The Y-system for Scattering Amplitudes

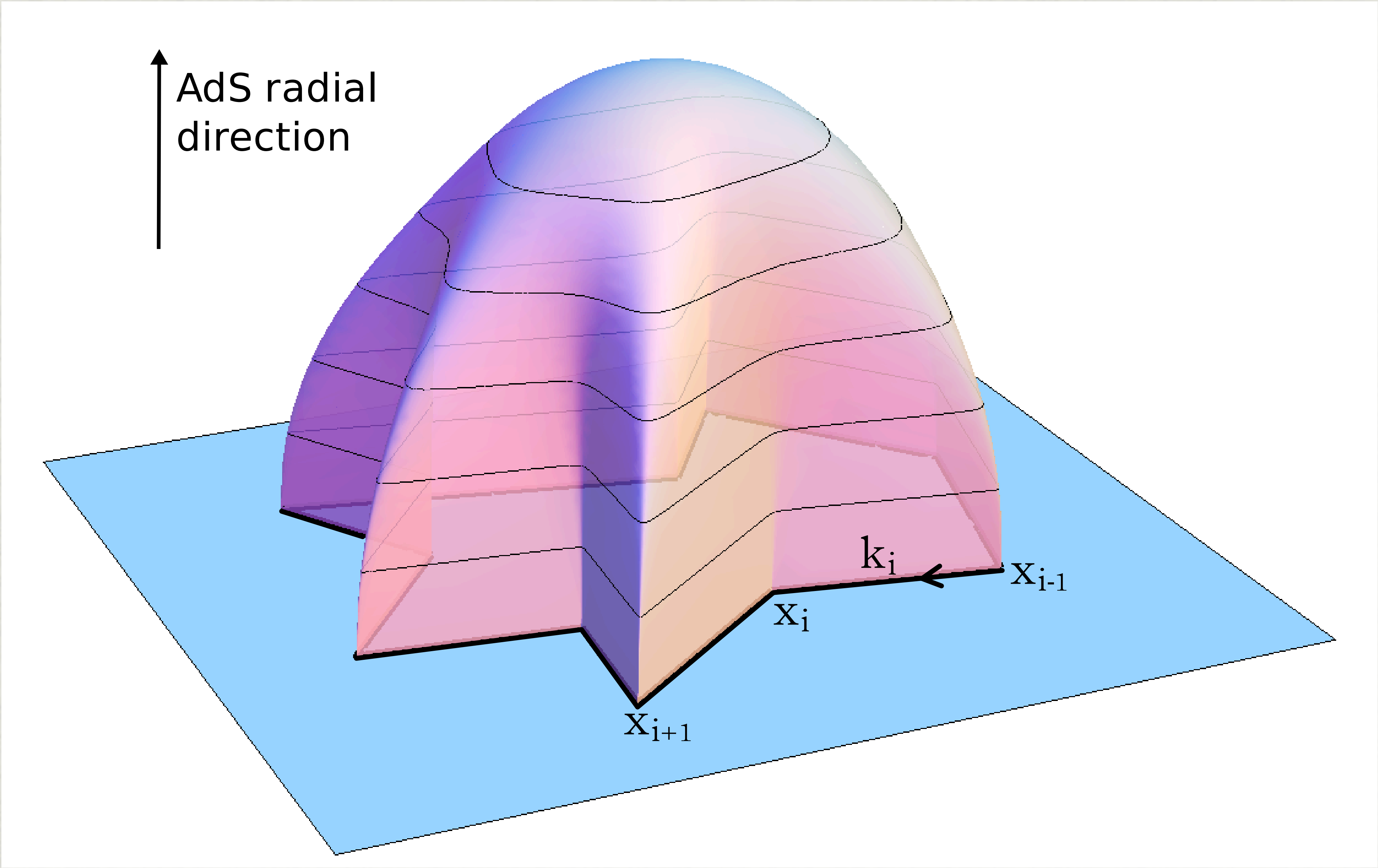
Amit Sever
Perimeter Institute

with **F. Alday, J. Maldacena, P. Vieira**
[arXiv:1002.2459]

Planar gluon scattering amplitudes (MHV) are given by

Null Polygonal Wilson Loops

[Alday,Maldacena; Drummond,Henn,Korchemsky,Sokatchev; Berkovitz,Maldacena; Beisert,Ricci,Tseytlin,Wolf; ...]

