RWQF7

1) $$u_r^\dagger(\vec{p})\, u_s(\vec{p}) = \frac{E+mc^2}{2mc^2}\left(\phi_r^\dagger \phi_s + \phi_r^\dagger \frac{\sigma^k p^k c}{E+mc^2}\frac{\sigma^j p^j c}{E+mc^2}\phi_s\right)$$

where I used $(\sigma^k)^\dagger = \sigma^k$. Then by using $\sigma^k \sigma^j = i\epsilon^{kjl}\sigma^l + \delta^{kj}$, I get $\sigma^k \sigma^j p^k p^j = |\vec{p}|^2$ as the ϵ-term vanishes when contracted with a symmetric object, such as $p^j p^k$. Then

$$u_r^\dagger(\vec{p})\, u_s(\vec{p}) = \frac{E+mc^2}{2mc^2}\phi_r^\dagger \phi_s\left(1 + \frac{|\vec{p}|^2 c^2}{(E+mc^2)^2}\right)$$

Then by using $\phi_r^\dagger \phi_s = \delta_{rs}$ and $E^2 = \vec{p}^2 c^2 + m^2c^4$

$$u_r^\dagger(\vec{p})\, u_s(\vec{p}) = \delta_{rs}\frac{E+mc^2}{2mc^2}\,\frac{E^2 + 2Emc^2 + m^2c^4 + |\vec{p}|^2 c^2}{(E+mc^2)^2}$$

$$= \delta_{rs}\frac{E}{mc^2}$$

$$v_r^\dagger(\vec{p})\, v_s(\vec{p}) = \frac{E+mc^2}{2mc^2}\Big(\chi_r^\dagger \chi_s^\dagger +$$

$$+ \chi_r^\dagger \frac{\sigma^k p^k c}{E+mc^2} \frac{\sigma^j p^j c}{E+mc^2} \chi_s \Big)$$

and from here the calculation is identical to the previous case.

The barred spinor is defined as $\bar{u} = u^\dagger \gamma^0$. Thus

$$\bar{v}_r(\vec{p})\, u_s(p) = \frac{E+mc^2}{2mc^2} u_r^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} u_s =$$

$$= \frac{E+mc^2}{2mc^2}\Big(\phi_r^\dagger \phi_s - \phi_r^\dagger \frac{\sigma^k p^k c}{E+mc^2} \frac{\sigma^j p^j c}{E+mc^2} \phi_s \Big)$$

$$= \frac{E+mc^2}{2mc^2} \delta_{rs} \left(1 - \frac{|\vec{p}|^2 c^2}{(E+mc^2)^2} \right) =$$

$$= \delta_{rs} \frac{E^2 + 2Emc^2 + m^2c^4 - |p|^2 c^2}{2mc^2 (E+mc^2)} = \delta_{rs}$$

similarly

$$\bar{v}_r(\vec{p})\, v_s(p) = \frac{E+mc^2}{2mc^2} \left(\frac{|\vec{p}|^2 c^2}{(E+mc^2)^2} - 1 \right) \chi_r^\dagger \chi_s = -\delta_{rs}$$

b) There are (at least) two approaches. The direct calculation by choosing values for a, b.

For instance

$$\sum_{r=1}^{2}\left[\bar{u}_{r1}(\vec{p})\, u_{r1}(\vec{p}) - \bar{v}_{r1}(\vec{p})\, v_{r1}(\vec{p})\right] =$$

$$= \frac{E+mc^2}{2mc^2}\left\{ \sum_{r=1}^{2}\left[\phi_{r1}\,\phi_{r1} - \left(\frac{\sigma^k p^k c}{E+mc^2}\chi_r\right)_1 \left(\chi_r^{+}\frac{\sigma^j p^j c}{E+mc^2}\right)_1\right]\right\}$$

$r=1 \qquad r=2 \qquad r=1$

$$= \frac{E+mc^2}{2mc^2}\left[1 - \frac{p^1 - i p^2}{E+mc^2}\,\frac{p_1 + i p^2}{E+mc^2}c^2 + \frac{(p^3 c)^2}{E+mc^2}\right] = 1,$$

and similarly for the other components. Another approach is to use the completness of u_r and v_r in $\mathbb{C}^4$; i.e. any vector w can be written as

$$w = \sum_{r=1}^{2} c_r\, v_r(\vec{p}) + d_r\, u_r(\vec{p}).$$ Then by using the previous results we have

$$\sum_{r=1}^{2}\left[u_{ra}(\vec{p})\,\bar{u}_{rb}(\vec{p}) - v_{ra}(\vec{p})\,\bar{v}_{rb}(p)\right] w_b =$$

$$= \sum_{r=1}^{2} c_r\, v_{ra}(\vec{p})\,\delta_{ab} + d_r\, u_{ra}(p)\,\delta_{ab} = w_a$$

Since this holds $\forall\, w$, then the matrix in the square parenthesis must be the identity.

(2) $$S_{KG} = \int d^4x \left[\overline{\partial_\mu \phi}\, \eta^{\mu\nu}\, \partial_\nu \phi - m^2 |\phi|^2\right]$$

Then by using $\partial_\mu \to \partial_\mu - i A_\mu$, we have

$$S_{KG}^{(A)} = \int d^4x \left[(\partial_\mu + i A_\mu)\,\bar{\phi}\;\eta^{\mu\nu}\,(\partial_\nu - i A_\nu)\,\phi - m|\phi|^2\right]$$

(Notice that $\overline{\partial_\mu \phi} = \partial_\mu \bar{\phi}$, but it's important to start from this formulation so as to obtain a real action when $A \neq 0$).

The linear term in A is

$$i A_\mu \left[\bar{\phi}\,\partial^\mu \phi - (\partial^\mu \bar{\phi})\,\phi\right]$$

The object in the square parenthesis is the Noether current for the symmetry $\phi \to e^{i\vartheta}\phi$

$$\delta S_{KG} \sim -i \int d^4x \, \partial_\mu \vartheta \left[\bar{\phi} \partial^\mu \phi - (\partial^\mu \bar{\phi}) \phi \right]$$

when $A_\mu = B \delta_{\mu 2} X^1$ we have

$$S^{(A)}_{KG} = \int d^4x \left[\sum_{\mu \neq 2} \partial_\mu \bar{\phi} \, \partial^\mu \phi - m^2 |\phi|^2 - \right.$$

$$\left. (\partial_2 + i B x^1) \bar{\phi} \, (\partial_2 - i B x^1) \phi \right]$$

Now let's take the Euler-Lagrange variation with respect to $\bar{\phi}$

$$\frac{\partial}{\partial x^\mu} \frac{\delta S^{(A)}_{KG}}{\delta \partial_\mu \bar{\phi}} - \frac{\delta S^{(A)}_{KG}}{\delta \bar{\phi}} = + m^2 \phi +$$

$$+ \partial_\mu \partial^\mu \phi + i \partial_2 (B x^1 \phi) + i B x^1 (\partial_2 - i B x^1) \phi$$

$$= \partial_\mu \partial^\mu \phi + m^2 \phi + 2 i B x^1 \partial_2 \phi + (B x^1)^2 \phi = 0$$