Gauge Theory Amplitudes from Strings in Twistor Space

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String theory: long list of field theory implications/techniques

- correlation functions (gauge/string duality)
- exact effective superpotentials (topological strings)
- new perturbative techniques for gauge theories and gravity

Latest: Witten (2003)

The perturbative expansion of $\mathcal{N} = 4$ SYM is equivalent to the non-perturbative topological string field theory of the B-model on $\mathbb{P}^{3|4}$; a D1-instanton of genus g contributes to amplitudes with at most g loops and an instanton of degree d contributes to an amplitude with $d + L - 1$ negative helicity gluons.

Yesterday: the twistor string in the connected prescription

The Connected Prescription

(Spradlin, Volovich, RR)

$$A = \sum_{q|H_r^A=0} \delta^4(H_1^A)\, J\, \frac{(\det F)^4}{\det(\partial \widehat{H}_r^A/\partial q_s)}$$

where

$$H_r^A = \begin{cases} -\lambda_i^\alpha + \xi_i \sum_{k=0}^d \sigma_i^k a_k^\alpha \quad , & A = \alpha = 1,2, \quad i = 1,\ldots,n \\ \sum_{i=1}^n \xi_i \sigma_i^k \bar{\lambda}_i^{\dot\alpha} \quad , & A = \dot\alpha = 1,2, \quad k = 0,\ldots d \end{cases}$$

$$q_s = (\sigma_i, \xi_i, a_k^\alpha) \qquad \# = (n + n + 2(d+1)) - 4$$

$$J = \frac{1}{\prod \xi_i} \frac{\text{GL(2) gauge fixing}}{\prod(\sigma_i - \sigma_{i+1})}$$

$$F_i{}^k = \xi_i \sigma_i^k \quad ; \quad i \text{ runs over negative hel. gluons}$$

Slight sleight of hand in relating it to the twistor string

Plan

- The connected prescription: explicit tests and properties
- BST, twistors and regularization: can CSG be decoupled?
- CSG and the twistor string
- The (mostly) connected prescription and loop amplitudes

Explicit Tests and Properties

- $d = 1$ $\longrightarrow$ mostly $(+)$ MHV amplitudes

  essentially equivalent to Nair's and Witten's derivations

- $d = n - 3$ $\longrightarrow$ mostly $(-)$ MHV amplitudes

General? not analytically...

Obs: Constraint equations H_r^A depend only on n and d but not on the specific helicity assignment $\longrightarrow$ all amplitudes for fixed n and d emerge by summing over the same roots

- numerically: all 6-point NMHV amplitudes

General: More profitable to show that the connected prescription has the general properties of field theory amplitudes

- Cyclicity:

$$A(2,3,\ldots,n,1) = A(1,2,\ldots,n).$$

- Reflection:

$$A(n,n-1,\ldots,1) = (-1)^n A(1,2,\ldots,n).$$

- Dual Ward (or Sub-Cyclic) Identity:

$$\sum_{C(1,\ldots,n-1)} A(1,2,3,\ldots,n) = 0,$$

n – fixed; $C(1,\ldots,n-1)$ set of cyclic permutations of $\{1,\ldots,n-1\}$

- Generalized dual Ward identity:

$$\sum_{\mathrm{Perm}(i,j)} A(i_1,\ldots,i_m,j_1,\ldots,j_k,n+1) = 0, \qquad 1 \leq m \leq n-1, \qquad m+k=n,$$

where the sum is taken over permutations of the set $(i_1,\ldots,i_m,j_1,\ldots,j_k)$ which preserve the order of the $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_k)$ separately.

