On N=4 Superconformal Amplitudes

– towards exact results for on-shell amplitudes in N=4 superconformal YM

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On-shell Methods for Scattering Amplitudes

- Off-shell quantities contain more information than on-shell objects
  Scattering amplitudes A_n usualy obtained from off-shell Green's functions G_n

$$(\mathrm{LSZ}) \cdot G_n \,|_{\mathrm{on-shell}} \longrightarrow A_n$$

- Off-shell methods:
  Feynman diagrams are designed to calculate off-shell Green's functions (perturbatively)

- On-shell methods: Calculate on-shell scattering amplitudes on-shell.
  Use on-shell building blocks in new types of diagrammatic techniques

  MHV-rules

  On-shell recursion relations

  Generalised unitarity

- Use standard techniques to streamline the calculation

  Colour-ordered amplitudes: Amplitude $= \sum$ Group-factor $\times$ Colour-ordered-Amplitude

  Spinor helicity approach

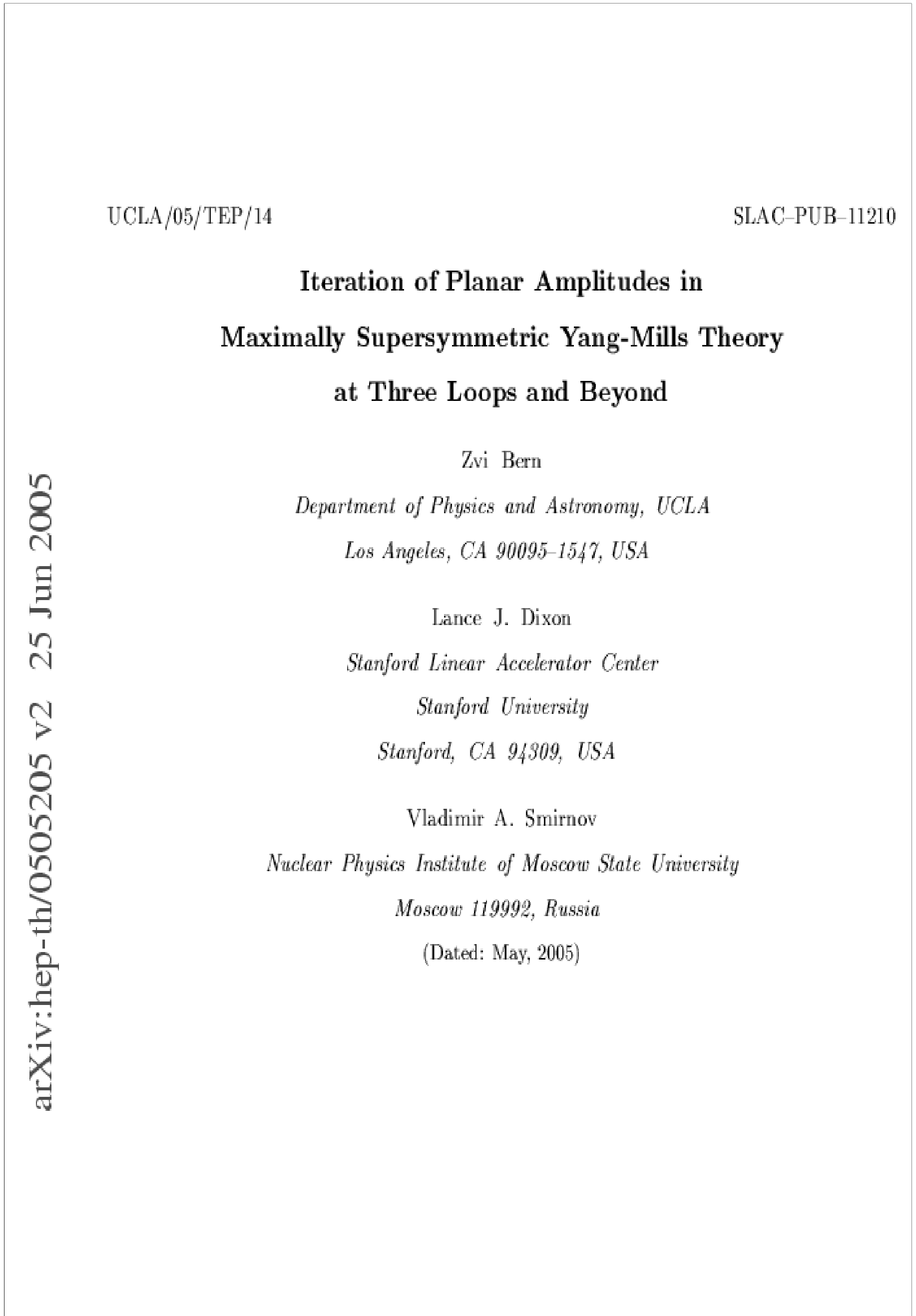
On-shell methods for on-shell amplitudes