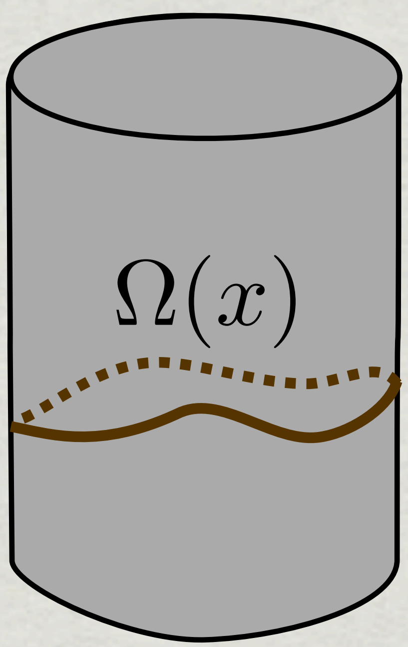


Classical Integrability at strong coupling

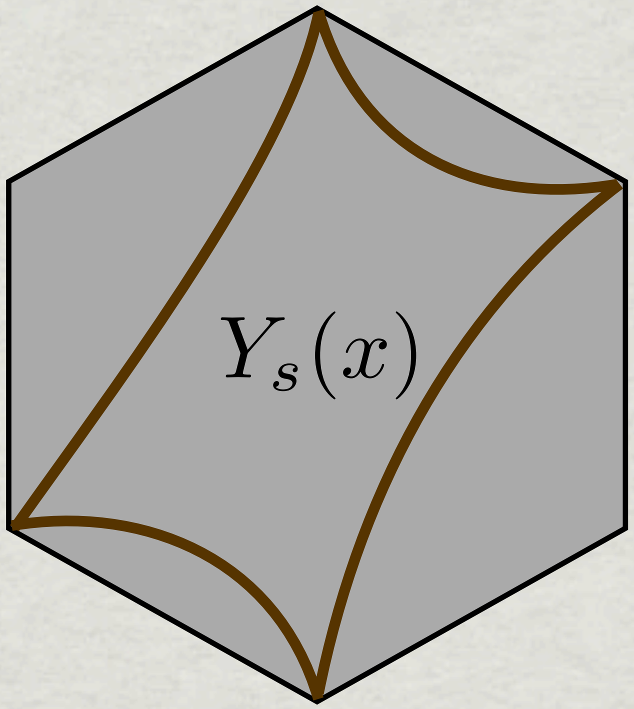
Flat connection

$$A = \frac{J + *J}{x-1} + \frac{J - *J}{x+1}\ , \qquad J = -g^{-1}dg$$

Spectrum



Amplitudes



Summary of results

$$Area_{reg} = \sum_k \int d\theta \; m_k \cosh\theta \log(1+Y_k(\theta)) \qquad e^{2\theta} = \zeta^2 = \frac{x-1}{x+1}$$