Consequences of the correlation function of vertex operators

$$J = \frac{1}{\prod \xi_i} \frac{\text{GL(2) gauge fixing}}{\prod(\sigma_i - \sigma_{i+1})}$$

- Conjugation/CPT: invariance under $+ \leftrightarrow -$ with simultaneous $\lambda \leftrightarrow \bar{\lambda}$.

$$A(\lambda_i, \bar{\lambda}_i, \eta_{iA}) = \int d^{4n}\psi \ \text{exp} \left[i \sum_{i=1}^{n} \eta_{iA} \psi_i^A \right] A(\bar{\lambda}_i, \lambda_i, \psi_i^A).$$

- Obscured by treating λ and $\tilde{\lambda}$ nondemocratically

- Can be proved by elementary means

(Spradlin, Volovich, RR)
related proof by Witten

- the solutions of the constraint equations $H_r^A(q) = 0$ for any n and d are in one to one correspondence with the solutions to the constraints $\widetilde{H}_r^A(\widetilde{q}) = 0$ for any n and $\widetilde{d} = n - d - 2$

- The proof is constructive. The solution for the remaining curve moduli can be easily calculated in both cases and the remaining constraints are mapped into each other by

$$\tilde{\sigma} = \sigma \qquad\qquad \tilde{\xi}_i = \frac{1}{\xi_i \prod_{j \neq i} (\sigma_i - \sigma_j)}$$

- Transformation of Jacobian: takes into account the flip of helicity

- Soft-Gluon Limit: in the limit $p_1 \to 0$ any amplitude behaves as

$$A(1^+, 2, \ldots, n) \longrightarrow \frac{\langle n\, 2\rangle}{\langle n\, 1\rangle \langle 1\, 2\rangle} A(2, \ldots, n).$$

- Collinear Limit I: same helicity ($p_1 \to zp$ and $p_2 \to (1-z)p$ with $p^2 = 0$)

$$A(1^+, 2^+, 3, \ldots, n) \longrightarrow \frac{1}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2\rangle} A(p^+, 3, \ldots, n).$$

Important observation: ▪ The singular behavour is captured by one (of the many) solution(s) of the constraint equations

▪ This solution has $\sigma_1 \simeq \sigma_2$

- Collinear Limit II: opposite helicity

  ($p_1 \rightarrow zp$ and $p_2 \rightarrow (1-z)p$ with $p^2 = 0$)

$$A(1^+, 2^-, 3, \ldots) \longrightarrow \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{[1\,2]} A(p^+, 3, \ldots) + \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{\langle 1\,2 \rangle} A(p^-, 3, \ldots)$$

- Multi-particle Poles: Color-ordered amplitudes can only have poles in channels corresponding to a sum of cyclically adjacent momenta going on-shell.

$$A_n(1, \ldots, n) \longrightarrow \sum_{\chi=\pm} A_{m+1}(1, \ldots, m, p^{\chi}) \frac{i}{p_{1,m}^2} A_{n-m+1}(m+1, \ldots, n, p^{-\chi}).$$

where $p_{1,m} = p_1 + p_2 + \cdots + p_m$ with $p_{1,m}^2 \rightarrow 0$

- Hard to prove in connected language

- Some arguments from relation with the disconnected prescription

Gukov, Motl, Nietzke

Connected vs. Disconnected?

- Connected instantons appear to give the YM amplitudes
- Cachazo, Svrcek, Witten: disconnected instantons do the same

$$T = \int d^4x_A^{a\dot{a}} d^4x_B^{b\dot{b}} \int \frac{dl'}{2\pi} d^2m_1'^{\dot{a}} d^2m_2'^{\dot{b}} V(\lambda_A, l') V(\lambda_B, l') I_\eta(x_A - x_B) G(m'_{12})$$
$$\times \prod_{i\in A} \delta^2(\mu_{Ai}^{\dot{a}} - x_A^{a\dot{a}} \lambda_{Ai\,a}) \delta^2(m_1'^{\dot{a}} - x_A^{a\dot{a}} l'_a) \prod_{i\in B} \delta^2(\mu_{Bi}^{\dot{b}} - x_B^{b\dot{b}} \lambda_{Bi\,b}) \delta^2(m_2'^{\dot{a}} - x_A^{a\dot{a}} l'_a)$$

- Bena, Bern, Kosower: result from partly connected instantons
- Gukov, Motl, Nietzke: All prescriptions localize on the intersection of the various components of the moduli space of curves
- Witten: Residue theorem: concise way of phrasing this

