The two-loop six-point Wilson loop in N=4 SYM

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Going beyond one-loop

- The aim of perturbation theory is the computation of physical observables at weak coupling, that can be compared to experiment.

- In QCD, only very few two-loop results are available:
  - ➡ Use simpler theory as a playground to understand multi-loop amplitudes.

- MHV amplitudes in planar N=4 SYM:

$$A_n^{MHV}(\epsilon) = A_n^{(0)}\, M_n(\epsilon) \qquad M_n(\epsilon) = 1 + \sum_{\ell=1}^{\infty} a^\ell\, M_n^{(\ell)}(\epsilon)$$

ABDK iteration

- Two-loop amplitudes are determined by the ABDK iteration

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + R_n^{(2)}(u_{ij}) + \mathcal{O}(\epsilon)\,,$$

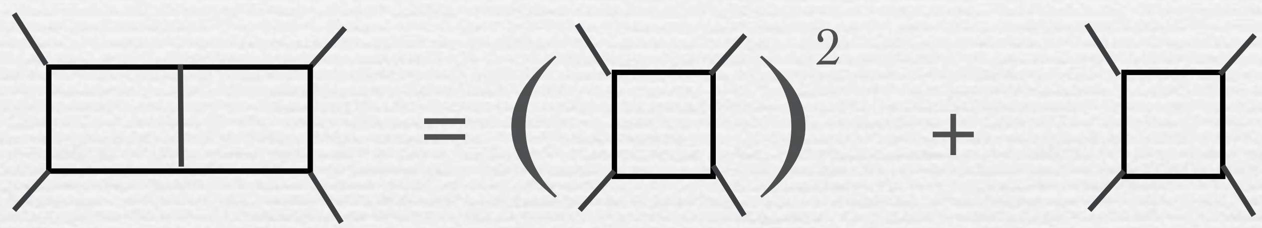
$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \qquad\qquad C^{(2)} = -\zeta_2^2/2$$

[Anastasiou, Bern, Dixon, Kosower]

- Remainder function is zero for $n = 4, 5$, but non zero starting from $n = 6$.

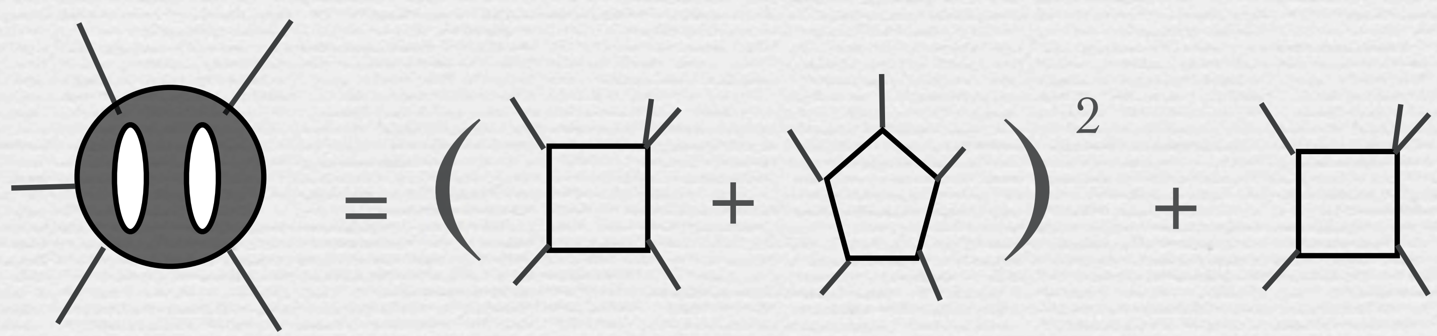
ABDK iteration

- The ABDK iteration is enough to fix completely the four-point amplitude



- Starting from five points, something new happens:

  The two-loop amplitude develops a parity-odd contribution!



- The parity-odd contirbutions however cancel in the logarithm.