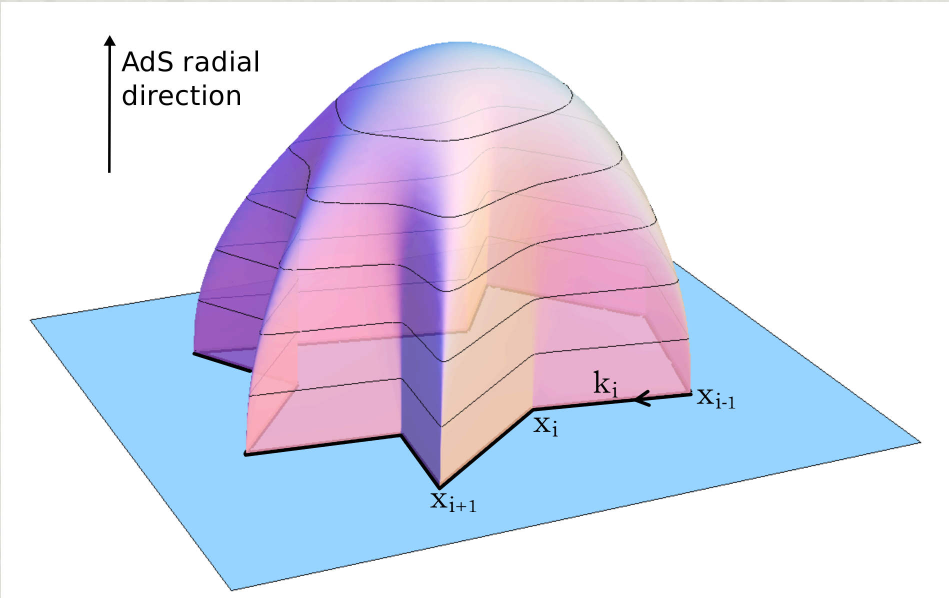
$$[\text{Cross ratio}]_s = Y_s(0)$$

$$\log Y_k(\theta) = -m_k \cosh\theta + C_k + K_{k,k'} \star \log(1+Y_{k'})$$

Simple integral equation for a set of 3n-15 generalized cross ratios $Y_k(\theta)$ in terms of 3n-15 parameters $m_k, \; C_k$

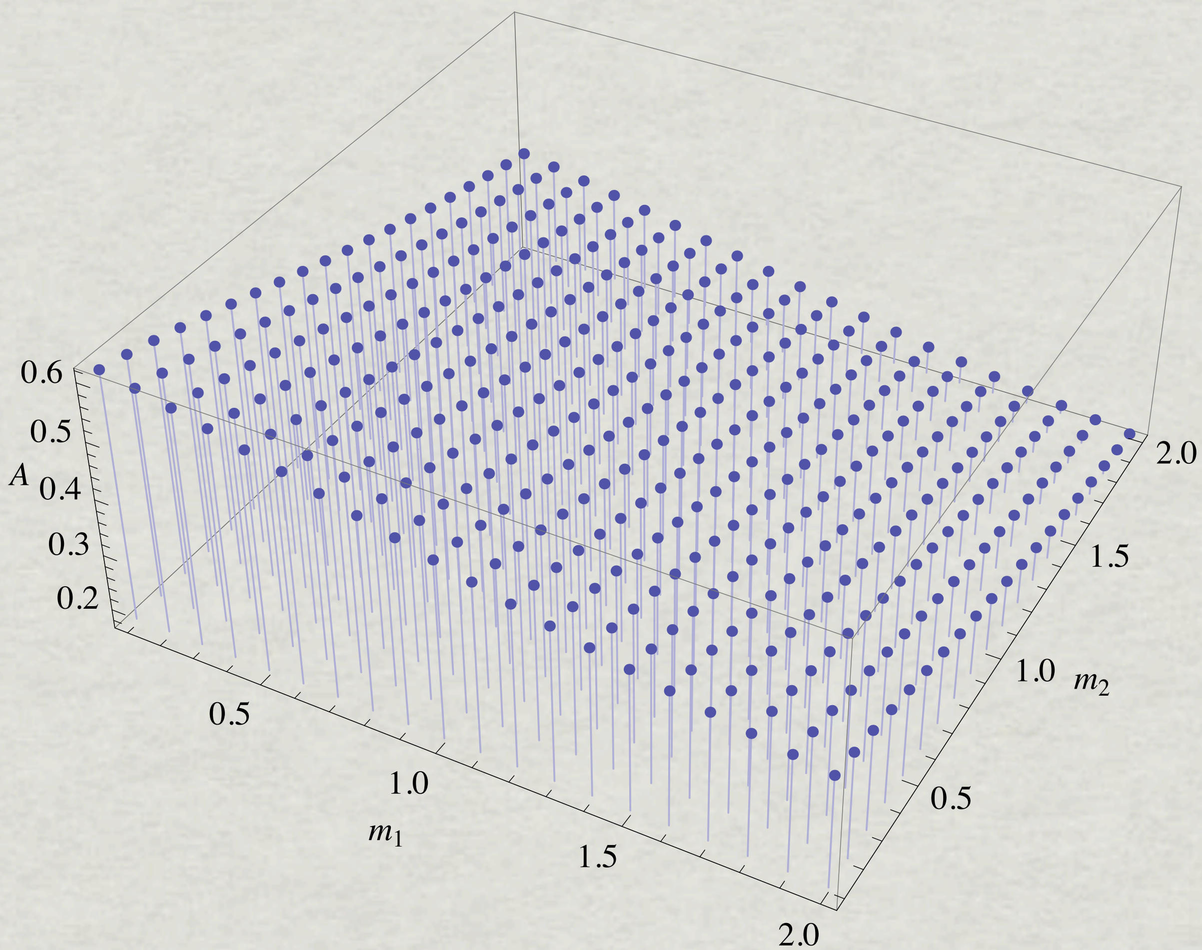
Takes the form of **Thermodynamic Bethe Ansatz equations** for an integrable model with relativistic particles of masses given by the m's. The Area is the corresponding **Free Energy!**

Very suggestive....