(details to be worked out)

$$\sum_{\{f_j(z_i)=0\}} \frac{1}{\det \partial f_j / \partial z_i} = 0$$

Agreement :	roots at fin. dist.
Degenerate curves :	roots at infinity

Loops

Formally:

$$A = \int d\mu_{\text{curves of degree } d \text{ and genus } g \leq L} \prod_{i=1}^{n} \Phi(Z(\sigma_i))\, d\sigma_i \, \langle J(\sigma_1) \ldots J(\sigma_n) \rangle_g$$

- Modification of earlier discussion:

  - $\prod_i \frac{1}{\sigma_i - \sigma_{i+1}} \quad \longrightarrow \quad \prod_i f(\vartheta(\tau|\sigma_i - \sigma_{i+1}),\ \text{aux. gauge field})$
  - Integration measure complicated, but computable/guessable
  - Analysis of resulting expression : challenging

- Sometimes necessary curves do not exist → simplifications
- Generally: similar to usual string theory in curved background

More complications: additional states – CSG

$$V_f: \quad (0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2) \qquad\qquad V_g: \quad (-2, -\tfrac{3}{2}, -1, -\tfrac{1}{2}, 0)$$

$$f^I(Z(\sigma))Y_I(\sigma) \qquad\qquad g_I(Z(\sigma))\partial Z^I(\sigma)$$

Wave fcts.:

$$f^{\dot\alpha}(\lambda,\mu,\psi) = \frac{\widetilde{\pi}^{\dot\alpha}}{\xi^2}\,\delta^2(\pi^\alpha - \xi\lambda^\alpha(\sigma))\,e^{i\xi[\mu(\sigma),\widetilde{\pi}]}u(\xi\psi(\sigma))$$

$$g_\alpha(\lambda,\mu,\psi) = \xi\lambda_\alpha(\sigma)\,\delta^2(\pi^\alpha - \xi\lambda^\alpha(\sigma))\,e^{i\xi[\mu(\sigma),\widetilde{\pi}]}u(\xi\psi(\sigma))$$

Zero-th order question: Assuming that CSG can be decoupled for some amplitude, do they remain decoupled under transformations mapping YM amplitudes into themselves? Are the CSG amplitudes recognizable after being transformed? If not, clearly they cannot be decoupled.

$$A_{n_f,m_g} = i^{|\mathcal{F}|}\int\frac{\{d\beta da_k^\alpha d\sigma d\xi\}}{Vol(GL(2))}\prod_{i=1}^{n+m}\delta^2(\pi_i^\alpha - \xi_i\lambda^\alpha(\sigma_i))\prod_{k=0}^{d}\delta^2(\sum_{i=1}^{m+n}\xi_i\sigma_i^k\widetilde{\pi}_i^{\dot\alpha})$$

CSG, connected prescription

$$\prod_{l\in\mathcal{G}}\pi_{i\alpha}\partial\lambda^\alpha(\sigma_l)\prod_{k\in\mathcal{F}}\sum_{\substack{i=1\\ i\neq k}}^{n+m}\frac{[k,i]}{\sigma_k-\sigma_i}\frac{\xi_i}{\xi_k^2}\prod_{i=1}^{n+m}u(\xi_i\psi(\sigma_i))$$

\# V_g vertices = 1+degree of curve; recover MHV, etc

Parity conjugation: $V_f \longleftrightarrow V_g$

$$A_{n_f,m_g} = i^{|\mathcal{G}|}\int \frac{\{d\beta da_k^{\dot\alpha} d\sigma d\tilde\xi\}}{Vol(GL(2))} \prod_{i=1}^{n+m} \delta^2(\tilde\pi_i^{\dot\alpha} - \tilde\xi_i \mu^{\dot\alpha}(\sigma_i)) \prod_{k=0}^{n-d-2} \delta^2(\sum_{i=1}^{m+n} \tilde\xi_i \sigma_i^k \pi_i^{\alpha})$$
$$\prod_{l\in\mathcal{F}} \tilde\pi_{i\dot\alpha}\partial\mu^{\dot\alpha}(\sigma_l) \prod_{k\in\mathcal{G}} \sum_{\substack{i=1\\ i\neq k}}^{n+m} \frac{\langle k,i\rangle}{\sigma_k-\sigma_i}\frac{\tilde\xi_i}{\tilde\xi_k^2} \prod_{i=1}^{n+m} u(\tilde\xi_i\psi(\sigma_i))$$

The conclusion: The coordinate transformation

$$\tilde\sigma_i = \sigma_i \qquad\qquad \tilde\xi_i = \frac{1}{\xi_i \prod_{j\neq i}(\sigma_i-\sigma_j)}$$

maps an amplitude into its party conjugate

Same pattern as for gluon amplitudes: the transformation of the constraiont equations (the analog of the Jacobian) compensates for the flip of the helicities of the external particles.