There has been a remarkable progress in calculations of scattering amplitudes in gauge theory

- at tree-level and
- at 1-loop level
- ranging from N=4 SYM to N=0 gauge theory – QCD

reviewed in talks of Dixon, Bjerrum-Bohr, Dunbar, Glover, Roiban, Britto, Risager, Kosower, Brandhuber

- One of these exciting new developments is the remarkable proposal of Bern, Dixon and Smirnov for MHV amplitudes in N=4 SYM to all orders in perturbation theory



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Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond

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Bern, Dixon and Smirnov in hep th/0505205 wrote down a formula for all-orders perturbative planar MHV amplitude M_n in N=4 SYM

This groundbreaking result is a conjecture based on their computation of 4-point amplitudes at 3-loop level

Also based on earlier work:
Anastasiou, Bern, Dixon, Kosower;
Bern, Rosowsky, Yan ‘97; …
Smirnov ’99 and ‘2003, …

and consistent with IR-resummation/ factorisation of amplitudes:
Sterman, Tejeda-Yeomans ‘2003
Catani ’98; Gile, Glover ’92 …

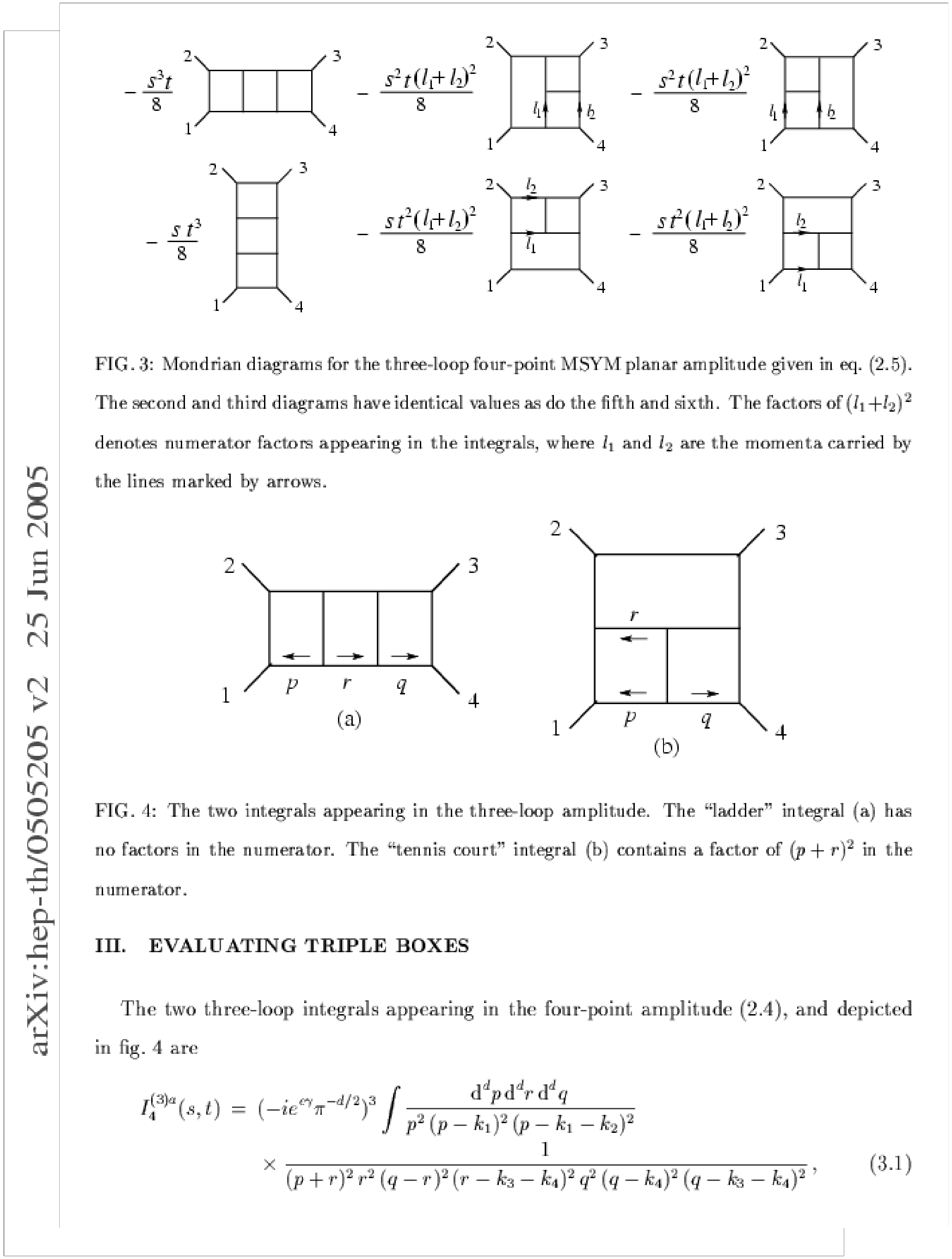