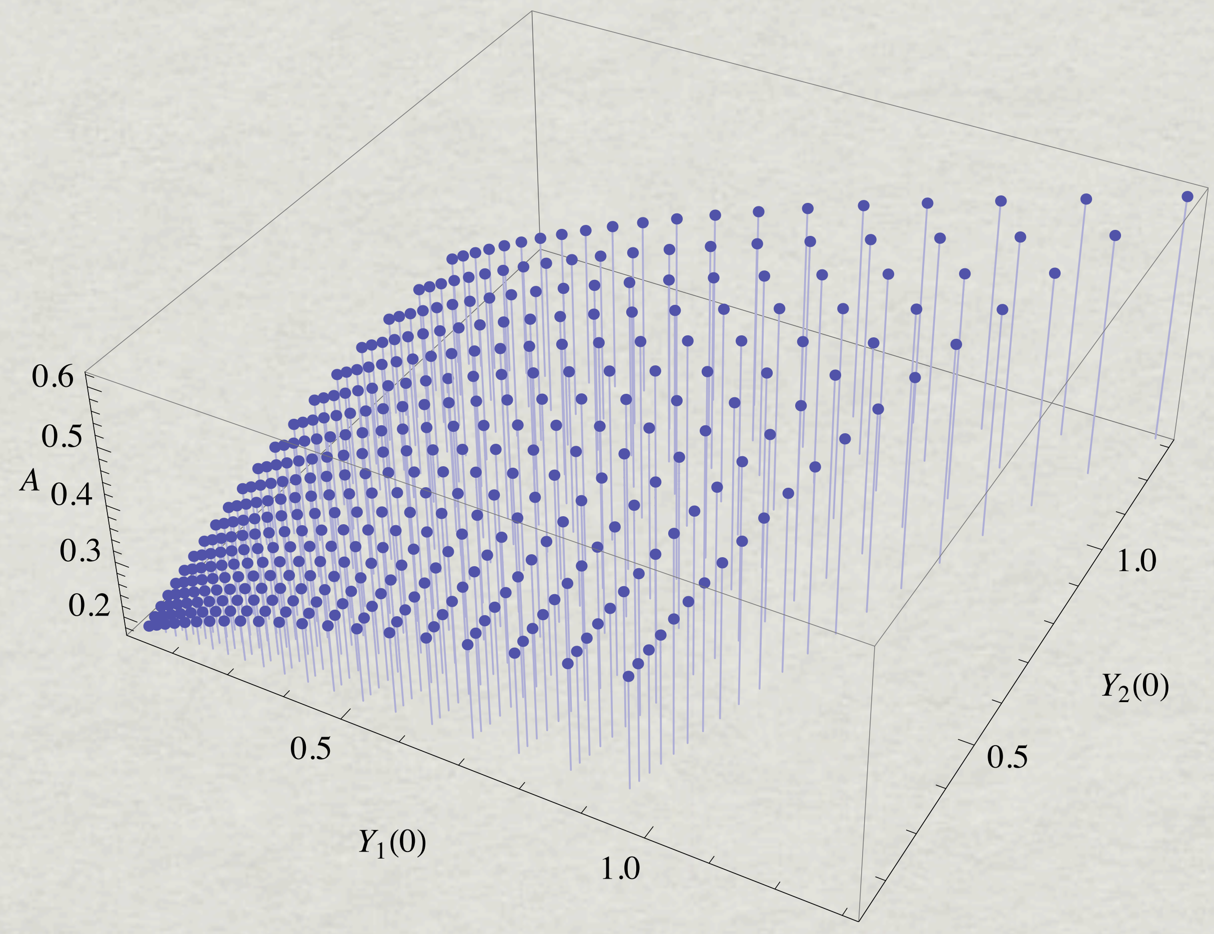


Example (in $R^{1,1}$ kinematics)