- Zero-th order condition checks out

Back to the explicit calculation of amplitudes

- Technically less challenging: use disconnected prescription
  - unclear what is the right $i\epsilon$ prescription
- Gauge theory input: try to glue MHV vertices into loops
  - using the same rules as for tree level
  - reconstruct string theory interpretation, $i\epsilon$ prescription and info about divergences by transforming to (λ, μ) space

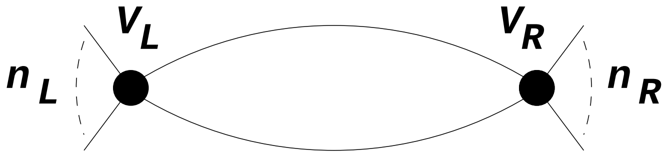
Gauge theory calculation is successful

- 1-loop MHV work out properly — Brandhuber, Spence, Travaglini
- Later work suggests that this works for all 1-loop amplitudes

- Twistor space interpretation

Bena, Bern, Kosower, RR

Step 1: Identify divergent diagrams



divergent if $n_L = 2$ or $n_R = 2$

divergences: only in 2-particle invariants, at || loop and external momenta

$n_L = 2$ momentum conservation $\Rightarrow$ 2 overlapping || divergences: not integrable

Step 2: Do the Fourier transform

- Convergent diagrams: expected structure
  - $i\epsilon$ prescription $\leftrightarrow$ specific integration over moduli of curves

$$
\begin{aligned}
\mathcal{A}_1 \quad = \quad & \int d^4 x_A^{a\dot{a}} d^4 x_B^{b\dot{b}} \int \frac{dl_1'}{2\pi}\frac{dl_2'}{2\pi} \int d^2 m'^{\dot{a}}_{A1} d^2 m'^{\dot{a}}_{A2} \int d^2 m'^{\dot{b}}_{B1} d^2 m'^{\dot{b}}_{B2} \\
& \times V_L(l_1', \lambda_A, l_2') V_R(l_2', \lambda_B, l_1') I_\eta(x_A - x_B) G_1(m'_{A1,B1}) G_2(m'_{A2,B2}) \\
& \times \prod_{i\in A} \delta^2(\mu^{\dot{a}}_{Ai} - x_A^{a\dot{a}}\lambda_{Ai\,a}) \delta^2(m'^{\dot{a}}_{A1} - x_A^{a\dot{a}} l'_{1\,a}) \delta^2(m'^{\dot{b}}_{B1} - x_B^{b\dot{b}} l'_{1\,b}) \\
& \times \prod_{i\in B} \delta^2(\mu^{\dot{b}}_{Bi} - x_B^{b\dot{b}}\lambda_{Bi\,b}) \delta^2(m'^{\dot{a}}_{A2} - x_A^{a\dot{a}} l'_{2\,a}) \delta^2(m'^{\dot{b}}_{B2} - x_B^{b\dot{b}} l'_{2\,b}).
\end{aligned}
$$

  - I_η related to $i\epsilon$ prescription and the off-shell continuation
  - G_1 and G_2 are propagator-like objects

- Divergent diagrams: need regularization: several ways

1) Relax momentum conservation

$$\delta^4(l_1 + l_2 + P_L) \to \Delta^4_\xi(l_1 + l_2 + P_L) \text{ with } \Delta_\xi(x) = \xi/(\xi^2 + x^2)$$

2) Invent a regulator reproducing the results of dim.reg.

