contribution like

$$\frac{\partial}{\partial \overline{z}} \frac{1}{z - b} = 2\pi \delta(z - b)$$

- Applied to our case, we see that there is a contribution from the pole at

$$|\lambda\rangle = |K|\eta], \qquad \Rightarrow |\widetilde{\lambda}] = |K|\eta\rangle$$

- Reading out the residue of this pole we finally get

$$\Delta I_2 \quad = \quad K^2(-\text{residue}) = -K^2 \left(\frac{[\eta\,\widetilde{\lambda}]}{\langle\lambda|K|\widetilde{\lambda}]} \right)_{|\lambda\rangle = |K|\eta]}$$

$$= \quad -\frac{K^2}{K^2} = -1$$

Cut contribution of three-mass triangle

$$\Delta I_3 \quad = \quad \int d^4\ell\, \delta(\ell^2) \frac{\delta((\ell - K_1)^2)}{(\ell + K_3)^2}$$

- After the t-integration:

$$\Delta I_3 \quad = \quad \int \langle \lambda\, d\lambda \rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | K_1 | \widetilde{\lambda} \right] \left\langle \lambda | Q | \widetilde{\lambda} \right]}$$

where $Q_{a\dot{a}} = \frac{K_3^2}{K_1^2}(K_{1,a\dot{a}}) + (K_{3,a\dot{a}})$ and $Q^2 = \frac{K_3^2 K_2^2}{K_1^2}$.

- Introduce a Feynman parameter:

$$\Delta I_3 \quad = \quad \int_0^1 dz \int \langle \lambda\, d\lambda \rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | (1 - z) K_1 + z Q | \widetilde{\lambda} \right]^2}$$

- Now we know how to do this integral (CSW):

$$\Delta I_3 = -\int_0^1 dz \frac{1}{((1-z)K_1 + zQ)^2}$$
$$= -\int_0^1 dz \frac{1}{K_1^2 + 2z((K_1 \cdot Q) - K_1^2) + z^2(Q-K_1)^2}$$

- Defining $a = (Q-K_1)^2$, $b = 2((K_1 \cdot Q) - K_1^2)$ and $c = K_1^2$, the result is

$$\Delta I_3 = \frac{1}{\sqrt{\Delta_{3m}}} \ln \left(\frac{2az + b - \sqrt{\Delta_{3m}}}{2az + b + \sqrt{\Delta_{3m}}} \right)$$

with

$$\Delta_{3m} = (K_1^2)^2 - 2K_1^2 K_2^2 - 2K_3^2 K_1^2 + (K_2^2 - K_3^2)^2$$

- The square root $\sqrt{\Delta_{3m}}$ (Gram determinant) functions as the signature of the three-mass triangle function.

General Setting

Now we can give the general outline for calculating coefficients of bubbles and triangles.

- First recall that

$$\text{Cut} = c_2 \Delta I_2 + \sum c_3 \Delta I_3^i + c_4 \Delta I_4^i$$

  where for a given (double) cut, there is only one bubble contribution but several triangle and box contributions.

- However, by our previous calculations, we know that $\Delta I_2 = -1$ while both ΔI_3^i and ΔI_4^i are logarithmic.

- This means that to read off the coefficient c_2, we just need to separate the rational part from Cut.

- To read off c_3, we need to find the logarithmic piece with signature

$$\ln\left(\frac{2az + b - \sqrt{\Delta_{3m}}}{2az + b + \sqrt{\Delta_{3m}}}\right)$$

General Cut Integrals

$$C_{i,\dots,j} = \int d\mu\, A^{\rm tree}(-\ell_1, i, \dots, j, -\ell_2) A^{\rm tree}(\ell_2, j{+}1, \dots, i{-}1, \ell_1)$$

$$d\mu = d^4\ell_1\, d^4\ell_2\, \delta^{(4)}(\ell_1 + \ell_2 - P_L)\, \delta^{(+)}(\ell_1^2)\, \delta^{(+)}(\ell_2^2)$$

- Express amplitudes in terms of spinor products.
- Use $\delta^{(4)}(\ell_1 + \ell_2 - P_L)$ to eliminate ℓ_2 in favor of ℓ_1. To do this, first rewrite the tree amplitudes to depend only on the vector ℓ_2.

$$\langle \bullet, \ell_2 \rangle = \frac{\langle \bullet, \ell_2 \rangle [\ell_2, \ell_1]}{[\ell_2, \ell_1]} = \frac{\langle \bullet | \ell_2 | \ell_1]}{[\ell_2, \ell_1]} = \frac{\langle \bullet | P_{ij} | \ell_1]}{[\ell_2, \ell_1]}.$$

The factors $[\ell_2, \ell_1]$ and $\langle \ell_2, \ell_1 \rangle$ all pair up in the end.

