Non-Locality in the Wavefunction of a Single Particle

Detector Screen

Detection HERE

Forbids the detection of the particle HERE

If a wavefunction were a local disturbance, then detections at both places (or nowhere) could occur — would seem to imply superluminal communication

Twistor Theory

Space-time M (Minkowski)

coordinates \((r^0, r^1, r^2, r^3)\)

Twistor space \(\mathbb{PN}\)

homogeneous coordinates \(Z^0 : Z^1 : Z^2 : Z^3\)

Incidence

\[
\left( \frac{Z^0}{Z^3} \right) = \frac{i}{\sqrt{2}} \left( r^0 + r^3, r^1 + i r^2, r^0 - i r^2 \right) \left( \frac{Z^3}{Z^0} \right)
\]

PT+

PT−

Eqn. of \( \mathbb{PN} \):

\[
Z^0 \overline{Z}^2 + Z^1 \overline{Z}^2 + \overline{Z}^2 Z^0 + Z^2 \overline{Z} = 0
\]
2-spinor Notation

Vector/tensor indices: $a, b, c, \ldots, a_0, a_1, \ldots$
4-dimensional indices $0, 1, 2, 3$

2-spinor indices: $A, B, C, \ldots, A_0, A_1, \ldots$
Abstract indices: $\alpha = A A', \beta = B B', \gamma = C C', \ldots, \alpha_0 = A A', \alpha_1 = A A'$

Complex conjugate: $c = \bar{c}$

Standard Coordinates:

$V_{\alpha A'} = \left( V_{00}^{\alpha}, V_{01}^{\alpha} \right) = \frac{1}{\sqrt{2}} \left( V_0 + V_3, V_1 + iV_2 \right)$

Raise/lower indices

$g_{ab}, \ g^{ab}$: $E_{AB}, E_{A'B'} E_{A'B'}$

Symmetric

$g_{ab} = g_{AA'} BB' = E_{AB} E_{A'B'}$

Anti-symmetric

Interpretation of $\eta^A$: [Diagram]
Twistor - Strings

New development, only about 2 years old (primarily due to E. Witten) in which twistor and string ideas are combined to give new insights and calculational techniques for calculating Yang-Mills (gluon) scattering amplitudes.

At first sight, this marriage seems improbable, owing to some basic incompatibilities:

- String theory's supra-dimensionality
- Twistor theory's holomorphicity

Dimension: twistor 1+3
String 1+25, 1+9, 1+10, 2+10, 2+2

NB: Spinors/twistors are complex only for signature $s-t \equiv 2 \pmod{4}$ ($s =$ space dim., $t =$ time dim.)
$s-t \pmod{8}$ $0 1 2 3 4 5 6 7$

number syst. $\mathbb{R} \mathbb{C} \mathbb{H} \mathbb{H} \mathbb{C} \mathbb{R}$

NB: Strings are Riemann surfaces, but p-branes not hol.

But in twistor string theory one puts the strings in twistor space whose dimensions are NOT additional to those of space-time; they are "already there" according to ordinary space-time geometry.

Some Remarks on Twistor/Spinor conventions

- In ++++ or +++-- signature we have a complex conjugation $z^a \rightarrow \bar{z}^a$ rather than $\bar{z}^a$ (Lorentzian case) so does not agree with canonical conjugation, so wavefunctions are not holomorphic.

- In +++-- we can consider $\delta$-fns. on real $\alpha$-planes. In [CM], these would be hyperfunctional objects

::: cohomology elements in space-time
Compare with Hodges elemental states

- Up/down nature of spinor parts $(\omega^A, \pi^A)$ fits in with conformal trans

\[ \begin{align*}
\mathcal{A}_a &= \Omega^a_{\alpha} \mathcal{A}_a \\
T_{\lambda} &= \lambda \mathcal{A}_a \\
\Gamma_a &= \pi^A_{\alpha} + \omega^a \\
\end{align*} \]
Quantum Twistor Theory

$Z^\alpha$ and $\bar{Z}_\alpha$ become non-commuting

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0$$
$$Z_\alpha Z_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$
$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar \delta^\alpha_\beta$$

so $Z^\alpha$ and $\bar{Z}_\alpha$ are canonical conjugate variables (as well as complex conjugate).

