MSci PHY7005/QMUL PHY-415 Relativistic Waves and Quantum Fields

Problem Set 6

In the following set $\hbar = c = 1$.

1. Consider the space-like components of the Lorentz generators $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and identify

$$\Sigma^{23} \to S^1 \ , \quad \Sigma^{31} \to S^2 \ , \quad \Sigma^{12} \to S^3 \ .$$

   Show that the S^j's satisfy the angular momentum commutation relations

$$[S^j, S^k] = i\epsilon^{jkl} S^l \ .$$

[5 marks]

2. Check explicitly that the current $\bar{\psi}\gamma^\mu\psi$ for the Dirac field $\psi(x)$ is conserved. **[5 marks]**

3. The non-relativistic limit of the Dirac equation is given by the Pauli equation

$$i\frac{\partial\phi}{\partial t} = \left(\frac{(\widehat{\vec{p}} + q\vec{A})^2}{2m} + \frac{q}{2m}\vec{\sigma}\cdot\vec{B} - qA^0 \right)\phi \,,$$

   where $\widehat{\vec{p}} = -i\vec{\nabla}$. Turn on a weak and homogeneous magnetic field with vector potential $\vec{A} = \frac{1}{2}\vec{B}\times\vec{x}$ where $\vec{B}$ is constant. Neglecting terms quadratic in $\vec{A}$ show that the Pauli equation can be written in the form

$$i\frac{\partial\phi}{\partial t} = \left(\frac{(\widehat{\vec{p}})^2}{2m} + \frac{q}{2m}(\widehat{\vec{L}} + 2\widehat{\vec{S}})\cdot\vec{B} - qA^0 \right)\phi \,,$$

   where $\widehat{\vec{L}} = \vec{x}\times\widehat{\vec{p}}$ is the standard angular operator in quantum mechanics. **[5 marks]**

4. Consider that the "second-quantised" Hamiltonian

$$\hat{H} = \int d^3x \ \hat{\phi}^\dagger(x) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_1(x) \right] \hat{\phi}(x) + \frac{1}{2}\int d^3x d^3y \ \hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y) V_2(x,y)\hat{\phi}(x)\hat{\phi}(y) \ ,$$

   where $V_1(x)$ and $V_2(x,y)$ are standard functions. Show that $\hat{H}$ commutes with the numer operator $\hat{N} = \int d^3x \hat{\phi}^\dagger(x)\hat{\phi}(x)$. **[5 marks]**