Ricci-flat supertwistor spaces

U.L., M.Rocek and R.v.Unge

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Plan of the talk

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Background

M.Rocek, Talk at Simons workshop 2004

N=2 supersymmetry is closely related to complex geometry:

The target space ***T*** of N=2 supersymmetric nonlinear sigma models is hyperkähler and thus associated with an $S^2 \sim P^1$ of complex structures. The twistor space ***Z*** of ***T*** is ***T***xP^1.

The graded abelian subspaces that define Projective Superspace are associated with extra coordinates that live on a P^1, the same P^1, and hypermultiplet actions are naturally defined on ***Z*** .

When the N=2 sigma models are coupled to supergravity, the target space geometry is different, it is Quarternion Kähler (QK).

N=2 supergravity in the N=2 conformal formalism is even more restrictive: Hypermultiplets lie on a Hyperkähler Cone (HKC) over a Quarternion Kähler space

Symmetries of the HKC are SU(2) and homothety.
They allow various quotients.

We shall eventually be interested in a supervarieties, but first look at the bosonic geometry.

HKC and quotients

B.deWit, M. Rocek and S. Vandoren hep-th/0101161

The homothetic conformal Killing vector

$$\nabla_A \chi^B = \delta_A^B \qquad \bar{X}^A \equiv \bar{J}_B^A \chi^B$$

$$\Rightarrow \nabla_{(A} \bar{X}_{B)} = \bar{0} \ .$$

gives rise to a hyperkähler potential

$$\chi \equiv \frac{1}{2} \chi^A g_{AB} \chi^B \quad \Rightarrow \quad g_{AB} = D_A \partial_B \chi$$

The Killing vectors $\bar{X}$ generate rotations of the complex structures, which are thus isometries. The homothety together with the Killing vector defined w.r.t. any of the complex structures define the complexified action of a *U(1)* subgroup of the *SU(2)* isometry group*.* This is the structure needed for a symplectic reduction. The quotient takes us to a Kähler manifold ***Z*** with one complex dimension less

$$X^A\partial_A = \partial_z + \bar{\partial}_{\bar{z}} \Leftarrow \ \chi = e^{(z+\bar{z}+K(u^i,\bar{u}^{\bar{\jmath}}))}$$

$$g_{a\bar{b}} = \partial_a\partial_{\bar{b}}\chi = \chi \begin{pmatrix} K_{i\bar{\jmath}} + K_i K_{\bar{\jmath}} & K_i \\ K_{\bar{\jmath}} & 1 \end{pmatrix}$$

The metric satisfies the Monge-Ampère equation

$$det(g_{a\bar{b}}) = 1$$

modulo Kähler gauge transformations. In our coordinates:

$$\det(g_{a\bar{b}}) = \chi^{2n} \det(K_{i\bar{\jmath}})$$

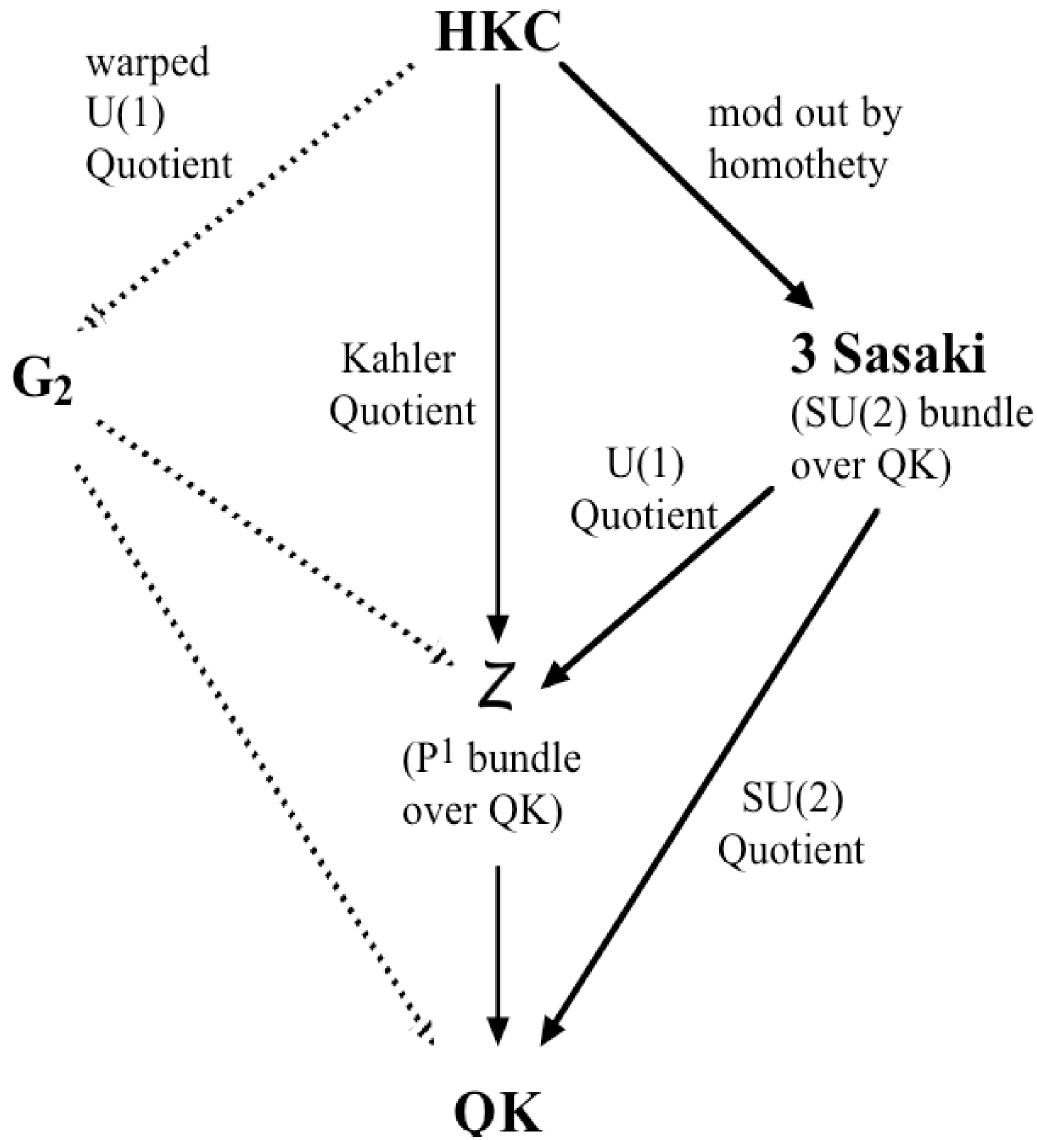
or

$$\det(K_{i\bar{\jmath}}) = e^{-2nK}$$

Since for a Kähler manifold,

$$R_{i\bar{j}} = [ln\ det(K_{m\bar{n}})]_{,i\bar{j}}$$

this means that the twistor space ***Z*** is Einsteinian with cosmological constant ***2n***. It is a P_1-bundle over a QK manifold which may be found by a further projection.



(Lhs: eight real dimensional HKC only)

Super-variety

We repeat the previous discussion for a HKC of Complex dimension **(2n|2m).** The super Monge-Ampère equation implies that the HKC is super Ricci-flat. For ***Z*** we now find:

$$\mathrm{sdet}(g_{a\bar{b}}) = \chi^{2(n-m)}\,\mathrm{sdet}(K_{i\bar{\jmath}})$$

$$\Rightarrow$$

$$\mathrm{sdet}(K_{i\bar{\jmath}}) = e^{-2(n-m)K}$$

For n=m the supertwistor space ***Z*** is super Ricci-flat and thus super Calabi Yau. ***Dim Z =(2n-1,2n).***