- CSW: $\ell_1 = \ell = t\lambda\tilde{\lambda}$.

$$C_{i,\dots,j} = \int_0^\infty t dt \langle\lambda, d\lambda\rangle[\tilde\lambda, d\tilde\lambda]\delta^{(+)}(t\lambda_a\tilde\lambda_{\dot a}P_{ij}^{a\dot a} - P_{ij}^2)G(\lambda,\tilde\lambda,t).$$

Perform the t integral using the delta function:

$$C_{i,\dots,j} = P_{ij}^2 \int \frac{\langle\lambda, d\lambda\rangle[\tilde\lambda, d\tilde\lambda]}{(P_{ij}^{a\dot a}\lambda_a\tilde\lambda_a)^2} G\left(\lambda, \tilde\lambda, \frac{P_{ij}^2}{P_{ij}^{a\dot a}\lambda_a\tilde\lambda_a}\right).$$

The spinors $\lambda, \tilde\lambda$ are homogeneous coordinates, but we must require that $\ell = t\lambda\tilde\lambda$ is invariant under scalings.

In particular, $G(\lambda, \tilde\lambda, t)$ is of degree zero in $\tilde\lambda$.

Thus it can be written as a sum of terms like

$$\frac{\prod_l [A_l, \tilde\lambda]}{\prod_i \left\langle\lambda|Q_i|\tilde\lambda\right] \prod_j [A_j, \tilde\lambda]} g(\lambda).$$

Now we will introduce Feynman parameters.

Feynman Parameters

Replace every factor $\frac{1}{[A_i,\tilde\lambda]}$ by $\frac{-\langle A_i,\lambda\rangle}{\langle\lambda|A_i|\tilde\lambda]}$; then combine all factors in the denominator with Feynman parameters $x_1,\ldots,x_{m+2}$.

$$\int\prod_{i=1}^{m+2} dx_i\,\delta\left(\sum_{j=1}^{m+2} x_j - 1\right)\int\frac{\langle\lambda,d\lambda\rangle[\tilde\lambda,d\tilde\lambda]}{\langle\lambda|T(x_1,\ldots,x_{m+2})|\tilde\lambda]^{m+2}}\prod_{l=1}^{m}[A_l,\tilde\lambda]\tilde g(\lambda)$$

The integrand can be written as a total derivative in $\tilde\lambda$.

$$\frac{[\tilde\lambda\,d\tilde\lambda]\prod_{i=1}^{j}[A_i\,\tilde\lambda][\eta\,\tilde\lambda]^{m-j}}{\langle\lambda|T|\tilde\lambda]^{m+2}} =$$

$$[d\tilde\lambda\,\partial_{\tilde\lambda}]\left[\frac{\prod_{i=1}^{j}\langle\lambda|T|A_i]}{\langle\lambda|T|\tilde\lambda]^{m+1}}\left(\sum_{k=0}^{j}\frac{(-1)^{j-k}(j-k)!}{(m+1-j)\ldots(m+1-k)}g_k[x_i]\frac{[\eta\,\tilde\lambda]^{m+1-k}}{\langle\lambda|T|\eta]^{j+1-k}}\right)\right].$$

Here, η is an arbitrary but fixed spinor and

$$g_k[x_s] = \sum_{i_1<i_2<\ldots<i_k} x_{i_1}\ldots x_{i_k}\quad\text{with}\quad x_i = \frac{[A_i,\tilde\lambda]}{\langle\lambda|T|A_i]}.$$

Refinement: Only One Parameter

We can do better by doing some preliminary simplification with Schouten's identity!

$$\frac{[a_1\ \widetilde{\lambda}]}{[b_1\ \widetilde{\lambda}][b_2\ \widetilde{\lambda}]} \times \frac{[b_1\ b_2]}{[b_1\ b_2]} = \frac{1}{[b_1\ b_2]}\frac{[a_1\ b_1][\widetilde{\lambda}\ b_2] + [a_1\ b_2][b_1\ \widetilde{\lambda}]}{[b_1\ \widetilde{\lambda}][b_2\ \widetilde{\lambda}]}$$

This splitting leads to the natural separation of box, triangle, and bubble contributions.