The five-point amplitude at two loops

- The appearance of these new pieces adds a new step in complexity!
- Several analytic representations are known that all indicate that the pentagon contribution has a very complicated analytic structure:
  - ➡ High energy limit: Representation through Appell and Kampé de Fériet function; expansion in terms of Goncharov polylogarithms in complicated arguments:

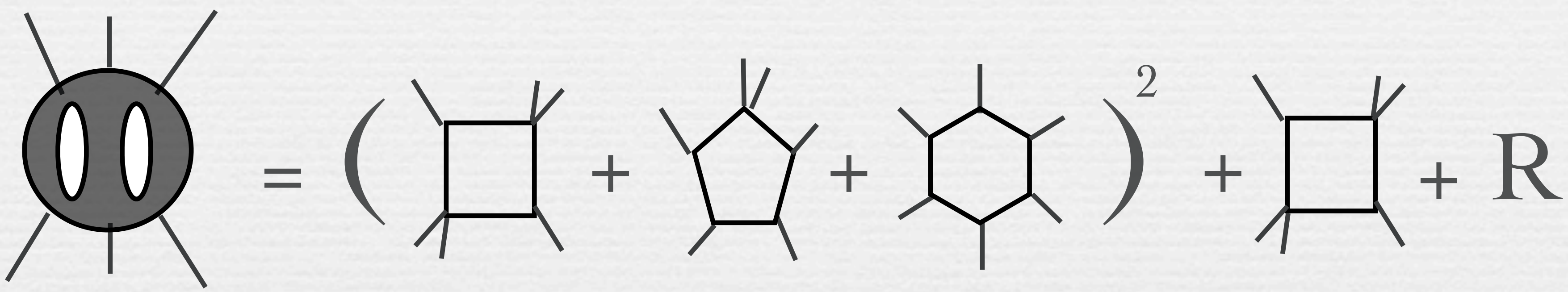
$$\lambda_1(x_1,x_2) = \frac{1}{2}\left(1+x_1-x_2-\sqrt{\lambda_K}\right)$$
$$\lambda_1(x_1,x_2) = \frac{1}{2}\left(1+x_1-x_2+\sqrt{\lambda_K}\right)$$
$$\lambda_K(x_1,x_2) = \lambda(x_1,x_2,-1) = 1+x_1^2+x_2^2+2x_1+2x_2-2x_1x_2.$$

[Del Duca, CD, Glover, Smirnov]

  - ➡ General kinematics: Representation through Appell functions. Expansion presently unknown. [Kniehl, Tarasov]

Going beyond five points

- The first non trivial place where the remainder function appears is n=6:



- Two-loop six-point integral basis is known, so we could in principle extract the remainder function from the amplitude.
- This would require the analytic computation of all two-loop master integrals, as well as of the hexagon to higher orders in epsilon.
- All of these nasty contributions cancel in the logarithm, so it would be desirable to directly compute the logarithm.

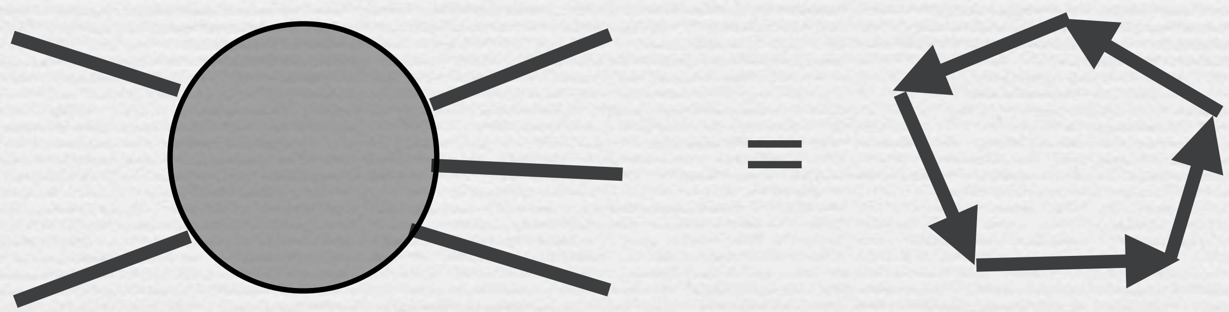
- Wilson loops and the remainder function
- Regge exactness of Wilson loops - Or to compute perturbative Wilson loops efficiently
- The two-loop six-point remainder function
- Some selected results