Examples

Ex1. The supertwistor space $\mathbb{CP}^{3|4}$

$$HKC : \mathbb{C}^{4|4} \longrightarrow Z : \mathbb{CP}^{3|4} \longrightarrow QK : S^{4|4}$$

Ex 2. The HKC $\mathbb{C}^{8|6}///U(1)$

We coordinatize $\mathbb{C}^{8|6} \equiv \mathbb{C}_{+}^{4|3} \times \mathbb{C}_{-}^{4|3}$ using $z_{\pm}, \psi_{\pm}$

The hyperkähler quotient is then obtained by the restrictions

$$\frac{\partial}{\partial V}\hat{\chi} \;=\; 0\,, \qquad z_{+}z_{-} + \psi_{+}\psi_{-} \;=\; 0$$

where

$$\hat{\chi} \;=\; e^{V}(z_+\bar{z}_+ + \psi_+\bar{\psi}_+) + e^{-V}(\bar{z}_-z_- + \bar{\psi}_-\psi_-)$$

We find the twistor space ***Z*** as a U(1) Kähler quotient of the resulting HKC. To see its structure, start from the gauged action

$$\widehat{\tilde{\chi}} = e^{\tilde{V}}\left[e^{V}(z_+\bar{z}_+ + \psi_+\bar{\psi}_+) + e^{-V}(\bar{z}_-z_- + \bar{\psi}_-\psi_-)\right] - \tilde{V}$$

and change variables to $V_\pm = \tilde{V} \pm V$

$$\widehat{\tilde{\chi}} = e^{V_+}(z_+\bar{z}_+ + \psi_+\bar{\psi}_+) + e^{V_-}(\bar{z}_-z_- + \bar{\psi}_-\psi_-) - \frac{(V_+ + V_-)}{2}$$

This gives the Kähler quotient $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$

Hence the twistor space *Z* is given by the quadric

$$z_+ z_- + \psi_+ \psi_- = 0 \qquad \text{in} \qquad \mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$$

This *ambitwistor space* has SU(4|3) symmetry and has been proposed as an alternate twistor space relevant to N=4 SYM.

Deformations

We would like to find new examples of Ricci-flat super-manifolds. (Note that existence of a globally defined top-form does not imply Ricci-flatness for supermanifolds, as shown in M. Rocek and N. Wadhwa, hep-th/0408188 and hep-th/0410081). We deform the existing models while preserving Ricci-flatness.

Ex. 1 Infinitesimal deformations of $\mathbb{CP}^{n-1|n}$

$$K = \ln(1 + z\bar{z} + \psi\bar{\psi}) + \Delta = K_0 + \Delta$$

The blocks of the metric are

$$\begin{aligned} A &\equiv K_{z\bar{z}} \\ B &\equiv K_{z\bar{\psi}} \\ C &\equiv K_{\psi\bar{z}} \\ D &\equiv K_{\psi\bar{\psi}} \end{aligned}$$

and we need to calculate the superdeterminant

$$\mathrm{sdet}(\partial\bar{\partial}K) = \frac{\det(A - BD^{-1}C)}{\det D}$$

When $\Delta = 0$ we verify that this expression equals 1.
For nonzero Δ we expand

$$A = A_0 + A_1, B = B_0 + B_1, \ldots$$

The condition that the superdeterminant is one reads

$$\det D = \det\left(A - BD^{-1}C\right)$$

which may be solved to lowest order to give

$$\mathrm{Tr} D_1 = \mathrm{Tr} A_1 + zA_1\bar{z} + zB_1\bar{\psi} + \psi C_1\bar{z} + \psi D_1\bar{\psi}$$

This is thus the condition for a linearized deformation to preserve Ricci-flatness.

An example of a solution to this equation is given by U(1) deformations of $\mathbb{CP}^{3|4}$.

A general problem with these deformations is that they seem to break the superconformal (SU(4)) symmetry of the models.

