Queen Mary
University of London

BSc and MSci Examination

Monday 14th May 2012 14:30 - 17:00

PHY7004 Relativistic Waves and Quantum Fields Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Answer THREE Questions

CALCULATORS ARE NOT PERMITTED IN THIS EXAMINATION.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

IMPORTANT NOTE:
THE *ACADEMIC REGULATIONS* STATE THAT POSSESSION OF UNAUTHORISED MATERIAL AT ANY TIME WHEN A STUDENT IS UNDER EXAMINATION CONDITIONS IS AN ASSESSMENT OFFENCE AND CAN LEAD TO EXPULSION FROM QMUL.

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EXAM PAPERS CANNOT BE REMOVED FROM THE EXAM ROOM.

Examiners:
Dr. R. Russo and Dr. G. Travaglini

Question 1: The Dirac equation

a) The Langrangian density for the Dirac field Ψ is

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi \ .$$

Write the action in four space-time dimensions and use the variational principle to derive the Dirac equation.

[3 marks]

b) Find all plane wave solutions of the Dirac equation for a particle at rest, i.e. $\vec{p} = \vec{0}$. State two alternative methods to generate solutions with arbitrary spatial momentum $\vec{p}$.

[6 marks]

c) Consider the covariant form of the Dirac equation. Assume that Ψ transforms under a Lorentz transformation $x' = \Lambda x$ as $\Psi(x) \to \Psi'(x') = \Lambda_s \Psi(x)$, with Λ_s being a four-by-four (constant) matrix. Show that the Dirac equation is form invariant (and hence covariant) if

$$\Lambda_s^{-1} \gamma^\nu \Lambda_s = \Lambda^\nu{}_\mu \gamma^\mu \ .$$

[6 marks]

d) Write the Dirac equation in five space-time dimensions and provide an explicit form for all the Gamma matrices (i.e. γ^μ, with $\mu = 0, \ldots, 4$). Comment on whether this representation of the five dimensional Lorentz algebra is reducible.

[5 marks]

Question 2: The Dirac field (in this problem keep explicitly all factors of $\hbar$ and c)

a) Start from Dirac's equation and motivate the relations

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}\mathbb{I}_4 \ , \quad \{\alpha^i, \beta\} = 0 \ , \quad \beta^2 = \mathbb{I}_4 \ .$$

[4 marks]

b) Show that it is not possible to satisfy these relations with Hermitean 2×2 or 3×3 matrices.

[6 marks]

c) Consider a Dirac field of the form

$$\Psi = \mathrm{e}^{-\frac{i}{\hbar}mc^2 t} \begin{pmatrix} \phi(t, x^i) \\ \chi(t, x^i) \end{pmatrix} ,$$

where ϕ and χ denote two component column spinors with space-time dependence. Start from Dirac's equation and decompose it into two coupled equations for ϕ and χ.

[3 marks]

d) In the non-relativistic limit one can use $1/c$ as an expansion parameter.

i) Assume that, in such an expansion, $\phi(t, x^i)$ is of order one and show that $\chi(t, x^i)$ is of order $1/c$. Write the two equations obtained in Question 2(c) to leading order in the $1/c$ expansion obtaining an equation for $\chi(t, x^i)$ and one for $\partial\phi(t, x^i)/\partial t$ in terms of the space derivatives of $\phi(t, x^i)$.

[2 marks]

ii) Push these two expansions up to the first subleading correction in $1/c$.

[5 marks]

Question 3: Symmetries and gauge fields

a) Show that the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, with χ an arbitrary, real function of the space-time coordinates.

[2 marks]

b) The Lagrangian density for the Dirac field Ψ is

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi \ .$$

i) Show that the transformation $\Psi \to \mathrm{e}^{i\chi}\Psi$, now with a real constant parameter χ, is a global symmetry of $\mathcal{L}$ and find the corresponding conserved current.

[6 marks]

ii) Write the minimal coupling of this current to the gauge field A_μ and obtain the QED Lagrangian density. By identifying the symmetry parameter χ with that of Question 3(a) show that the QED action enjoys a local (gauge) invariance.

[5 marks]

c) The Chern-Simons Lagrangian density in three space-time dimensions is

$$\mathcal{L}_{CS} = \frac{k}{2}\epsilon^{\rho\mu\nu} A_\rho F_{\mu\nu} \ ,$$

where $\epsilon^{\rho\mu\nu}$ is antisymmetric in the exchange of any two indices and $\epsilon^{012} = 1$. Show that $\mathcal{L}_{CS}$ changes by a total derivative under the gauge transformation defined in Question 3(a).

[2 marks]

d) Derive the transformation properties of $\mathcal{L}_{CS}$ under the proper Lorentz transformations and under the improper ones (such as space reflection).

[5 marks]

Question 4: The neutral Klein-Gordon

a) The Lagrangian density for a free, real Klein-Gordon field ϕ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \ .$$

Obtain the field equation for ϕ.