FIG. 3: Mondrian diagrams for the three-loop four-point MSYM planar amplitude given in eq. (2.5). The second and third diagrams have identical values as do the fifth and sixth. The factors of $(l_1+l_2)^2$ denotes numerator factors appearing in the integrals, where l_1 and l_2 are the momenta carried by the lines marked by arrows.

FIG. 4: The two integrals appearing in the three-loop amplitude. The "ladder" integral (a) has no factors in the numerator. The "tennis court" integral (b) contains a factor of $(p+r)^2$ in the numerator.

III. EVALUATING TRIPLE BOXES

The two three-loop integrals appearing in the four-point amplitude (2.4), and depicted in fig. 4 are

$$I_4^{(3)a}(s,t) = (-ie^{\epsilon\gamma}\pi^{-d/2})^3 \int \frac{d^dp\, d^dr\, d^dq}{p^2\,(p-k_1)^2\,(p-k_1-k_2)^2} \times \frac{1}{(p+r)^2\, r^2\, (q-r)^2\, (r-k_3-k_4)^2\, q^2\, (q-k_4)^2\, (q-k_3-k_4)^2}, \qquad (3.1)$$

Bern, Dixon and Smirnov in hep th/0505205 wrote down a formula for all-orders perturbative planar MHV amplitude M_n in N=4 SYM

This groundbreaking result is a conjecture based on their computation of 4-point amplitudes at 3-loop level

So far it was (is) impossible to independently confirm or falsify the BDS conjecture for M^{MHV}_n nor to use it to derive methods for calculating non-MHV amplitudes to all-orders (or at least beyond 1-loop).