- twistor space: x integral enforces momentum conservation

$\longrightarrow$ can regulate the x integral or the l integrals ▪ choose the latter

- general result:

$$\text{div} \sim \frac{1}{\epsilon^2}(-s_{12})^{-\epsilon} f(s_{12}, \cdots ; \epsilon),$$

- Regulator: $R_\epsilon(l, \lambda) = \left(\frac{\langle l_1 l_2 \rangle}{\langle l_1 \lambda_1 \rangle \langle l_2 \lambda_2 \rangle} \right)^\epsilon \frac{1}{[\tilde{\lambda}_1, \tilde{\lambda}_2]^\epsilon}$

$$\delta^2(m'^{\dot\alpha}_{A1} - x^{a\dot\alpha}_A l'_a)\delta^2(m'^{\dot\alpha}_{A2} - x^{a\dot\alpha}_A l'_a) \longrightarrow R_\epsilon(l, \lambda, m) \propto \frac{\epsilon}{(2\pi)^3} [m'_{A1} - x^a_A l'_{1\,a}, m'_{A2} - x^a_A l'_{2\,a}]^{\epsilon - 2}$$

$$\mathcal{A}_1 = \frac{\epsilon\Gamma(2-\epsilon)}{(2\pi)^3\Gamma(1+\epsilon)} \int d^4x_A^{a\dot{a}} d^4x_B^{b\dot{b}} \int \frac{dl'_1}{2\pi}\frac{dl'_2}{2\pi} \int d^2m'^{\dot{a}}_{A1} d^2m'^{\dot{a}}_{A2} \int d^2m'^{\dot{b}}_{B1} d^2m'^{\dot{b}}_{B2}$$

$$\times R_\epsilon(l_1, l_2, \lambda_1, \lambda_2, m'_{A1}, m'_{A2}) I_\eta(x_A - x_B)$$

$$\times V_L(l'_1, \lambda_A, l'_2) V_R(l'_2, \lambda_B, l'_1) G_1(m'_{A1,B1}) G_2(m'_{A2,B2})$$

$$\times \prod_{i\in A} \delta^2(\mu^{\dot{a}}_{Ai} - x_A^{a\dot{a}}\lambda_{Ai\,a})\delta^2(m'^{\dot{b}}_{B1} - x_B^{b\dot{b}} l'_{1\,b})$$

$$\times \prod_{i\in B} \delta^2(\mu^{\dot{b}}_{Bi} - x_B^{b\dot{b}}\lambda_{Bi\,b})\delta^2(m'^{\dot{b}}_{B2} - x_B^{b\dot{b}} l'_{2\,b})$$

- In line with the original intuition
- CSG modes are absent
- Projected out by propagators between degree 1 instantons

◇ Single-trace structure of the amplitude is crucial

◇ Suggests that, while the the twistor string contains CSG states, on an amplitude-by-amplitude basis they can be eliminated

- What about the connected prescription?
  - Not clear in general; successful for partly connected
  - focus on divergences and regularization
- MHV: $d = 2$ and $g = 1 - \not\exists\, \mathbb{P}^3$; but $d = 2$ and $g = 0$ w/ propagator

Idea: Use the pole at infinity to cast the amplitude in a form comparable to the disconnected prescription (similar to Gukov, Motl, Nietzke)

- start with $(h_0, -_1, -_2, ++ \cdots + (-h)_{n+1})$

$$\mathcal{A}_{tree} = \int \int [d\mu_{curves\ \ d=2}] \prod_{i=0}^{n+1} \delta^7 \left(\frac{Z_i^A}{Z_i^1} - \frac{Z^A(\sigma_i)}{Z^1(\sigma_i)} \right) \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$

$$\mathcal{A}_{1-loop} = \int dZ_0^A dZ_{n+1}^A G_1(Z_0^A, Z_{n+1}^B) \times$$
$$\times \int \int [d\mu_{curves\ \ d=2}] \prod_{i=0}^{n+1} \delta^7 \left(\frac{Z_i^A}{Z_i^1} - \frac{Z^A(\sigma_i)}{Z^1(\sigma_i)} \right) \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$

