Queen Mary
University of London

MSci EXAMINATION

PHY-415 (MSci 4242) Relativistic Waves and Quantum Fields

Time Allowed: 2 hours 30 minutes

Date: 4^{th} May, 2011

Time: 14:30 - 17:00

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work comprises 10% of the final mark.**

Throughout the paper units are used such that $\hbar = c = 1$ (Natural Units). A FORMULA SHEET is provided at the end of the questions paper.

Numeric calculators are not permitted in this examination. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A. Brandhuber (MO), Dr. R. Russo (DMO)

QUESTION 1: The Klein-Gordon and Dirac equations

(a) Give an informal derivation of the Dirac equation and motivate the form of its ansatz. Derive the continuity equation of the Dirac equation and show that the probability density is given by $\rho = \Psi^\dagger\Psi$. What is the main difference between the probability densities of the Klein-Gordon equation and the Dirac equation? **[7]**

(b) Find all plane wave solutions of the Dirac equation for a particle at rest, i.e. $\vec{p} = 0$. (Hint use the explicit form of the Dirac matrices given in the formula sheet.) Give a physical interpretation of the solutions. State two alternative methods to generate solutions with arbitrary spatial momentum $\vec{p}$. **[4]**

(c) Consider the covariant form of the Dirac equation. Assume that Ψ transforms under a Lorentz transformation as $\Psi(x) \to \Psi'(x') = S(\Lambda)\Psi(x)$, with $x' = \Lambda x$ and $S(\Lambda)$ a four-by-four matrix. Show that the Dirac equation is form invariant (and hence covariant) if

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu \gamma^\mu \,.$$

[6]

(d) Using the result from Q1(c) and the identity $S^{-1} = \gamma^0 S^\dagger \gamma^0$ find the transformation under Lorentz transformation of $\overline{\Psi}\Psi$ and $\overline{\Psi}\gamma^\mu\Psi$. **[3]**

QUESTION 2: Solutions of the Dirac equation

In the following use the *chiral representation* of the Dirac matrices

$$\beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} , \ \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} , \ i = 1, 2, 3 ,$$

where the σ^i denote the Pauli matrices.

(a) Define the helicity of a particle. Show that the helicity operator for a Dirac particle is

$$h(\vec{p}) = \frac{\vec{p} \cdot \vec{\Sigma}}{2|\vec{p}|} \ \text{ with } \ \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} , \ i = 1, 2, 3$$

[2]

(b) Describe (in words) how we have to modify the solutions of the Dirac equation to be able to describe massless neutrinos and anti-neutrinos. **[3]**

(c) Consider the Dirac equation (**using the above Dirac matrices**)

$$\Psi = \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} ,$$

where ϕ and χ denote two component column spinors with spacetime dependence. Derive differential equations for ϕ and χ. **[3]**

Now consider the case $m = 0$ and find positive energy plane wave solutions for ϕ and χ separately, i.e. setting

$$\phi(x) = e^{-ip\cdot x}\phi_0 \ \text{ and } \ \chi(x) = e^{-ip\cdot x}\chi_0 \ ,$$

where ϕ_0 and χ_0 are constant two-component spinors. Find equations for ϕ_0 and χ_0 and, hence, find the helicities of these spinors. Hence explain why "mass couples particles of opposite helicities".

[6]

(d) Consider the Dirac Hamiltonian $\hat{H}$ (**using the above Dirac matrices**), assuming that $\hat{H}$ is acting on a plane wave solution so that the momentum operator can be replaced by its eigenvalues. Calculate the commutator of this matrix with the spin operator matrix $\frac{1}{2}\Sigma^i$. For this you may use $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ without proof. Hence find the unique (up to an overall numerical factor) linear combination of the spin operators which does commute with the Hamiltonian. **[6]**

QUESTION 3: Symmetries and gauge fields

(a) State what is meant by covariance (form invariance) of a relativistic wave equation under symmetry transformations. Symmetries of relativistic wave equations (and quantum field theories) can be continuous global, continuous local or discrete; give one example of each type. **[4]**

(b) Show that the electromagnetic field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under the gauge transformation $A^\mu \to A^\mu + \partial^\mu \chi$, with χ an arbitrary, real function of the space-time coordinates. How must the complex field Φ transform under a gauge transformation, in order that the combined transformation of A^μ and Φ preserves the scalar QED Lagrangian (proof required)

$$\mathcal{L}_{sQED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} (D_\mu \Phi)^\dagger D^\mu \Phi \ ,$$

with $D_\mu \Phi = \partial_\mu \Phi - iqA_\mu \Phi$. **[6]**

(c) Consider the Lagrangian density $\mathcal{L}_{sQED}$ from Q3(b) and find the Euler-Lagrange equations of motion for A_μ, Φ and $\Phi^\dagger$. For the purpose of deriving the equations of motion you may treat Φ and $\Phi^\dagger$ as independent quantities. Which gauge might you use for the 4-vector potential (gauge field) A_μ to simplify the equations of motion? The Lagrangian contains a term of the form $A_\mu B^\mu$ which is linear in the gauge field A_μ. What is the interpretation of B^μ in field theory? **[10]**

QUESTION 4: The neutral Klein-Gordon field

The quantised free, neutral Klein-Gordon field $\phi = \phi^\dagger$ field may be expanded in the form

$$\phi(x) = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x} \right] ,$$

with $E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}$ and $k \cdot x = E_{\vec{k}} t - \vec{k} \cdot \vec{x}$. Wherever needed you may use the commutation relations for the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ given on the FORMULA SHEET without proof.