Parametric solution (10 gluons)



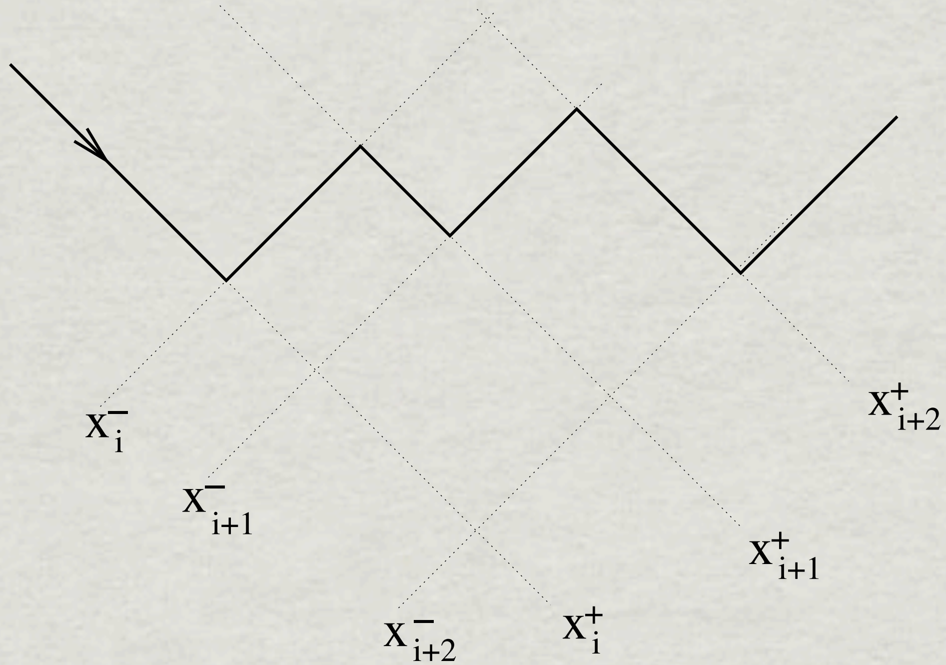
In terms of masses



In terms of cross ratios

$$\begin{bmatrix} \text{Area } (m) \\ \text{Cross ratios } (m) \end{bmatrix} \rightarrow \text{Area (Cross ratios)}$$

AdS_3 preliminaries



Restrict to simple $R^{1,1}$ kinematics,
i.e. surfaces in AdS_3

Flat SL(2) connection $\mathcal{A}(\zeta) = A + \frac{1}{\zeta}\Phi_z dz + \zeta \Phi_{\bar{z}} dz$, $\quad [D_z, D_{\bar{z}}] = 0$

$\Rightarrow \quad p(z) = \mathrm{Tr}\Phi_z^2/2$ Holomorphic function ($T(z)=0$)