Wilson loops in N=4 SYM

- Definition of a Wilson loop:

$$W[\mathcal{C}_n] = \mathrm{Tr}\,\mathcal{P}\,\exp\left[ig\oint \mathrm{d}\tau\,\dot{x}^\mu(\tau) A_\mu(x(\tau))\right]$$

- It is conjectured that Wilson loop along an *n*-edged polygon is equal to an *n*-point MHV scattering amplitude:



$$p_i = x_{i,i+1} = x_i - x_{i+1}$$

[Alday, Maldacena;
Drummond, Korchemsky, Sokatchev]

- Proven analytically at one-loop for arbitrary *n*, and at two-loops for *n* = 4, 5, 6.

[Drummond, Henn, Korchemsky, Sokatchev;
Brandhuber, Heslop, Spence]

Wilson loops in N=4 SYM

- Wilson loops possess a conformal symmetry, and it was shown that a solution to the corresponding Ward identities is the BDS ansatz, e.g., at two-loops,

[Drummond, Henn, Korchemsky, Sokatchev]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} \qquad\qquad + \mathcal{O}(\epsilon)\,,$$

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$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_n^{(2)}(u_{ij}) + \mathcal{O}(\epsilon)\,,$$

- ... but we can always add a arbitrary function of conformal invariants and we still obtain a solution to the Ward identities!

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

The remainder function

- Cyclic symmetry of the amplitude implies symmetries for the remainder function.
  - ➡ For $n = 6$, the remainder function is completely symmetric.

- Multi-collinear limits:

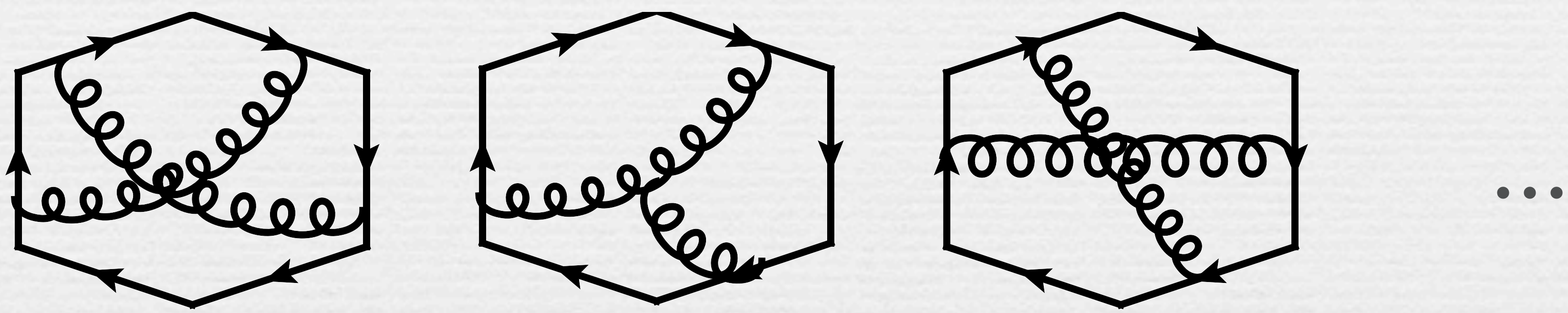
$$\mathcal{R}_n \to \mathcal{R}_{n-k} + \mathcal{R}_{k+4}$$ [Heslop, Khoze]

  - ➡ For $n = 6$, the remainder function vanishes in the two-particle collinear limits.

- It vanishes in the multi-Regge limit (in the Euclidean region).

How can we compute this function?

- Anastasiou, Brandhuber, Heslop, Khoze, Spence and Travaglini worked out the two-loop Wilson loop diagrams:

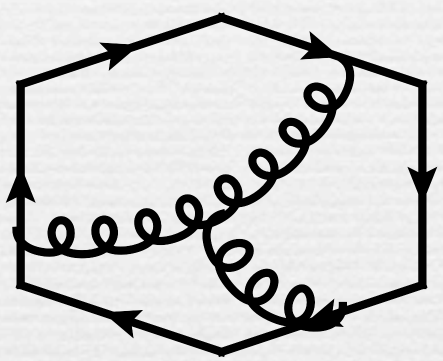


- Each of these diagrams is an integral, similar to a Feynman parameter integral.