Choose $\hbar = 1$, for convenience. We find

$$P_a = \pi_a \pi_a$$
$$M^{ab} = i \omega^{(a b)} A_a B_b - i E A B \omega^{(a b)}$$
undisturbed by factor ordering, but

$$S = \frac{1}{4} (Z^\alpha \bar{Z}_\alpha + Z_\alpha Z^\alpha)$$

The standard commutators for $P_a$ and $M^{ab}$ follow

$$[P_b, P_a] = 0$$
$$[P_b, M^{bc}] = i (g^{bc} p^a - g^{ca} p^b)$$
$$[M^{ab}, M^{cd}] = i (g^{bd} M^{ac} + g^{ad} M^{bc} - g^{cd} M^{ab})$$

Lie algebra generators for the Poincaré group.

Twistor Wavefunctions

Since $Z^\alpha$ and $\bar{Z}_\alpha$ are conjugate variables, a twistor wavefunction $f$ ought to depend on either $Z^\alpha$ or $\bar{Z}_\alpha$, but not both. But what does it mean to say that $f(Z^\alpha)$ does not depend on $\bar{Z}_\alpha$? The condition is

$$\frac{\partial f}{\partial \bar{Z}_\alpha} = 0$$

the Cauchy–Riemann equations asserting that $f$ is holomorphic in $Z^\alpha$.

Helicity eigenstates

These are eigenstates of the helicity operator $S = \frac{\hbar}{2} (-Z^\alpha \partial_\alpha Z^\alpha)$.

But $\frac{\partial}{\partial Z^\alpha}$ is the Euler homogeneity operator, whose eigenstates are homogeneous functions, with eigenvalue = degree of homogeneity.

Take the integer $n = 2S/\hbar$ to represent the helicity; then $f$ is homogeneous (holomorphic) in $Z^\alpha$ of degree $-2 - n$. Alternatively use $f(W_\alpha)$, with $W_\alpha = Z_\alpha$. Then $f$ is holomorphic of degree $-2 + n$. 
How does this relate to an ordinary space-time description?

\[ n = 0: \quad \nabla_a \nabla^a \phi = 0 \]

with \[ \nabla_a \left( \nabla_a \phi \right) = \frac{\partial}{\partial x^a} \]

\[ n < 0: \quad \nabla^A A' \psi_{AB...L} = 0 \]

\[ \psi_{AB...L} = \psi^r_{(AB...L)} \text{ symmetry brackets} \]

\[ n > 0: \quad \nabla^{A'} A' \phi_{A'B'...L'} = 0 \]

\[ \phi_{A'B'...L'} = \phi^{r(AB'...L')} \]

Free-field Maxwell equations

\[ F_{ab} = \psi_{AB} \varepsilon_{A'B'} + \varepsilon_{A'B'} \phi_{A'B'} \]

implies \[ F_{ab} = -F_{ba} \]

and \[ \nabla^A A' \psi_{AB} = 0, \quad \nabla^A A' \phi_{A'B'} = 0 \]

imply \[ \nabla_{[a} F_{bc]} = 0, \quad \nabla^a F_{ab} = 0 \]

and conversely. \[ dF = 0, \quad d^*F = 0 \]

Linearized source-free Einstein theory

\[ K_{abcd} = \psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \varepsilon_{AB} \varepsilon_{C'D'} \psi_{A'B'C'D'} \]

implies \[ K_{abcd} = -K_{bacd} = -K_{cdab} = K_{dabc} \]

and \[ K_{abcd} + K_{bacd} + K_{cadb} = 0, \quad K_{abc} = 0; \]

moreover \[ \nabla^A A' \psi_{ABCD} = 0, \quad \nabla^A A' \psi_{ABCD} = 0 \]

imply (Bianchi) \[ \nabla_a K_{bcde} + \nabla_b K_{cade} + \nabla_c K_{dae} = 0 \]

and conversely.