There are only two types of integrals that will remain:

$\int \frac{\langle\lambda,d\lambda\rangle[\tilde{\lambda},d\tilde{\lambda}]}{(P_{ij}^{a\dot{a}}\lambda_a\tilde{\lambda}_a)^2} H(\lambda)$ requires no Feynman parameters and produces a rational function.

$\int \frac{\langle\lambda,d\lambda\rangle[\tilde{\lambda},d\tilde{\lambda}]}{\langle\lambda|P|\tilde{\lambda}]\langle\lambda|Q_r|\tilde{\lambda}]} H(\lambda)$ requires one Feynman parameter and produces only logarithms.

Canonical Decomposition

Let's look more closely at the type of term yielding logarithms:

$$\int \frac{\langle \lambda, d\lambda \rangle [\tilde{\lambda}, d\tilde{\lambda}]}{\left\langle \lambda | P | \tilde{\lambda} \right] \left\langle \lambda | Q_r | \tilde{\lambda} \right]} H(\lambda)$$

Combine the denominators with a Feynman parameter x. There are contributions from (1) poles in $H(\lambda)$ and (2) the pole $\langle \lambda | xP + (1-x) Q | \eta]$.

The first kind gives linear factors of x in the denominator. Upon integration, this gives logarithms, which are the discontinuities of 1m-, 2m-, and 3m- (finite) scalar box integrals.

The second contribution has the same form as our bubble example:

$$\int_0^1 dx \frac{1}{(xP + (1-x)Q)^2} H(\lambda(x))$$

Here $\lambda(x)$ is the solution of $\langle\lambda|xP + (1-x)Q|\eta] = 0$.

When the discriminant $\Delta = 4(((P-Q)\cdot Q)^2 - (P-Q)^2Q^2)$ is not a perfect square, then $\sqrt{\Delta}$ is the signature square root of a 3m-triangle or 4m-box.

Suppose that we have identified all box coefficients by quadruple cuts.

We will look for discriminants of the form

$$\Delta_{3m} = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2K_1^2 K_2^2 - 2K_3^2 K_1^2 - 2K_2^2 K_3^2.$$

If $P = K_1$, then we need to see $Q_{a\dot{a}} = \frac{K_3^2}{K_1^2}(K_1)_{a\dot{a}} + (K_3)_{a\dot{a}}$.

We can use our triangle result to stop short of the full integration.

Simply identify the terms of the form $c_3^{3m} \int_0^1 dx \frac{1}{(xP+(1-x)Q)^2}$ and read off the coefficients!

Example

The cut C_{234} of $A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$.

- After the t integration, we are left with

$$C \int \langle \ell \, d\ell \rangle \, [\ell \, d\ell] \frac{P_{561}^2}{\langle \ell | P_{561} | \ell]^2} \frac{[3 \, \ell]}{[4 \, \ell]} \frac{\langle \ell \, 1 \rangle \, \langle \ell | P_{561} | 3]}{\langle \ell \, 5 \rangle \, \langle \ell | P_{561} | 2]}$$

- Split into two terms:

$$\int \langle \ell \, d\ell \rangle \, [\ell \, d\ell] \frac{C P_{561}^2 \, \langle \ell \, 1 \rangle \, \langle \ell | P_{561} | 3]}{\langle \ell \, 5 \rangle \, \langle \ell | P_{561} | 2] \, \langle \ell | P_{561} | 4]} \left(\frac{[3 \, 4]}{[4 \, \ell] \, \langle \ell | P_{561} | \ell]} + \frac{\langle \ell | P_{561} | 3]}{\langle \ell | P_{561} | \ell]^2} \right)$$

- The second term is of the type I_2 and gives a rational function.

$$C\left(\frac{\langle 1|6|5]\,\langle 5|P_{561}|3]^2}{\langle 5|P_{561}|2]\,\langle 5|P_{561}|4]\,\langle 5|P_{561}|5]}+\frac{\langle 1\,2\rangle\,[2\,3]^2P_{561}^2}{[2\,4]\,\langle 5|P_{561}|2]\,\langle 2|P_{561}|2]}\right.$$
$$\left.+\frac{\langle 1\,4\rangle\,[3\,4]^2P_{561}^2}{[4\,2]\,\langle 5|P_{561}|4]\,\langle 4|P_{561}|4]}\right)$$

which is, in fact, the coefficient of the bubble integral $I_{2;(234)}$.

