RWQF 8

1) From $\gamma^1 = U \hat{\gamma}^1 U^{-1}$ we see that $[U, \sigma^2] = 0$.

Thus let's try $U = \left(a \mathbb{1} + b \sigma^2\right)$ for some $a, b \in \mathbb{C}$. Since U is unitary, we have $U U^\dagger = \mathbb{1}$

$$U U^\dagger = \left(a \mathbb{1} + b \sigma^2\right)\left(\bar{a} \mathbb{1} + \bar{b} \sigma^2\right) = \mathbb{1}$$

if $|a|^2 + |b|^2 = 1$ and $\mathrm{Re}(a \bar{b}) = 0$.

Then from $\gamma^0 = U \hat{\gamma}^0 U^{-1}$, we have

$$\sigma^3 = \left(a \mathbb{1} + b \sigma^2\right) \sigma^1 \left(\bar{a} \mathbb{1} + \bar{b} \sigma^2\right)$$

$$= \left(|a|^2 - |b|^2\right) \sigma^1 + i \left(a \bar{b} - b \bar{a}\right) \sigma^3$$

Thus we have to choose $a = \frac{i}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$ and

$$U = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

This is not the only possible solution for U.

2) $D_\mu \psi = (\partial_\mu - iA_\mu)\psi$ which under a gauge transformation becomes

$$D_\mu \psi \to (\partial_\mu - iA_\mu - i\partial_\mu \vartheta)\, e^{i\vartheta}\psi =$$

$$= e^{i\vartheta}\left[i\psi \cancel{\partial_\mu \vartheta} + \partial_\mu \psi - iA_\mu \psi - i\psi \cancel{\partial_\mu \vartheta} \right]$$

$$= e^{i\vartheta} D_\mu \psi$$

3) We need to use the commutation relation

$$[a(\vec{p}_1), a^\dagger(\vec{p}_2)] = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

and the definition of the vacuum

$$a(\vec{p}_1)|0\rangle = 0 \quad ; \quad \langle 0| a^\dagger(\vec{p}_1) = 0$$

Then we have

$$\langle 0| a(\vec{p}_1) a^\dagger(\vec{p}_2)|0\rangle = \langle 0| [a(\vec{p}_1), a^\dagger(\vec{p}_2)] |0\rangle$$

$$= (2\pi)^3 \delta^3(\vec{p}_1 - \vec{p}_2) \langle 0|0\rangle = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

Similarly we have

$$\langle 0|\, a(\vec{p}_4)\, a(\vec{p}_3)\, a^\dagger(\vec{p}_2)\, a^\dagger(\vec{p}_1)\, |0\rangle =$$

$$(2\pi)^3\, \delta^{(3)}(\vec{p}_3 - \vec{p}_2)\, \langle 0|\, a(\vec{p}_4)\, a^\dagger(\vec{p}_1)\, |0\rangle \; +$$

$$\langle 0|\, a(p_4)\, a^\dagger(\vec{p}_2)\, a(p_3)\, a^\dagger(\vec{p}_1)\, |0\rangle$$

where we used $a(p_3)\, a^\dagger(p_2) = (2\pi)^3 \delta^{(3)}(\vec{p}_3 - \vec{p}_2) + a^\dagger(\vec{p}_2)\, a(\vec{p}_1)$.

Then by using again the commutation relations we get

$$= (2\pi)^6 \left[\delta^{(3)}(\vec{p}_3 - \vec{p}_2)\, \delta^{(3)}(\vec{p}_4 - \vec{p}_1) + \delta^{(3)}(\vec{p}_4 - \vec{p}_2)\, \delta^{(3)}(\vec{p}_3 - \vec{p}_1) \right]$$

3) The Noether current J^μ is

$$J^\mu = -i\, \eta^{\mu\nu} \left[(\partial_\nu \bar{\phi})\, \phi - \phi\, \partial_\nu \bar{\phi} \right]$$

The mode expansion is

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a(\vec{p})\, e^{-ipx} + b^\dagger(\vec{p})\, e^{ipx} \right)$$

where $E_p = +\sqrt{\vec{p}^{\,2} + m^2}$

Thus we have

$$j^\mu = -i\,\eta^{\mu\nu}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{1}{\sqrt{2E_{p_1}2E_{p_2}}}\cdot\Big\{$$

$$\left(i p_{1\nu}\, a^\dagger(\vec{p}_1)\, e^{ip_1x} - i p_{1\nu}\, b(\vec{p}_1)\, e^{-ip_1x}\right)$$

$$\left(a(\vec{p}_2)\, e^{-ip_2x} + b^\dagger(\vec{p}_2)\, e^{ip_2x}\right) -$$

$$\left(a^\dagger(\vec{p}_1)\, e^{ip_1x} + b(\vec{p}_1)\, e^{-ip_1x}\right)\cdot$$

$$\left(-i p_{2\nu}\, a(\vec{p}_2)\, e^{-ip_2x} + i p_{2\nu}\, b^\dagger(\vec{p}_2)\, e^{ip_2x}\right)\Big\}$$

$$= -i\eta^{\mu\nu}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{1}{\sqrt{2E_{p_1}2E_{p_2}}}\Big\{a^\dagger(\vec{p}_1)\, a(\vec{p}_2)\, e^{i(p_1-p_2)x}$$

$$(ip_{1\nu}+ip_{2\nu}) - b(\vec{p}_1)\, b^\dagger(\vec{p}_2)\, e^{-i(p_1-p_2)x}(ip_{1\nu}+ip_{2\nu}) +$$

$$a^\dagger(\vec{p}_1)\, b^\dagger(\vec{p}_2)\, e^{i(p_1+p_2)x}(ip_{1\nu}-ip_{2\nu}) - b(\vec{p}_1)\, a(\vec{p}_2)\, e^{-i(p_1+p_2)x}(ip_1-ip_2)\Big\}$$

This is what was required. For the sake of curiosity you can derive the charge from j^0

$$Q = \int d^3\vec{x}\; j^0(x)$$

Then

$$\int d^3\vec{x}\, e^{\pm i (p_1 - p_2) x} = \int d^3\vec{x}\, e^{\pm i (E_{p_1} - E_{p_2}) t} e^{\mp i (\vec{p}_1 - \vec{p}_2) \vec{x}}$$

$$= e^{\pm i (E_{p_1} - E_{p_2}) t} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2) = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

Notice that the t-dependence drops since the δ-function on $\vec{p}$ implies that $E_{p_1} = E_{p_2}$. Similarly

$$\int d^3\vec{x}\, e^{\pm i (p_1 + p_2) x} = e^{\pm i (E_{p_1} + E_{p_2}) t} (2\pi)^3 \delta^{(3)}(p_1 + p_2)$$

which again implies $E_{p_1} = E_{p_2}$. Then only the first and the last term in Q survive

$$Q = \eta^{\mu\nu} \int \frac{d^3\vec{p}_1}{(2\pi)^3} \left[a^+(\vec{p}_1) a(p_1) - b(\vec{p}_1) b^+(\vec{p}_1) \right]$$