Examples of deformations

Deformations of the chiral twistor space $\mathbb{CP}^{3|4}$

It is natural to think of the HKC over ***Z***= $\mathbb{CP}^{3|4}$
as a hyperkähler quotient w.r.t. SU(2) and U(1)'s
(see L. Anguelova, M. Rocek and S. Vandoren hep-th/0202149).
This suggests deformations via couplings to the fermions.

$$\int d^4\theta\, \psi e^{\sigma V}\bar{\psi} e^{q\sum_i V_i} + \bar{\tilde{\psi}} e^{-\sigma V}\tilde{\psi} e^{-q\sum_i V_i} + \left[\int d^2\theta\, \sigma\psi\phi\tilde{\psi} + q\psi\tilde{\psi}\sum_i \phi_i + c.c.\right]$$

Putting $\sigma = 0$ we find a U(1) deformation and verify that the deformed space is a HKC, writing the potential as

$$e^{u+\bar{u}+K(z,\bar{z},\psi,\bar{\psi})}$$

Finally, we find ***K*** to be deformed by a Δ that satisfies the correct equation for a linear deformation.

Putting $q = 0$ and $\sigma = 1$ we find the SU(2) deformation. It is not “small”. We again verify that the deformed space is a HKC and find the ***K*** of the corresponding ***Z***. In the absence of fermions, the manifest O(5) is enhanced to an SU(4) symmetry. We were hoping to see the manifest Osp(5|2) enhanced to SU(4|2) in this case, but found no evidence neither for nor against.

Deformations of ambitwistor space

$$z_+ z_- + \psi_+ \psi_- = 0 \quad \text{in} \quad \mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$$

Ex.
Modify the U(1) charges in the quotient construction.
We choose

$$\frac{\partial}{\partial V}\hat{\chi} \;=\; 0\,, \qquad z_+ z_- + \psi_+ \psi_- \;=\; 0$$

with

$$\hat{\chi} \;=\; e^V(z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) + e^{-V}(\bar{z}_- z_- + \bar{\psi}_- \psi_-) + \psi_0 \bar{\psi}_0$$

This gives a HKC with potential

$$\chi = 2\sqrt{(z_+\bar{z}_+ + \psi_+\bar{\psi}_+)(\bar{z}_- z_- + \bar{\psi}_-\psi_-)} + \psi_0\bar{\psi}_0$$

The twistor space ***Z*** is found by taking a U(1) quotient. Its Kähler potential is found to be

$$K = \frac{1}{2}\ln(M_+) + \frac{1}{2}\ln(M_-) + \ln\left(1 + \frac{\psi_0\bar{\psi}_0}{2\sqrt{M_+M_-}}\right)$$

where

$$M_+ = (z_+\bar{z}_+ + \psi_+\bar{\psi}_+) \quad , \quad M_- = (\bar{z}_- z_- + \bar{\psi}_-\psi_-) \ .$$

The deformed space has symmetry SU(4|n)xSU(6-2n),
Where n is the number of charged doublets $\psi_\pm$

Other choices of the U(1) action on the fermions give other deformations with different residual symmetries.

There are many other deformations available. They typically preserve superconformal symmetries, but also introduce additional symmetries.

Some of these deformations are based on the equivalent descriptions of the underlying bosonic quaternion Kähler manifold, which implies equivalence of the corresponding HKC's. Other are achieved by curving the initial flat space whose U(1) hyperkähler quotient gives the HKC, in analogy to the cases discussed in L. Anguelova, M. Rocek and S. Vandoren, hep-th/0402132.

Conclusions

- Supertwistor spaces constructed from hyperkähler cones with equal number of bosonic and fermionic coordinates are super-Ricci-flat.
- This may be used to discuss deformations of supertwistor spaces.
- Deformations of chiral supertwistor spaces seem to typically break superconformal invariance.
- It is simple to deform ambitwistor space in ways that preserve a variety of superconformal symmetries.

Outlook

- Do any of the deformations of ambitwistor space arise in superconformal Yang-Mills theories?
- Could some of the deformations of chiral twistor space describe a non-conformal phase of N=4 super Yang-Mills theory?