[2 marks]

b) Obtain the conjugate variable to the field ϕ and write the canonical commutation relations.

[3 marks]

c) Use the Fourier expansion

$$\phi(x^\mu) = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x} \right] ,$$

and derive the commutation relation among the Fourier modes.

[7 marks]

d) Now set the mass to zero, $m = 0$.

i) Show that the commutator of two fields at *generic* space-time points x and y

$$i\Delta(x-y) = [\phi(x), \phi(y)]$$

vanishes outside the light-cone (i.e. when $(x-y)^2 \neq 0$).

[3 marks]

ii) Derive an explicit expression for $\langle 0|\phi(x)\phi(y)|0\rangle$ and show that it does not vanish outside the light-cone.

[5 marks]

Hint: the following integrals may be of help

$$\int_0^\infty x\, \mathrm{e}^{i\alpha x} dx = -\frac{1}{\alpha^2} \ , \qquad \int_0^\infty \mathrm{e}^{i\alpha x} dx = \frac{i}{\alpha}$$

Question 5: The S-matrix

a) Consider the theory of one real ϕ_r and one complex scalar field ϕ_c defined by the action

$$S = \int \left\{ \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r + \partial_\mu\phi_c^\dagger\partial^\mu\phi_c - \frac{1}{2}m^2\phi_r^2 - \lambda\phi_c^\dagger\phi_c\phi_r \right\} d^4x \ .$$

Define the free Lagrangian density $\mathcal{L}_0$ and the interaction part $\mathcal{L}_{\text{int}}$. Give the physical units for the ϕ's and λ so as to make the action dimensionless (use the conventions $c = \hbar = 1$).

[4 marks]

b) Write the Dyson formula for the S-matrix in terms of $\mathcal{L}_{\text{int}}$.

[4 marks]

c) Use the Fourier expansions

$$\phi_r(x^\mu) = \int \frac{d^3k}{2E_{\vec{k}}(2\pi)^3} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x} \right] ,$$

$$\phi_c(x^\mu) = \int \frac{d^3k}{2|k|(2\pi)^3} \left[b(\vec{k})\, e^{-ik\cdot x} + c^\dagger(\vec{k})\, e^{ik\cdot x} \right] .$$

The field ϕ_c creates a type of particles (c-particles) and destroys another type of particles (b-particles); the field ϕ_r creates and destroys a-particles. Give at least one reason why b or c particles cannot decay into a-particles.

[4 marks]

d) Consider the decay of an a-particle of momentum $\vec{p}_3$ into a b-particle and c-particle of momenta $\vec{p}_1$ and $\vec{p}_2$ respectively.

i) Write down the in- and out-states using the Fourier modes and the vacuum state $|0\rangle$.

[3 marks]

ii) Focus on the first term in Dyson's formula that contributes to the decay process mentioned above. Write this term as a function of the Fourier modes and calculate the decay amplitude.

[5 marks]

End of Paper - An Appendix of 1 page follows

Appendix

Formula sheet (in units $\hbar = c = 1$)

4-vector notation:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu \eta_{\mu\nu} = a_\mu b_\nu \eta^{\mu\nu} \text{ with } \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (t, \vec{x}) \quad , \quad x_\mu = (t, -\vec{x})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \quad , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad , \quad \widehat{p}^\mu = i\partial^\mu \quad , \quad \widehat{p}_\mu = i\partial_\mu$$

Klein-Gordon equation: $(-\widehat{p} \cdot \widehat{p} + m^2)\psi = (\partial_\mu \partial^\mu + m^2)\psi = (\Box + m^2)\psi = 0$
Free Dirac equation in Hamiltonian form: $i\frac{\partial}{\partial t}\Psi = (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m)\Psi$, or in covariant form:

$$(i\partial\!\!\!/ - m)\Psi = (i\gamma^\mu \partial_\mu - m)\Psi = (\widehat{p\!\!\!/} - m)\Psi = (\gamma \cdot \widehat{p} - m)\Psi = (\gamma^\mu \widehat{p}_\mu - m)\Psi = 0$$

Dirac and Gamma matrices:

$$(\alpha^i)^2 = \mathbb{I}_4 \, , \, i = 1, 2, 3; \;\; \beta^2 = \mathbb{I}_4; \;\; \alpha^i \alpha^j + \alpha^j \alpha^i = 0 \, , \, i \neq j; \;\; \alpha^i \beta + \beta \alpha^i = 0 \, , \, i \neq j;$$
$$\gamma^0 = \beta, \; \gamma^i = \beta\alpha^i, \; \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_4 \, ,$$
$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Dirac matrices:

$$\alpha^i = \sigma^1 \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \; i = 1, 2, 3 \;\; , \quad \beta = \sigma^3 \otimes \mathbb{I}_2 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} ,$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Note that α^i, β and γ^0 are Hermitian, whereas the γ^i are anti-Hermitian. $\mathbb{I}_d$ represents the $d \times d$ identity matrix.