(a) A one particle state is given by $|p\rangle = a^\dagger(\vec{p})|0\rangle$. Hence, calculate the expression $\langle 0|\phi(x)|p\rangle$. **[3]**

(b) Calculate the commutator of two field operators at *general* space-time points x and y

$$i\Delta(x-y) = [\phi(x), \phi(y)] \ .$$

Note that you do not have to perform the final three-momentum integral explicitly. Show that the result is Lorentz invariant and vanishes for space like separations $(x-y)^2 < 0$. Discuss the physical significance of the latter property of $[\phi(x), \phi(y)]$. **[6]**

(c) Write down the *equal time commutator* (ETC) of Φ and Π where $\Pi = \dot{\phi}$ denotes the momentum canonically conjugate to ϕ. Show that the ETC follows from result of Q4(b) (after taking a suitable time derivative and performing the final three-momentum integral). **[5]**

(d) The *normal ordered* expression for the 4-momentum operator is

$$P_\mu = \int \frac{d^3p}{2E_{\vec{p}}(2\pi)^3} p_\mu a^\dagger(\vec{p}) a(\vec{p}) \ .$$

Hence show that

$$[P_\mu, \phi(x)] = -i\partial_\mu \phi(x) \ .$$

Explain the term normal ordering and discuss in words why it is used. **[6]**

QUESTION 5: The S-matrix

(a) Assume that the Hamiltonian H of a quantum field is split up in a free and interacting part as $H = H_0 + H_{int}$. The interaction Hamiltonian in the interaction picture is given as

$$H_I \equiv e^{iH_0 t}(H_{int})_S e^{-iH_0 t} .$$

Show that a state $|\psi(t)\rangle_I$ in the interaction picture obeys the Schrödinger equation

$$i\frac{d|\psi(t)\rangle_I}{dt} = H_I(t)|\psi(t)\rangle_I .$$

(Note that states and operators in the Schrödinger picture (subscript S) and in the interaction picture (subscript I) are related as: $|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S$ and $\mathcal{O}_I(t) = e^{iH_0 t}\mathcal{O}_S e^{-iH_0 t}$.) **[5]**

(b) Write the solution of the Schrödinger equation in Question 5(a) as

$$|\psi(t)\rangle_I = U(t, t_0)\, |\psi(t_0)\rangle_I ,$$

where $U(t, t_0)$ is the unitary time evolution operator with $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$. Hence, find a differential equation for $U(t, t_0)$ and from that the integral representation of this equation imposing the condition $U(t, t) = 1$. Using the ansatz

$$U(t, t_0) = 1 + f_1(t, t_0) + f_2(t, t_0) ,$$

solve the integral equation for $U(t, t_0)$ perturbatively up to second order in H_I, where f_1 is linear in H_I and f_2 is quadratic in H_I. Without proof, write down Dyson's formula for $U(t, t_0)$. **[11]**

(c) State how the S-matrix (operator) S is related to the operator $U(t, t_0)$. Give a definition of initial and final states in a scattering process and write S-matrix elements or scattering amplitudes in terms of S, initial states and final states. Give a qualitative description of how physical cross sections are obtained from a given S-matrix element or scattering amplitude. **[4]**

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FORMULA SHEET (in units $\hbar = c = 1$)

<u>4-vector notation</u>:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu \eta_{\mu\nu} = a_\mu b_\nu \eta^{\mu\nu} \text{ with } \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \ , \ x_\mu = (t, -\vec{x})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \ , \ \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \ , \ \widehat{p}^\mu = i\partial^\mu \ , \ \widehat{p}_\mu = i\partial_\mu$$

<u>Klein-Gordon equation</u>: $(-\widehat{p}\cdot\widehat{p} + m^2)\psi = (\partial_\mu\partial^\mu + m^2)\psi = (\Box + m^2)\psi = 0$

<u>Free Dirac equation</u> in <u>Hamiltonian form</u>: $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha}\cdot\widehat{\vec{p}} + \beta m)\Psi$, or in <u>covariant form</u>:

$$(i\partial\!\!\!/ - m)\Psi = (i\gamma^\mu\partial_\mu - m)\Psi = (\widehat{p\!\!\!/} - m)\Psi = (\gamma\cdot\widehat{p} - m)\Psi = (\gamma^\mu\widehat{p}_\mu - m)\Psi = 0$$

<u>Dirac and Gamma matrices:</u>

$(\alpha^i)^2 = \mathbb{I}_4 \, , \, i = 1,2,3; \ \beta^2 = \mathbb{I}_4; \ \alpha^i\alpha^j + \alpha^j\alpha^i = 0 \, , \, i \neq j; \ \alpha^i\beta + \beta\alpha^i = 0 \, , \, i \neq j;$
$\gamma^0 = \beta, \ \gamma^i = \beta\alpha^i, \ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_4 \, ,$
$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Dirac matrices:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \ i = 1,2,3 \ , \ \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} ,$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Note that α^i, β and γ^0 are Hermitian, whereas the γ^i are anti-Hermitian. $\mathbb{I}_d$ represents a $d \times d$ identity matrix.

<u>Commutation relations of the raising/lowering operators of the neutral Klein-Gordon field:</u>

$$\left[a(\vec{k}), a(\vec{k}')\right] = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right] = 0, \ \left[a(\vec{k}), a^\dagger(\vec{k}')\right] = (2\pi)^3 2E_{\vec{k}}\,\delta^{(3)}(\vec{k}-\vec{k}') \text{ with } E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}.$$