Z_2 projection $U\mathcal{A}(\zeta)U^{-1} = \mathcal{A}(-\zeta)$

The area $Area = 2\int d^2z\ \mathrm{Tr}(\Phi_z \Phi_{\bar{z}})$

Boundary conditions

- Simple $|z| \to \infty$ asymptotics as in four cusp solution
- Having 2n cusps

$\Rightarrow$ $A \sim 0$ at large $|z|$
$p = z^{n-2} + a_1 z^{n-3} + \ldots$

AdS_3 preliminaries - The linear problem

Flat section $[d+\mathcal{A}(\zeta)]\varphi_a = 0 \ , \qquad a=1,2$

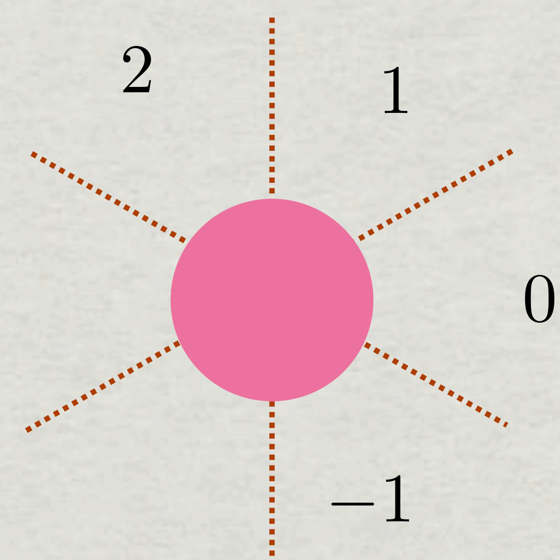
Spinor $\longrightarrow$ vector problems $Y_{a\dot{a}} = \varphi(1)\cdot M \cdot \varphi_{\dot{a}}(i)$ $\qquad \begin{vmatrix} Y_{-1}+Y_2 & Y_1 - Y_0 \\ Y_1+Y_0 & Y_{-1}-Y_2 \end{vmatrix} = -1$

SL(2) (traceless) connection $\Rightarrow$ $\langle \varphi, \psi \rangle \equiv \epsilon^{\alpha\beta}\varphi_\alpha \psi_\beta = \mathrm{const}$ (Wronskian)

At $|z| \to \infty$ $\qquad d+\mathcal{A}(\zeta) \sim d + \frac{dz}{\zeta}\begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix} + \zeta d\bar{z} \begin{pmatrix} \sqrt{\bar{p}} & 0 \\ 0 & -\sqrt{\bar{p}} \end{pmatrix} \ , \qquad dw = \sqrt{p}dz$

$w \longrightarrow z^{\frac{n}{2}} \quad \Rightarrow \quad \varphi \longrightarrow e^{\pm(z^{\frac{n}{2}}/\zeta + \bar{z}^{\frac{n}{2}}\zeta)}$ n Stokes sectors

s_i - small in sector i $\qquad b_i$ - big in sector i



The cross ratios $\qquad \frac{x^+_{ij}x^+_{kl}}{x^+_{ik}x^+_{jl}} = \frac{\langle s_i, s_j\rangle \langle s_k, s_l\rangle}{\langle s_i, s_k\rangle \langle s_j, s_l\rangle}(1)$

Black board derivation

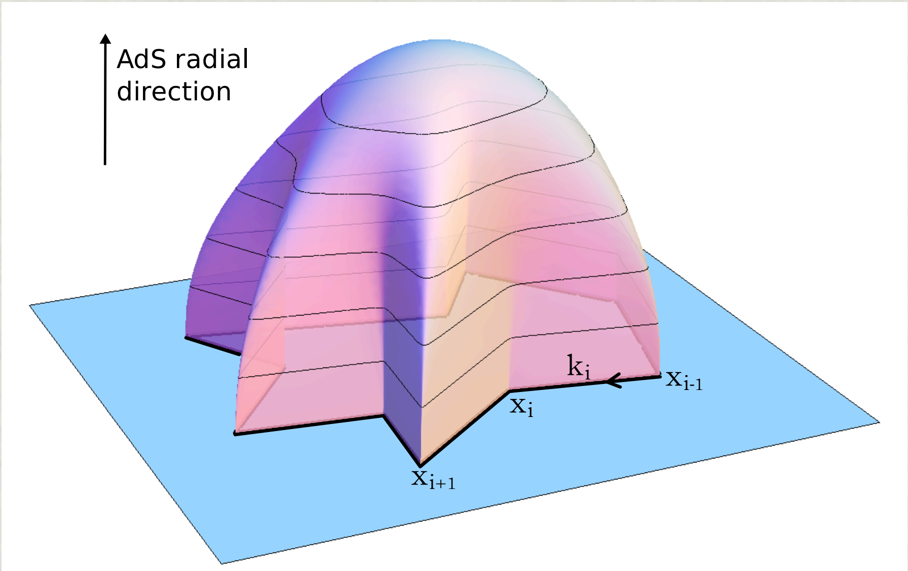
Conclusions

$$\mathcal{A} \sim e^{-\frac{\sqrt{\lambda}}{2\pi} Area}$$

Since the area is the free energy, this formula looks like we are computing the partition function of the system on a torus, where one of the sides has length proportional to $\sqrt{\lambda}$.

Future directions

- Continuous limit.
- The quantum problem.
- How to introduce a spectral parameter at weak coupling?
- Correlation functions.