- Numerical studies of these integrals confirmed all the properties of the remainder function.

How can we compute this function?

- For n = 6, many of the integrals can be computed explicitly, but one is particularly 'hard':



$$f_H(p_1, p_2, p_3; Q_1, Q_2, Q_3) := \frac{\Gamma(2-2\epsilon_{\mathrm{UV}})}{\Gamma(1-\epsilon_{\mathrm{UV}})^2} \int_0^1 \Big(\prod_{i=1}^3 d\tau_i\Big) \int_0^1 \Big(\prod_{i=1}^3 d\alpha_i\Big) \delta(1-\sum_{i=1}^3 \alpha_i)\ (\alpha_1\alpha_2\alpha_3)^{-\epsilon_{\mathrm{UV}}} \frac{\mathcal{N}}{\mathcal{D}^{2-2\epsilon_{\mathrm{UV}}}}\,,$$

$$\begin{aligned}\mathcal{N} &= 2(p_1p_2)(p_1p_3)\Big[\alpha_1\alpha_2(1-\tau_1)+\alpha_3\alpha_1\tau_1\Big] + 2(p_1p_3)(p_2p_3)\Big[\alpha_3\alpha_1(1-\tau_3)+\alpha_2\alpha_3\tau_3\Big] \\ &+ 2(p_1p_2)(p_2p_3)\Big[\alpha_2\alpha_3(1-\tau_2)+\alpha_1\alpha_2\tau_2\Big] + 2\alpha_1\alpha_2\Big[2(p_1p_2)(p_3Q_3)-(p_2p_3)(p_1Q_3)-(p_3p_1)(p_2Q_3)\Big]\end{aligned}$$

$$+\ldots$$

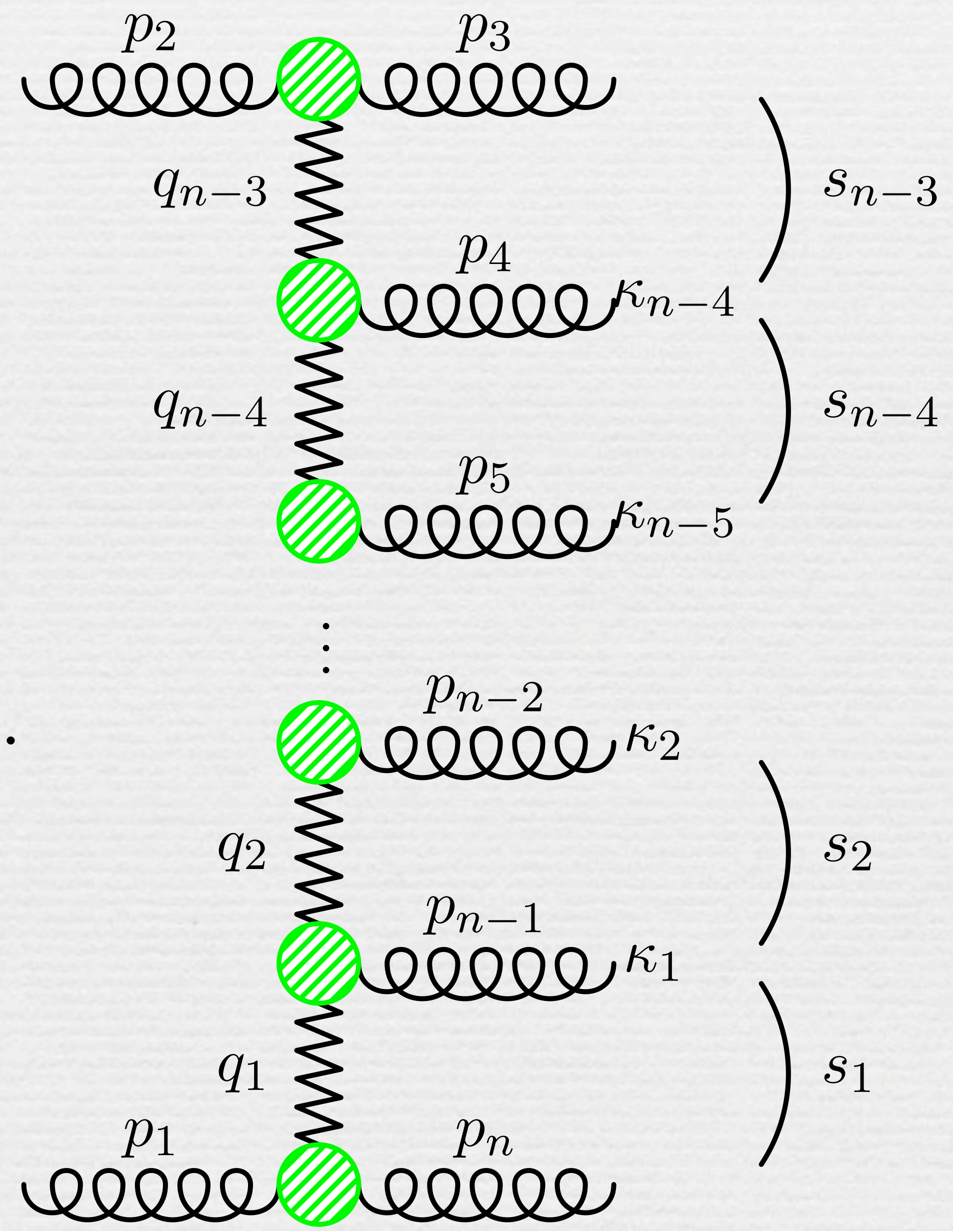
- The integrals do not explicitly depend on conformal ratios.
- But is all this complexity really needed..?
- Could we go to simplified kinematics?