- The first term is the type of I_3 and we need to perform the Feynman parameter integrations. The result is

$$\frac{\langle 1\,5\rangle\,\langle 1|P_{561}|3]^2\,\langle 5|P_{561}|3]}{[2\,3]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|P_{561}|2]\,\langle 5|P_{561}|4]^2}\times\ln\left(\frac{P_{234}^2}{P_{234}^2-P_{23}^2}\,\frac{-P_{45}^2}{P_{234}^2-P_{61}^2}\right)$$
$$-\frac{\langle 1|P_{234}|2]\,\langle 1|P_{234}|3]^2}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,[4\,2]^2\,\langle 5|P_{234}|2]\,P_{234}^2}\times\ln\left(\frac{P_{23}^2P_{34}^2}{(P_{234}^2-P_{23}^2)(P_{234}^2-P_{34}^2)}\right)$$

It can be checked that they are the imaginary parts of box contributions.

Another Example

Let's see how to get a three-mass triangle coefficient. Here is one of the terms of the cut C_{23} of $A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$.

The integrand is given by

$$C g(\ell) \frac{\langle 1|\ell_2 + 4 + 5|5]}{(\ell_2 + 4 + 5)^2} \frac{[3\,\ell]}{[2\,\ell]}$$

where

$$\begin{aligned} g(\ell) &= -\frac{\langle \ell|P_{23}|5]}{\langle \ell|P_{23}|4]} \frac{(\langle 1|6|5]\,\langle 2\,\ell\rangle + \langle 2|3|5]\,\langle 1\,\ell\rangle)^2}{\langle \ell|P_{61}|5]\,\langle \ell|P_{23}P_{45}|6\rangle} \\ C &= \frac{1}{[4\,5]\,\langle 6\,1\rangle\,\langle 2\,3\rangle} \end{aligned}$$

After the t-integration and splitting, we end up with

$$C_{23}^{(3)} = \quad C \int \langle \ell \, d\ell \rangle \, [\ell \, d\ell] \frac{P_{23}^2}{\langle \ell | P_{23} | \ell]^2} \frac{[3 \; \ell]}{[2 \; \ell]} \frac{-\langle 1 | P_{61} | 5] \langle \ell | P_{23} | \ell] + P_{23}^2 \langle 1 | \ell | 5]}{\langle \ell | Q | \ell] P_{23}^2} g(\ell)$$

where $Q = \frac{P_{61}^2 P_{23} + P_{23}^2 P_{61}}{P_{23}^2}$.

Using our splitting procedure we end up with

$$\begin{aligned} C_{23}^{(3)} \quad = \quad & C \frac{1}{\langle \ell | P_{23} | \ell] \, [2 \; \ell]} \frac{g(\ell)[3 \; 2]}{\langle \ell | Q | 2]} \left(- \langle 1 | 6 | 5] + \frac{\langle 3 \; 2 \rangle \, \langle 1 \; \ell \rangle \, [2 \; 5]}{\langle \ell \; 3 \rangle} \right) \\ & - C \frac{1}{\langle \ell | P_{23} | \ell]^2} \frac{g(\ell) P_{23}^2 \, \langle 1 \; \ell \rangle \, \langle \ell | P_{23} | 5] \, \langle \ell \; 2 \rangle}{\langle \ell \; 3 \rangle \, \langle \ell | P_{23} Q | \ell \rangle} \\ & - C \frac{1}{\langle \ell | P_{23} | \ell] \, \langle \ell | Q | \ell]} \frac{g(\ell) \, \langle \ell | Q | 3]}{\langle \ell | Q | 2] \, \langle \ell | P_{23} Q | \ell \rangle} A \end{aligned}$$

where $A = \big(\langle 1 | 6 | 5] \, \langle \ell | P_{23} Q | \ell \rangle + P_{23}^2 \, \langle 1 \; \ell \rangle \, \langle \ell | Q | 5] \big)$.