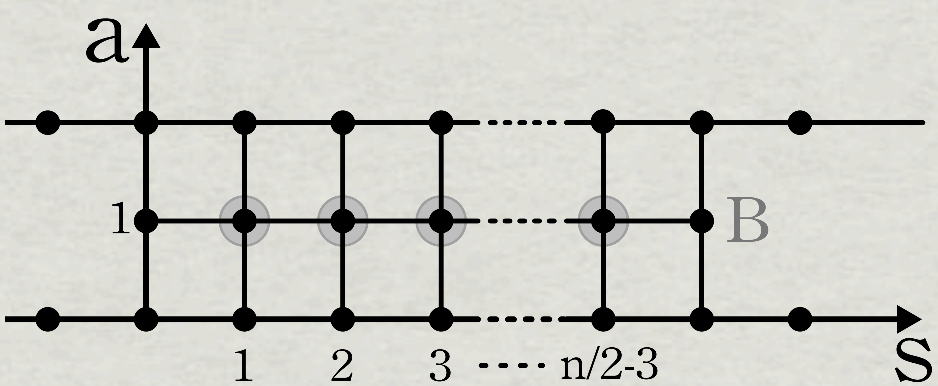
Thank you

Generic AdS$_5$ Kinematics

AdS$_3$ → AdS$_5$

two component spinors → four component Hodge momentum twistors

$Y_s \,, \qquad s = 1, \ldots, n/2 - 3$ → $Y_{a,s} \,, \qquad a = 1, 2, 3, \qquad s = 1, \ldots, n - 5$



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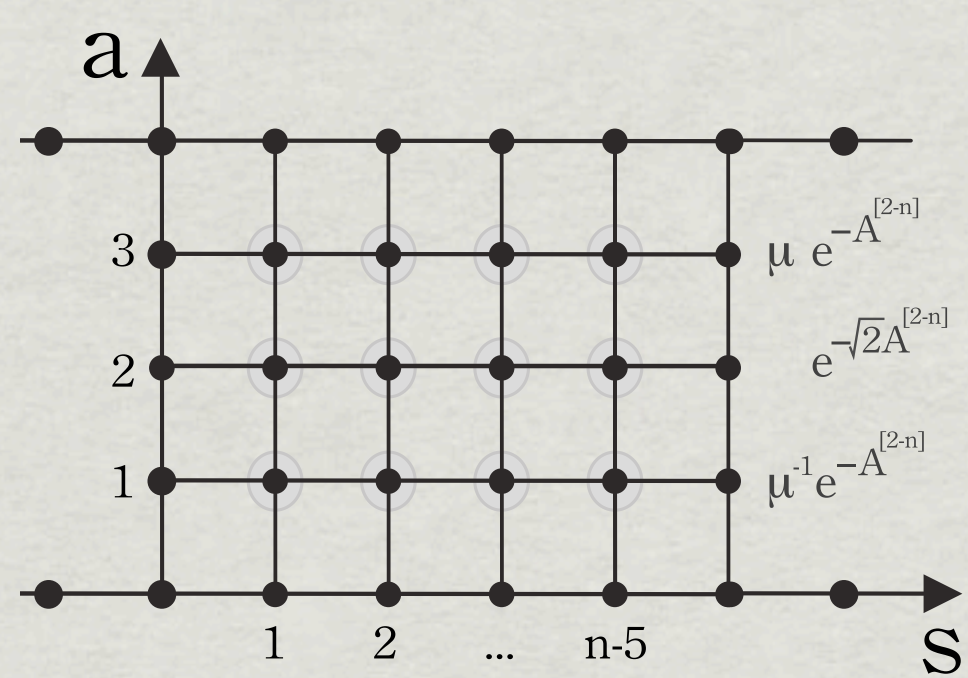


$K_1 \equiv \dfrac{1}{2\pi \cosh\theta}$ → $K_1 \equiv \dfrac{1}{2\pi \cosh\theta} \,, \quad K_2 = \dfrac{\sqrt{2}\cosh\theta}{\pi \cosh 2\theta} \,, \quad K_3 = \dfrac{i}{\pi} \tanh 2\theta \,.$

