MSci PHY7005/QMUL PHY-415 Relativistic Waves and Quantum Fields

Problem Set 2

(1) The harmonic oscillator Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Rewrite $\hat{H}$ in terms of the operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\hbar\omega}} \ , \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}}{\sqrt{2m\hbar\omega}} \ .$$

[2 marks]

Show that the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ implies $[\hat{a}, \hat{a}^\dagger] = 1$ **[2 marks]**

Write and solve the Heisenberg equations of motion for $\hat{a}$ and $\hat{a}^\dagger$ **[5 marks]**

(2) A Lorentz boost acting on a time-like vector v_μ yields the vector v'_μ. Show that if v_0 is positive, then also v'_0 is positive. Is this true for all Lorentz transformations? **[4 marks]**

(3) Suppose that the field $\phi(x)$ is a scalar, i.e. under a Lorentz transformation ($(x')^\mu = \Lambda^\mu{}_\nu x^\nu$ with ${}^t\Lambda\eta\Lambda = \eta$) we have the simple trasformation property $\phi(x') = \phi(x)$. Derive the transformations properties of the field

$$\psi_\mu(x) = \frac{\partial\phi(x)}{\partial x^\mu} \ ,$$

[2 marks]

Now work in three space dimensions, i.e. the indices $\mu, \nu, \ldots$ can take the values $0, 1, 2$. Derive the transformation properties of $\epsilon^{\mu\nu\rho} A_\nu(x)\psi_\rho(x)$, where A_μ transforms as ψ_μ and ϵ is the Levi-Civita symbol. **[5 marks]**