An excursion to multi-Regge kinematics

- Multi-Regge kinematics are defined by

$$y_3 \gg y_4 \gg \ldots \gg y_{n-1} \gg y_n$$

$$|p_{3\perp}| \simeq |p_{4\perp}| \simeq \ldots \simeq |p_{n-1\perp}| \simeq |p_{n\perp}|,$$



- This implies a hierarchy of scales:

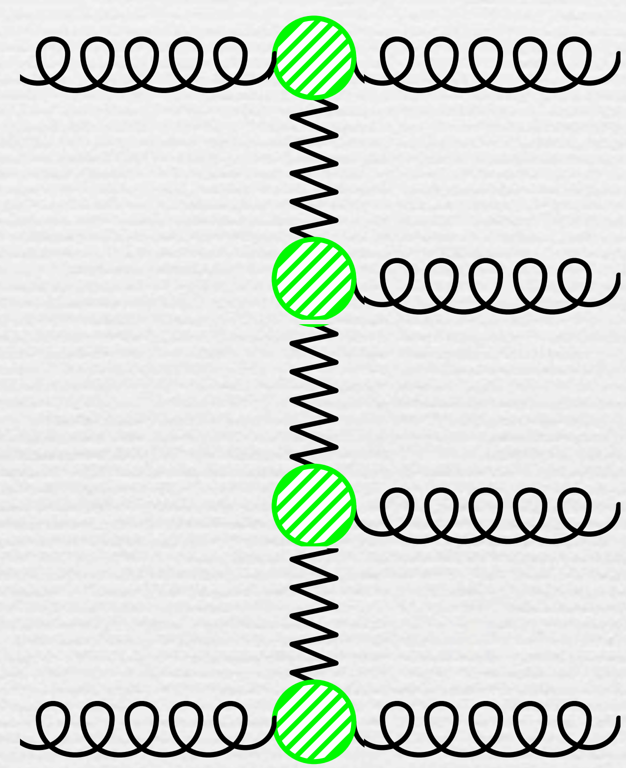
$$s \gg s_1,\ s_2, \ldots, s_{n-3} \gg -t_1,\ -t_2, \ldots, -t_{n-3}\,.$$