I. Perturbative Amplitudes in N=4 SYM

- N=4 SYM in the conformal case (no vevs, no masses) and in the planar limit (large N_c)
- All perturbative amplitudes are UV-finite
- All loop amplitudes are IR-divergent:

  internal lines can be soft, or collinear to external lines (internal loop momenta are integrated over)

- Work in $D=4-2\varepsilon$ dimensions and use susy preserving DimRed scheme to regulate IR-div's
- In $D=4-2\varepsilon$ the theory is not conformal, amplitudes M_n depend on ε and on the mass scale μ

$$M_n^{\text{loops}}\,(\text{kinematics}, \alpha\,;\,\varepsilon\,,\,\mu)$$

Want to use M_n to extract a `physical quantity' A_n such that:

- A_n is IR-finite -no poles in ε (also UV-finite)

  in $M_n^{1\text{-loop}}$: soft $1/\varepsilon^2$ and $1/\varepsilon$ poles and collinear $1/\varepsilon$ poles

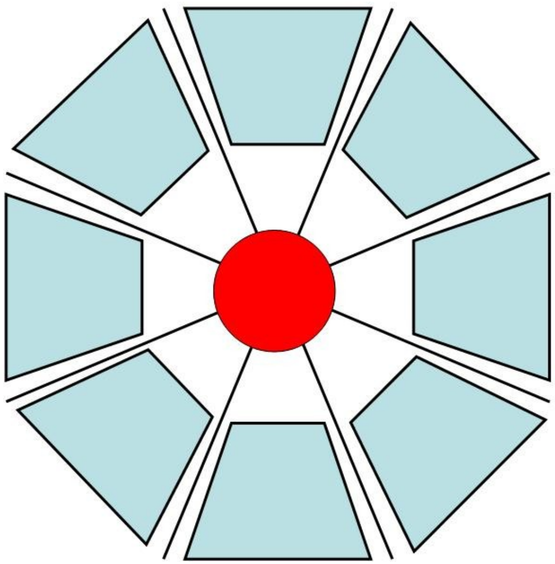
  in $M_n^{L\text{-loops}}$: soft and collinear IR-poles: $1/\varepsilon^{2L} + \ldots + 1/\varepsilon$
- A_n is independent of the μ – scale
- A_n can be attributed to the 4-dimensional conformal SYM

High degree of conformal/super-symmetry possibly can lead to all-orders compact answers for $A_n^{\text{All-loops}}$ in N=4

It likely is the appropriate quantity to address in dual theory

IR-resummation and amplitude factorization

$$\mathcal{M}_n = J\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) \times S\left(k_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) \times H_n\left(k_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) = W\, H_n$$



Consider a planar perturbative amplitude in generic YangMills

it contains the hard amplitude H_n which is IR-finite

and the IR-divergent long -range effects: soft and collinear radiation: S and J.
In planar limit J S = W where
W is the product of all wedges

Sterman, Tejeda-Yeomans '03;
Giele, Glover '80; Kunszt, Signer, Trocsanyi '94; Catani '98 …

$$M_n = \prod_{i=1}^{n} \left[\text{Sudakov} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times H_n \left(k_i, \mu, \alpha_s, \epsilon \right)$$

Wedge factor is half of the Sudakov formfactor for each 2 adjacent partons.

Wedge is universal (process-independent) -does not depend on helicities or types of external particles.

It is μ-dependent and IR-singular (ε-poles)

W-factor describes an embedding of hard partons into degenerate asymptotic states.

H_n is the scattering amplitude of hard partons.

It depends on helicities and types of particles: process—dependent.

It is IR-safe (no ε-poles)
In a generic theory it depends on the subtraction scale μ

H describes scattering at short distances – a `physical part' of the amplitude.

The IR-singular W-factor is simplified in conformal N=4 SYM : α_s = const, no running:

Bern-Dixon-Smirnov

$$\prod_{i=1}^{n}\left[\text{Sudakov}\left(\frac{s_{i,i+1}}{\mu^2},\alpha_s,\epsilon\right)\right]^{1/2} = \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty}a^l\left(\hat{\gamma}_K^{(l)}+2l\hat{\mathcal{G}}_0^{(l)}\epsilon\right)\frac{1}{(l\epsilon)^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon}\right]$$