Of these three terms, the first one gives imaginary parts of box functions. The second term contributes rational function which will be part of the coefficient of bubble. The third term is the one we are looking for and will give the coefficient of three-mass triangle.

Now let us see how to read off the coefficient of a three-mass triangle.

- The third term can be written as

$$C_{23}^{(3;3)} = -C\int_0^1 dz \int \langle \ell\, d\ell\rangle\,[\ell\, d\ell]\frac{1}{\langle \ell|P|\ell]^2}\frac{g(\ell)\,\langle \ell|Q|3]}{\langle \ell|Q|2]\,\langle \ell|P_{23}Q|\ell\rangle}A$$

  with $P \equiv (1-z)P_{23} + zQ$.

- The integration of $\lambda, \widetilde{\lambda}$ gives

$$C_{23}^{(3;3)} = C\int_0^1 dz \left[\frac{\langle \widetilde{\eta}|P|\ell]}{\langle \ell|P|\ell]\, P^2}\frac{g(\ell)\,\langle \ell|Q|3]}{\langle \ell\, \widetilde{\eta}\rangle\,\langle \ell|Q|2]\,\langle \ell|P_{23}Q|\ell\rangle}A\right]_{poles}$$

- Since the red part does not contain poles, we can do the $\int_0^1 dz$ integration first before taking residues of poles.
- The $\int_0^1 dz$ integration can be further simplified by a partial fraction expansion.

$$\begin{aligned}
& \int_0^1 dz \frac{(zc_1 + c_2)}{(a_0 z^2 + a_1 z + a_2)(zb_1 + b_2)} \\
= \quad & \int_0^1 dz \frac{b_1(-b_2 c_1 + b_1 c_2)}{(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)} \frac{1}{(zb_1 + b_2)} \\
+ \quad & \int_0^1 dz \frac{(b_2 c_1 - b_1 c_2)}{2(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)} \frac{(2za_0 + a_1)}{(a_0 z^2 + a_1 z + a_2)} \\
+ \quad & \int_0^1 dz \frac{(2a_2 b_1 c_1 - a_1 b_2 c_1 - a_1 b_1 c_2 + 2a_0 b_2 c_2)}{2(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)} \frac{1}{(a_0 z^2 + a_1 z + a_2)}
\end{aligned}$$

We have found the coefficient of the three-mass triangle.

$$c_3^{3m} = -C \sum_{i=1}^{7} \left[\langle \ell\, \ell_i \rangle \, \frac{g(\ell)\, \langle \ell | Q | 3]\, R_1(a_0, a_1, a_2, b_1, b_2, c_1, c_2)}{\langle \ell\, \widetilde{\eta} \rangle\, \langle \ell | Q | 2]\, \langle \ell | P_{23} Q | \ell \rangle} A \right]_{\ell \to \ell_i}$$

with

$$\begin{aligned} a_0 &= (Q - P_{23})^2, \quad a_1 = 2P_{23} \cdot (Q - P_{23}), \quad a_2 = P_{23}^2 \\ b_1 &= \langle \ell | (Q - P_{23}) | \ell] \,, \quad b_2 = \langle \ell | P_{23} | \ell] \\ c_1 &= \langle \widetilde{\eta} | (Q - P_{23}) | \ell] \,, \quad c_2 = \langle \widetilde{\eta} | P_{23} | \ell] \end{aligned}$$

and

$$R_1(a_0, a_1, a_2, b_1, b_2, c_1, c_2) = \frac{(2a_2 b_1 c_1 - a_1 b_2 c_1 - a_1 b_1 c_2 + 2a_0 b_2 c_2)}{2(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)}$$

Conclusions

- We have a systematic approach to evaluate finite unitarity cuts.
- It is good for $\mathcal{N} = 1$ Super-Yang-Mills at one loop (or also cut-constructible part of nonsupersymmetric YM).
- There is a modified basis of integrals for one-loop gluon amplitudes with finite boxes, 3-mass triangles, and bubbles.
- We use the phase space measure of CSW. This is particularly effective because the integral falls apart into a sum over poles.
- We arrange the remaining integration to involve at most one Feynman parameter.
- Coefficients are naturally separated and can thus be targeted independently.