Quasi-multi-Regge limits

- Multi-Regge kinematics

$$y_3 \gg y_4 \gg y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



- In the multi-Regge limit, the cross ratios become trivial:

$$u_1 = \frac{s_{12}\, s_{45}}{s_{345}\, s_{456}} \simeq 1$$

$$u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{456}} \simeq \mathcal{O}\left(\frac{t}{s}\right)$$

$$u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}} \simeq \mathcal{O}\left(\frac{t}{s}\right)$$

[Bartels, Lipatov, Vera; Brower, Nastase, Schnitzer; Del Duca, CD, Glover]

Quasi-multi-Regge limits

- Quasi-multi-Regge kinematics

$$y_3 \gg y_4 \simeq y_5 \gg y_6$$

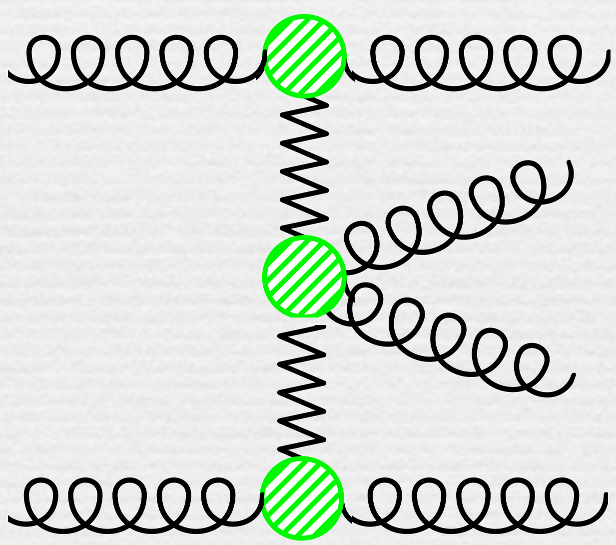
$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$

- In the quasi-multi-Regge limit, the cross ratios stay generic:

$$u_1^{\mathrm{QMRK}} = \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)}$$

$$u_2^{\mathrm{QMRK}} = \frac{|p_{3\perp}|^2 p_5^+ p_6^-}{(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)(p_4^+ + p_5^+) p_6^-}$$

$$u_3^{\mathrm{QMRK}} = \frac{|p_{6\perp}|^2 p_3^+ p_4^-}{p_3^+ (p_4^- + p_5^-)(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)}$$



[Bartels, Lipatov, Vera; Brower, Nastase, Schnitzer; Del Duca, CD, Glover]

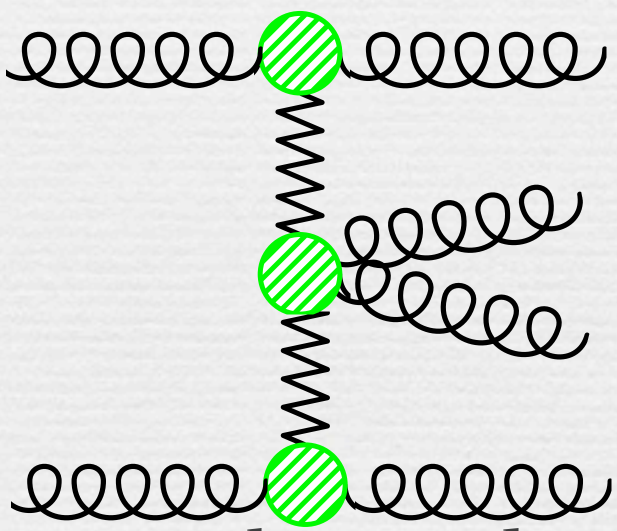
Regge-exactness of Wilson loops

- The result is in fact even stronger:

  The (logarithm of the) Wilson-loop is **Regge-exact** in this limit, i.e., it is the same in this special kinematics and in arbitrary kinematics

$$y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n$$

$$|p_{3\perp}|^2 \simeq \ldots \simeq |p_{n\perp}|^2$$



- This limit leaves the conformal cross ratios unchanged for an arbitrary number of edges.
- This result is in fact true for Wilson loops with an arbitrary number of edges and loops!