Sum over loops

coupling const $N_c\alpha_s/2\pi$

Sum over wedges

Soft (cusp) anomalous dimension

$$\hat{\gamma}_K^{(1)} = 4\,, \qquad \hat{\mathcal{G}}_0^{(1)} = 0\,,$$

$$\hat{\gamma}_K^{(2)} = -4\zeta_2 \qquad \hat{\mathcal{G}}_0^{(2)} = -\zeta_3\,,$$

$$\hat{\gamma}_K^{(3)} = 22\,\zeta_4 \qquad \hat{\mathcal{G}}_0^{(3)} = 4\zeta_5+\frac{10}{3}\zeta_2\zeta_3$$

The IR-singular W-factor is simplified in conformal N=4 SYM : α_s = const, no running:

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$$= \exp\left[\sum_{l=1}^{\infty} a^l\, f^{(l)}(\epsilon)\hat{I}_n^{(1)}(l\epsilon)\right]$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

$$\hat{I}_n^{(1)}(\epsilon) = -\frac{1}{2}\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon}$$

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)}$$

$$f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)}$$

$$\hat{\gamma}_K^{(1)} = 4\,,\qquad \hat{\gamma}_K^{(2)} = -4\zeta_2\qquad \hat{\gamma}_K^{(3)} = 22\,\zeta_4$$

$$\hat{\mathcal{G}}_0^{(1)} = 0\,,\qquad \hat{\mathcal{G}}_0^{(2)} = -\zeta_3\,,\qquad \hat{\mathcal{G}}_0^{(3)} = 4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3$$

- BDS conjectured an expression for the all-orders MHV amplitude in planar N=4 superconformal YM
- Factor out from this expression the known IR-singular Wedge factor
- Left with the Hard amplitude H_n^{MHV} evaluated in $D=4-2\varepsilon$ dimensions

$$M_n = \underbrace{\prod_{i=1}^{n}\left[\text{Sudakov}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon\right)\right]^{1/2}}_{\exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon) \hat{I}_n^{(1)}(l\epsilon)\right]} \times H_n\left(k_i, \mu, \alpha_s, \epsilon\right)$$

Furthermore, set $\varepsilon=0$, i.e. evaluate H_n^{MHV} directly in 4 dimensions

Resulting H_n^{MHV} at $\varepsilon=0$:

- is a finite amplitude of the 4-dimensional theory
- the μ-dependence disappears from H_n^{MHV} $(\varepsilon=0)$ - it is scale-invariant
- H_n^{MHV} $(\varepsilon=0)$ is an ideal candidate for the finite conformal amplitude A_n^{MHV}

  In fact, only for $\varepsilon=0$ the answer for H_n^{MHV} is fully under control [otherwise non-iterative unknown contr-s in BDS]
- H_n^{MHV} $(\varepsilon=0)$ is given by a remarkably simple expression valid to all orders in perturbation theory (see next slide)

  [This Hard Amplitude happens to agree with what BDS called Finite Reminders F]

$$A_n^{MHV} = H_n^{MHV}(\varepsilon=0) == A_n^{\text{tree MHV}} \exp\left[\frac{1}{4}\gamma_K(a) F_n^{(1)}(k_i) + C(a)\right]$$

Cusp anomalous dimension $\gamma_k = \sum_{l=1}^{\infty} \hat{\gamma}_K^{(l)} a^l = 4\,a - 4\zeta_2\,a^2 + 22\zeta_4\,a^3 + \cdots$

$$C = -\frac{1}{2}\zeta_2^2\,a^2 + \left[\left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\right] a^3 + \cdots$$

$F_n^{(1)}$ is the finite part of the 1-loop n-point MHV-amplitude.

Thus kinematic dependence in A_n^{MHV} at all-orders comes only through 1-loop and tree level

In the simplest case of a 4-point amplitude:

$$F_4^{(1)} = \frac{1}{2}\log^2\left(\frac{-t}{-s}\right) + \frac{2}{3}\pi^2 \quad = \frac{1}{2}\ln^2\left(\frac{-t}{-s}\right) + 4\zeta_2$$

$F_n^{(1)}$ is the finite part of the 1-loop n-point MHV-amplitude. For general case of n-external particles it is given by