Maxwell field (complex)

\[ S/h \quad \text{hom.} \quad Z^a \quad \text{hom.} \]

\[ F_{ab} = \psi_{AB} \varepsilon_{A'B'} + \varepsilon_{AB} \phi_{A'B'} \]

-1 -4 0

Lineairized gravity (complex 1st order curvature)

-2 +2 -6

K_{abcd} = \psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \varepsilon_{AB} \varepsilon_{C'D'} \psi_{A'B'C'D'}

-2 -6 +2

Massless neutrino

-3 -1

Massless anti-neutrino

-1 -3

Scalar wave

-2 -7 -3
Twistor description of single-photon states:

\[ f(Z^a) \]

- helicity: hom.deg. 0

\[ f = f_0 + f_{-4} \]

\[ S = \pm 1 \quad S = \mp 1 \]

\[ F_{ab} = \psi_{AB} E_{A'B'} + \epsilon_{AB} \phi_{A'B'} \]

abstract indices

\[ F_{ab}(x) = \int \left\{ \epsilon_{AB} \psi_{A'B'} \frac{\partial}{\partial \omega^a} \frac{\partial}{\partial \omega^b} + \epsilon_{AB} \pi^a_{A'B'} \right\} f(Z) d\pi \]

Integrate over region (dim.)

\[ \omega = ix \pi \text{ (incidence)} \]

Integrate out \( j \) left with \( F(x) \). Maxwell automatic

Contour Integral Expressions

(Whittaker, Bateman, RP, Hughston)

\[ \phi(x^0) = \text{con.x.} \int_0^\pi f(\omega^a, \pi^a) \sin \omega = i \pi \]

Incidence

Homogeneous version \((1-dim f)\)

\[ \delta_{\omega^a} = \pi A^a d\pi^a \]

Inhomogeneous version \((2-dim f)\)

\[ \delta_{\omega^a} = \pi A^a d\pi^a \]

Positive Helicity:

\[ \Phi_{A'B' \ldots L'}(x^0) = \text{con.x.} \int_0^\pi \pi_{A'} \pi_{B'} \ldots \pi_{L'} f(\omega, \pi) d\pi \]

Negative Helicity:

\[ \psi_{AB \ldots L}(x^0) = \text{con.x.} \int_0^\pi \pi_{A} \pi_{B} \ldots \pi_{L} f(\omega, \pi) d\pi \]

(cannonical case (elementary state):

\[ f = \frac{1}{(A_n Z^a B_n Z^a)} \]

General positive frequency:

\[ z^a = (\omega^a, \pi^a) \]

\[ d^2 \pi = d\pi^a d\pi^a \]

\[ \omega = \frac{i}{2} \epsilon_{AB} d\pi^a d\pi^a \]

\[ \text{Nierman sphere} \]

\[ \text{Lines in PT } \text{corr. pts. in the forward tube} \]

\[ \text{Regions of Singularity} \]

\[ \text{Pts. with past-pointing imag. part} \]
More generally, need

\[ f_{ij} = -f_{ji} \text{ on overlaps} \]

\[ f_{ij} + f_{jk} = f_{ik} \text{ on triple overlaps} \]

\[ f_{ij} \equiv f_{ij} + h_i - h_j \text{ if defined on } U_i \]

Branched contour $\$ \text{ Non-linear (e.g. building a manifold)}$

Cohomology:

a precise non-local measure — here of the degree of IMPOSSIBILITY
General relativity

Anti-self-dual curvature
("non-linear graviton")

\[ F = \frac{1}{\alpha^0} \frac{\partial \alpha^a}{\partial z^a} \]

Z "infinity twistor" gives general soln of asd. vacuum

Self-dual curvature — information coded (in strange way) by "googly"

\[ G = f^a_e z^a \frac{\partial}{\partial z^e} \]

For asymptotically flat curved vacuum space-times, the full gravitational field is encoded geometrically (but strangely) in the structure of a deformed twistor space.

