RWQF4

(1) The free part of the Lagrangian density contains all quadratic terms

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi_r \eta^{\mu\nu} \partial_\nu \phi_r - \frac{1}{2} m^2 \phi_r^2 +$$

$$\partial_\mu \overline{\phi}_c \eta^{\mu\nu} \partial_\nu \phi_c$$

The interacting part contains the remaining terms of the complete Lagrangian (density)

$$\mathcal{L}_i = -\lambda \overline{\phi}_c \phi_c \phi_r$$

The action should be dimensionless. Thus we see that ϕ_r and ϕ_c must have the same dimensions (in this way the dimensions of the two kinetic terms is the same). Then let's focus on the term of the action containing the

Kinetic term of ϕ_r

$$\int d^4x \frac{1}{2} \frac{\partial \phi_r}{\partial x^\mu} \eta^{\mu\nu} \frac{\partial \phi_r}{\partial x^\nu}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ L^4 & [\phi]/L & [\phi]/L \end{array}$$

$[\phi]$ indicates the dimension of ϕ_r and L is a length

Then $L^4\left(\frac{[\phi]}{L}\right)^2 =$ no dimensions

which implies $[\phi] = 1/L$. By using that $[m] = 1/L$ when $\hbar$ and c are set to 1 we see that also the mass terms is dimensionless

$$\int d^4x \; \frac{1}{2} m^2 \varphi_r^2$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ L^4 & 1/L^2 & 1/L^2 \end{array}$$

. Finally the term $\sim \lambda$ gives

$$\int \lambda \, \bar{\varphi}_c \varphi_c \varphi_r \, d^4x \Rightarrow [\lambda] = \frac{1}{L}$$

$$\begin{array}{ccccc} \downarrow & & & & \\ [\lambda] & 1/L & 1/L & 1/L & L^4 \end{array}$$

- Let's first take the variation with respect to φ_r

$$\frac{\partial}{\partial x^\mu} \frac{\delta S}{\delta \frac{\partial \phi_r}{\partial x^\mu}} - \frac{\delta S}{\delta \varphi_r} = 0 \quad \text{implies}$$

$$\frac{\partial}{\partial x^\mu}\left(\eta^{\mu\nu}\frac{\partial \phi_r}{\partial x^\nu}\right) - \left(-m^2\varphi_r - \lambda|\varphi_c|^2\right) = 0$$

$$\frac{\partial}{\partial x^\mu}\eta^{\mu\nu}\frac{\partial \varphi_r}{\partial x^\nu} + m^2\varphi_r + \lambda|\varphi_c|^2 = 0$$

Then the variation w.r.t. $\bar{\varphi}_c$ yields

$$\frac{\partial}{\partial x^\mu}\left(\eta^{\mu\nu}\frac{\partial \Phi_c}{\partial x^\nu}\right) - \left(-\lambda\varphi_c\varphi_r\right) = 0$$

$$\frac{\partial}{\partial x^\mu}\eta^{\mu\nu}\frac{\partial}{\partial x^\nu}\varphi_c + \lambda\varphi_c\varphi_r = 0$$

The variation w.r.t. φ_c yield the complex conjugate of the eq. above.

(2) The basic idea is to connect Dirac's eq. to the Klein-Gordon one. Let's start from the eq. given in the text and take the $\frac{\partial}{\partial t}$ derivative of both sides

$$i\hbar \frac{\partial^2}{\partial t^2}\psi = -i\hbar c \alpha^i \frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\psi + mc^2\beta\frac{\partial \psi}{\partial t}$$

Then we use again the original eq. to rewrite the $\frac{\partial}{\partial t}$ derivatives on the r.h.s. in terms of space derivatives:

$$i\hbar \frac{\partial^2 \psi}{\partial t^2} = -c\alpha^i \frac{\partial}{\partial x^i}\left(-i\hbar c \alpha^j \frac{\partial \psi}{\partial x^j} + mc^2\beta\psi\right) +$$

$$mc^2\beta\left(-c\alpha^j\frac{\partial \psi}{\partial x^j} + \frac{mc^2}{\hbar i}\beta\psi\right)$$

which should match K.G. equation

$$\frac{\partial}{\partial x^\mu} \eta^{\mu\nu} \frac{\partial}{\partial x^\nu} \varphi + \left(\frac{mc}{h}\right)^2 \varphi = 0 \quad \text{K.G. eq.}$$

Thus we need to get rid of all mixed terms (such as $\frac{\partial}{\partial t}\frac{\partial}{\partial x^i}$ or $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ with $i \neq j$)

and treat β, α^i as constant objects.

- $\{\alpha^i, \beta\} = \alpha^i\beta + \beta\alpha^i = 0$ kills the mixed terms $\frac{\partial}{\partial x^i}\frac{\partial}{\partial t}$

- $\{\alpha^i, \alpha^j\} = 0$ if $i \neq j$ kills the mixed terms $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$

Then the last eq in the previous page reads

$$i\hbar \frac{\partial^2 \psi}{\partial t^2} = + i\hbar c^2 \sum_{i=1}^{3} (\alpha^i)^2 \frac{\partial^2}{\partial x^{i2}} \psi - i \frac{(mc^2)^2}{\hbar} \beta^2 \psi$$

Then if $(\alpha^i)^2 = \beta^2 = 1$, we have

$$i\hbar c^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi-\frac{\partial^2}{\partial x^{i2}}\psi\right)+i\frac{(mc^2)^2}{\hbar}\psi=0$$

Then by using $x^0=ct$ we get

$$\left(\frac{\partial^2}{\partial x^{02}}-\frac{\partial^2}{\partial x^{i2}}\right)\psi+\left(\frac{mc}{\hbar}\right)^2\psi=0$$

which is the K-G. equation.

- If U is unitary then $U^\dagger=U^{-1}$. Thus

$$\{\tilde\alpha^i,\tilde\alpha^j\}=U\alpha^iU^\dagger U\alpha^jU^\dagger+U\alpha^jU^\dagger U\alpha^iU^\dagger$$
$$=U\{\alpha^i\alpha^j\}U^\dagger=2\delta^{ij}UU^\dagger$$
$$=2\delta^{ij}$$

where we used $U^\dagger U=UU^\dagger=1$. Similarly

$$\{\tilde\beta,\tilde\alpha^i\}=U\{\beta,\alpha^i\}U^\dagger=0$$ and

$$\tilde\beta^2=U\beta\cancel{U^\dagger U}\beta U^\dagger=UU^\dagger=1$$

- An explict representation of the β and α^i in terms of 4x4 matrices (Dirac representation) is

$$\beta = \sigma^3 \otimes \mathbb{1} \quad ; \quad \alpha^j = \sigma^1 \otimes \sigma^j$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad ; \quad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$$