Bern-Dixon-Dunbar-Kosower '94
Bern-Dixon-Smirnov '05

$$F_n^{(1)} = \frac{1}{2}\sum_{i=1}^{n} g_{n,i}$$

$$g_{n,i} = -\sum_{r=2}^{\lfloor n/2 \rfloor -1} \ln\left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}}\right)\ln\left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}}\right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2$$

$$D_{2m+1,i} = -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right),$$

$$L_{2m+1,i} = -\frac{1}{2}\ln\left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}}\right),$$

$$D_{2m,i} = -\sum_{r=2}^{m-2} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right) - \frac{1}{2}\mathrm{Li}_2\left(1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}}\right),$$

$$L_{2m,i} = -\frac{1}{4}\ln\left(\frac{-t_i^{[m]}}{-t_{i+m+1}^{[m]}}\right)\ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i+m}^{[m]}}\right).$$

- Non-MHV amplitudes are unknown beyond 1-loop
- However, the IR-singular factor is universal

Of course one can always move some finite terms from H to W. This reflects freedom in defining asymptotic states. We can stick to a particular choice of W and always use the same one.

$$M_n = \underbrace{\prod_{i=1}^{n} \left[\text{Sudakov}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2}}_{\exp\left[\sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon) \hat{I}_n^{(1)}(l\epsilon)\right]} \times H_n(k_i, \mu, \alpha_s, \epsilon)$$

set ε=0 to deduce

$A_n(k_i, \alpha)$

Finite, scale-invariant 4-dimensional amplitude for arbitrary external states and helicities

Formal definition of A_n →

Conjectures

- A_n is scale-invariant
- A_n can be attributed to the 4-dimensional conformal SYM
- A_n is the appropriate quantity to address in dual theories

Twistor space interpretation of all A_n

for MHV and non-MHV (expanded in powers of α)

$d = n_{(-)} + L - 1$ if twistor space is relevant

String-theory interpretation of all A_n

if there is a weakly-dual string

A_n should be related to integrability approach to N=4 SYM

Expansion in powers of α enters into A_n via cusp anomalous dimension γ_K

γ_K – appears naturally in the integrability of the Dilatation oper. approach

also in the AdS/CFT correspondence as cusps of classical strings

Appearance of γ_K

is a unique link between amplitudes and integrability + AdS/CFT

Conjectures

- Conformal symmetry (and N=4 supersymmetry) of A_n give reasons to expect that results may be compact to all orders in perturbation theory
- MHV-amplitudes A_n are extremely simple to all orders.

If they have twistor-space interpretation (as anticipated)

there are good reasons to expect new MHV-type-rules methods

to calculate non-MHV finite amplitudes from MHV ones at all orders

- A naïve cartoon of such new MHV-rules method would be a set of tree-level MHV-rules with vertices being $A_n^{\text{All-loops MHV}}$ connected by scalar propagators => $A_n^{\text{All-loops non-MHV}}$

Calculate IR-finite 4D amplitudes $A_n^{\text{All-loops non-MHV}}$ directly in terms of $A_n^{\text{All-loops MHV}}$ at tree-level. [Any loop-diagram involving A_n vertices will necessarily give IR-divergent terms, thus stick to tree-level in order to get finite $A_n^{\text{All-loops non-MHV}}$. The devil is as always in details …]

Does it make sense to discuss S-matrix in conformal SYM ?

- In conformal field theory there are problems with defining asymptotic states
- Fortunately, our decomposition of the amplitude attributes the details of the asymptotic state construction (as well as the questions of their existence) entirely to the IR-singular wedge factor. H_n and A_n are well-defined.

$$M_n = \prod_{i=1}^{n} \left[\text{Sudakov}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times H_n$$

$$\underbrace{\prod_{i=1}^{n} \left[\text{Sudakov}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2}} = \exp\left[\sum_{l=1}^{\infty} a^l \, f^{(l)}(\epsilon) \hat{I}_n^{(1)}(l\epsilon) \right]$$

- From the perspective of H_n and A_n conformal N=4 is not that different from QCD. In QCD quarks and gluons can not form true asymptotic states (which are hadrons) however this does not influence hard scattering amplitudes.