Not yet clear how to retrieve the space-time from the twistor space.
Positive/Negative Frequency Splitting

Riemann sphere $\mathbb{CP}^1$

- Frequency
- $\mathbb{C}$
- $\mathbb{D}$
- Real axis

Splitting of $H^3$ (ordinary functions) by holomorphic extension

Projective twistor space $\mathbb{PN} = \mathbb{CP}^3$

- $\mathbb{PN}$ (null twistors)
- $\mathbb{PN}$

This realized (finally!) one of the very early motivations behind twistor theory

Quantum Twistor Geometry

A form of non-commutative geometry?

$Z_\alpha$ and $\bar{Z}_\alpha$ do not commute:

$$Z_\alpha \bar{Z}_\beta - \bar{Z}_\beta Z_\alpha = \delta_\alpha^\beta$$

(taking $\hbar = 1$). Realized by $\bar{Z}_\alpha = -\frac{\partial}{\partial Z^\alpha}$.

Note that a satisfactory solution of the googly problem depends upon an isomorphism between the algebra (and geometry) of $Z_\alpha$

and that of $\frac{\partial}{\partial Z_\alpha}$ (whatever that means!)

In effect, this seems to come down to a proper understanding of the scalar product $\langle \ldots | \ldots \rangle$, as we have:

$$\langle Z^* f | g \rangle = - \langle f | \frac{\partial}{\partial Z^*} g \rangle$$

and

$$\langle \frac{\partial}{\partial Z^*} f | g \rangle = \langle f | g \rangle.$$
Twistor functions are normally taken to be homogeneous (to represent helicity states), and for homogeneity \( n > -4 \) we have expressions like

\[
\langle f_{\omega}, g_{\lambda} \rangle = \int \frac{d^4w, d^4z}{(2\pi)^4} \frac{1}{(z \cdot w)^{n+4}} f_\omega(z) g_\lambda(z) d^4w, d^4z
\]

But what we really need, instead of \( \frac{1}{x^2} \) (writing \( x = z \cdot w \)) is:

\[
\frac{1}{x}, -\frac{1}{x^2}, \frac{21}{x^3}, -\frac{31}{x^4}, \ldots
\]

to ensure \( z \cdot w = -\frac{2}{3} z \).

But we really shouldn't insist on homogeneity (e.g. plane-polarized photon, googlys etc.). So we require

\[
\frac{1}{x} - \frac{1}{x^2} + \frac{21}{x^3} - \frac{31}{x^4} + \ldots
\]

(noting that "cross terms" of wrong homogeneity all give zero in the \( \phi \)).

But \( \frac{1}{x} - \frac{1}{x^2} + \frac{21}{x^3} - \frac{31}{x^4} + \frac{41}{x^5} \ldots = F(x) \) diverges for all \( x \neq 0 \). Here Euler comes to our rescue!

Note (formally) \( \frac{df(x)}{dx} = f(x) - \frac{1}{x} \)

Define \( G(x) = e^x \int_0^\infty e^{-y} \frac{dy}{y} \)

then \( \frac{d}{dx} G(x) = G(x) - \frac{1}{x} \)

also \( G(x) = \int_x^\infty \frac{e^{-y}}{y} dy = \int_0^\infty \frac{e^{-u}}{x+u} du \)

Euler says: \( F(x) \equiv G(x) \)

Problem: understand equivalence

Use \( G \)

Hodges boundary on \( w \cdot z + k \)
Euler: \( \gamma = \text{Euler's constant} \)

\[ G(x) = -\gamma - \log x \]
\[ + x (1 - \gamma - \log x) \]
\[ + \frac{x^2}{2!} \left(1 + \frac{1}{2} - 2 - \log x\right) \]
\[ + \frac{x^3}{3!} \left(1 + \frac{1}{2} + \frac{1}{3} - 3 - \log x\right) \]
\[ + \frac{x^4}{4!} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - 4 - \log x\right) \]
\[ + \ldots \]

Note that 0th-order term vanishes at \( x = e^{-\gamma} \).

Contour integral \( \int \) Hodges k

Expressions can be made consistent via a "blow-down" only with boundary at \( x = k \).