- Gauge choices and change of variables

$$Z^1(\sigma_i) = \sigma_i \qquad \frac{\beta_0^2}{\beta_2^2} = 1 \qquad C = \sqrt{\beta_0^2 \beta_2^2}$$

$$\mathcal{A}_{1-loop} \propto \int dZ_0^A dZ_{n+1}^A G_1(Z_0^A, Z_{n+1}^B) \times$$
$$\times \int \frac{dC}{C^3} \prod_{A \neq 1} d\beta_1^A \prod_{A \neq 1,2} d\left(\frac{\beta_0^A}{\beta_0^2}\right) d\left(\frac{\beta_2^A}{\beta_2^2}\right) \prod_{A,i} \delta\left(\frac{Z_i^A}{Z_i^1} - \frac{Z^A(\sigma_i)}{Z^1(\sigma_i)}\right) \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$

- The embedding curve in these variables:

$$\beta_0^2 \frac{\beta_0^A}{\beta_0^2} \sigma^{-1} + \beta_1^A + \beta_2^2 \frac{\beta_2^A}{\beta_2^2} \sigma \to \beta_1^A + C\left(\frac{\beta_0^A}{\beta_0^2} \sigma^{-1} + \frac{\beta_2^A}{\beta_2^2} \sigma\right)$$

$\Longrightarrow C \to 0$ is the infinity in the moduli space of curves

- Many infinities depending on the scaling of σ_i

$$\sigma_i \to \begin{cases} \frac{\sigma_i}{\beta_2^2} & i \in (p \ldots 1) \qquad \beta_1^A + \frac{\beta_2^A}{\beta_2^2}\sigma + C\ \frac{\beta_0^A}{\beta_0^2}\sigma^{-1} \\ \frac{\beta_0^2}{\sigma_i} & i \in (2 \ldots p-1) \qquad \beta_1^A + \frac{\beta_0^A}{\beta_0^2}\sigma + C\ \frac{\beta_2^A}{\beta_2^2}\sigma^{-1} \end{cases} .$$

$$\prod_i \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}$$
$$\to C^2 \frac{d\sigma_p}{\sigma_p - \sigma_{p+1}} \cdots \frac{d\sigma_n}{\sigma_n - \sigma_1} \frac{d\sigma_1}{\sigma_1 \sigma_2 - C} \frac{d\sigma_2}{\sigma_2 - \sigma_3} \cdots \frac{d\sigma_{p-2}}{\sigma_{p-2} - \sigma_{p-1}} \frac{d\sigma_{p-1}}{\sigma_{p-1}\sigma_p - C}$$

The comparison:

– Integrate C $\to$ set $C = 0$ $\to$ same as in the disconnected prescription in which one propagator is replaced with the residue of the pole at infinity of the moduli space of disconnected curves

1) Fourier-transform: moduli integrals $\to$ linewise mom.conservation

$\Rightarrow$ IR divergence if two legs sit on one line; need regularization

2) C integral last/shift pole $\leftrightarrow$ delta-converging series

3) Dimensional regularization: aim at the same regulator as in d.p.

- Integrate C and replace 2 delta functions
- Replace localization on 4-vertex curve with regulator

The message: In the (partly) connected prescription and as far as IR divergences are concerned it is possible to isolate the gauge theory amplitude from the contributions of CSG multiplet.

How did it happen?

- In the disconnected picture this happened because the twistor space propagators propagate only YM states.

- In the (partly) connected this does happens for the same reason as at tree level: the correlation function of vertex operators yields a single-trace structure.

• field theory standpoint: (C)SG contributes subleadingly in $1/N$. Even though N is fixed, it is possible to isolate planar amplitudes on a case by case basis

Summary

The twistor string led to remarkable insight into the structure of scattering amplitudes in gauge theories. I have reviewed some results obtained at tree level in the connected prescription and briefly touched on some of the featues of the twistor string at loop level

- the connected prescription – a collection of algebraic operations. Starting from the expression of a general amplitude I have described some of its properties which make us confident the result is correct.

- I have described the twistor space image of loop calculations in the disconnected prescription (BST), emphasizing that it suggests that CSG modes can be eliminated on an amplitude-by-amplitude basis, pointed out regulators, and outlined a calculation in the connected prescription with essentially the same result

The twistor string may still have lessons to teach us!