- Can make this analogy more precise by deforming N=4 to a confining theory by giving masses to some of the superpartners. At energies below this mass-scale, the theory is confining and there are asymptotic colourless states. There is an S-matrix. One can now study scattering at energies high compared to masses, the amplitudes will approach those in N=4 superconformal theory.

- H_n and A_n are well-defined. The important difference with QCD is that in 4 dimensions H_n will not depend on the subtraction scale μ

What can we do with finite MHV-amplitudes alone? E.g. all A_4 amplitudes are MHV…

- What we cannot do is to calculate total cross-section via

$$\sigma^{tot} = 1/s \, \mathrm{Im}\, A_4 (k_1,k_2, k_1,k_2)$$

where $A_4 (k_1,k_2, k_1,k_2)$ is our finite conformal amplitude

since $\mathrm{Im}\, A_4 (k_1,k_2, k_1,k_2) = 0$

One has to use instead the full amplitude, M_4

$$\sigma^{tot} = 1/s \, \mathrm{Im}\, M_4 (k_1,k_2, k_1,k_2)$$

And this total partonic cross-section is IR-divergent!

There is no factorization for total cross-sections without an induced hard scale Q^2 in one of the external states. Just like in QCD.

What can we do with finite MHV-amplitudes alone?

- We can calculate differential cross-sections in N=4 SYM for processes with observed jets. MHV-amplitudes are sufficient for processes with n=4 and n=5.
- For general n>4 one can try a simple approximation of Kunszt and Stirling ‘88

$$\sigma^{gg\text{->}g\ldots g}_n = \text{const}\ \sigma^{MHV}_n \qquad (\text{const} = 1 \text{ for } n = 4\ ,\ 5)$$

where const is the number of all allowed helicity configurations over the number of MHV and MHVbar helicity conf-s.

$$\text{const} = (2^n - 2(n\text{-}1))\ /\ (n\ (n\text{-}1))$$

Just as a rough estimate, but to all-orders in α

Expressions for conformal MHV-amplitudes were derived in perturbation theory, i.e. in the weak-coupling regime:

$$A_n^{\mathrm{MHV}} = A_n^{\mathrm{tree\,MHV}} \exp\left[\frac{1}{4}\gamma_K(a) F_n^{(1)}(k_i) + C(a)\right]$$

$$F_4^{(1)} = \frac{1}{2}\log^2\left(\frac{-t}{-s}\right) + \frac{2}{3}\pi^2$$

$$a = \frac{\alpha N_c}{2\pi} \ll 1 \ : \ \gamma_K(a) = 4a - \frac{2}{3}\pi^2 a^2 + \mathcal{O}(a^3)$$

$$C(a) = -\frac{1}{72}\pi^4 a^2 + \mathcal{O}(a^3)$$

From the AdS/CFT correspondence cusp anomalous dimensions are also known in the opposite strong-coupling regime: (Gubser-Klebanov-Polyakov)

$$a \gg 1 \; : \; \gamma_K(a) = 2\sqrt{2}\sqrt{a} - \frac{6\log 2}{8\pi} + \mathcal{O}(a^{-1/2})$$

Does the formula:

$$A_n^{\mathrm{MHV}} = A_n^{\mathrm{tree\,MHV}} \exp\left[\frac{1}{4}\gamma_K(a) F_n^{(1)}(k_i) + C(a)\right]$$

$$F_4^{(1)} = \frac{1}{2}\log^2\left(\frac{-t}{-s}\right) + \frac{2}{3}\pi^2$$

make sense at strong-coupling?!

Beyond perturbation theory…

- What is the meaning of the amplitude at strong coupling?

  Naïvely it is either zero or infinite depending on kinematics

- Is there a finite radius of convergence of perturbation theory for N=4 conformal amplitudes?

  There are no renormalons in N=4

- What are instanton contributions to N=4 amplitudes?

  Do they have a twistor-space interpretation and what is it

  \- work in progress