AdS$_5$

$$\log(Y_{2,s}) = -m_s\sqrt{2}\cosh(\theta) - K_2 \star \alpha_s - K_1 \star \beta_s$$

$$\log(Y_{1,s}) = -m_s\cosh(\theta) - C_s - \frac{1}{2}K_2 \star \beta_s - K_1 \star \alpha_s - \frac{1}{2}K_3 \star \gamma_s$$

$$\log(Y_{3,s}) = -m_s\cosh(\theta) + C_s - \frac{1}{2}K_2 \star \beta_s - K_1 \star \alpha_s + \frac{1}{2}K_3 \star \gamma_s$$

where

$$\alpha_s \equiv \log\frac{(1+Y_{1,s})(1+Y_{3,s})}{(1+Y_{2,s-1})(1+Y_{2,s+1})}\ ,\ \gamma_s \equiv \log\frac{(1+Y_{1,s-1})(1+Y_{3,s+1})}{(1+Y_{1,s+1})(1+Y_{3,s-1})}$$

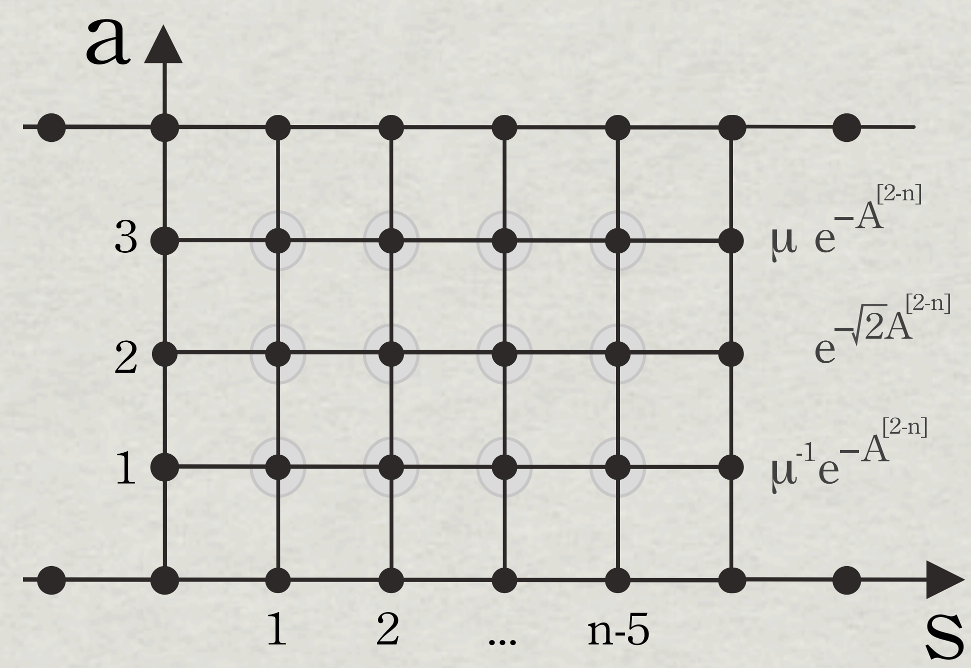
$$\beta_s \equiv \log\frac{(1+Y_{2,s})^2}{(1+Y_{1,s-1})(1+Y_{1,s+1})(1+Y_{3,s-1})(1+Y_{3,s+1})}\,,$$

and

$$K_1 \equiv \frac{1}{2\pi\cosh\theta}\ ,\ K_2 = \frac{\sqrt{2}\cosh\theta}{\pi\cosh 2\theta}\ ,\ K_3 = \frac{i}{\pi}\tanh 2\theta\,.$$

Finally

$$A_{free} = \sum_s \int \frac{d\theta}{2\pi}|m_s|\cosh\theta\log\left[(1+Y_{1,s})(1+Y_{3,s})(1+Y_{2,s})^{\sqrt{2}}\right](\theta+i\alpha_s)$$



Spacetime cross ratios:

$$U_s^{[r]} \equiv 1 + \frac{1}{Y_{2,s}}\Big|_{\theta = i\pi r/4}$$

Then

$$U_{2k-2}^{[0]} = \frac{\mathbf{x}^2_{-k,k}\mathbf{x}^2_{-k-1,k-1}}{\mathbf{x}^2_{-k-1,k}\mathbf{x}^2_{-k,k-1}}\ .$$