[Del Duca, CD, Smirnov]

Regge-exactness of Wilson loops

- The proof is very simple:

$$\ln W_n = \sum_{\ell=1}^{\infty} f_{WL}^{(\ell)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(\ell)} + R_n^{(\ell)}(u_{ij}) + \mathcal{O}(\epsilon)$$

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$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\mathcal{M}_n^{(1)}$$

[Brandhuber, Heslop, Travaglini]

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Structure of the one-loop amplitude:

$$\ln s_{ij} \qquad + \qquad \mathrm{Li}_2(1-u_{ij})$$

[Bern, Dixon, Dunbar, Kosower]

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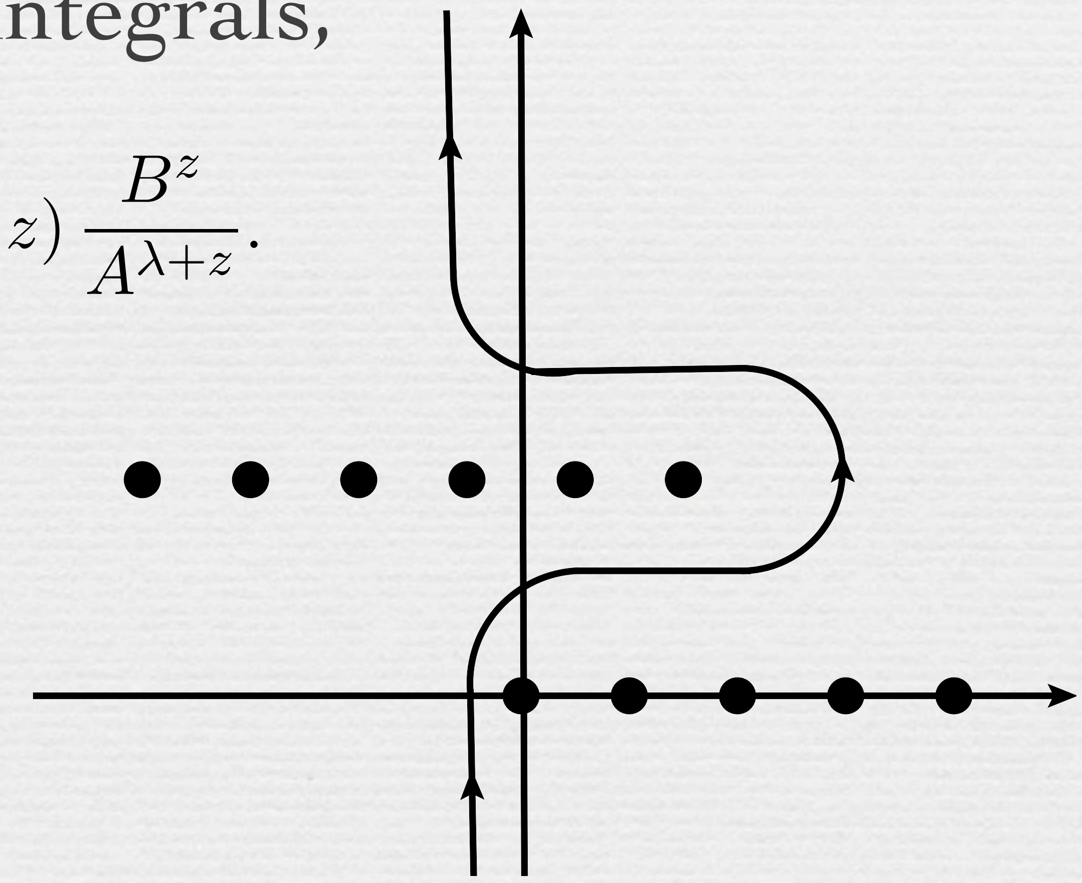
Log's are not power suppressed.

A Recipe to compute Wilson loops

- Step 1:

  We write down a Mellin-Barnes representation for each diagram, i.e., we replace denominators in the Feynman parameter integrals by contour integrals,

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\,\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\Gamma(-z)\,\Gamma(\lambda+z)\,\frac{B^{z}}{A^{\lambda+z}}.$$



- This turns the Feynman parameter integral into residue calculus:

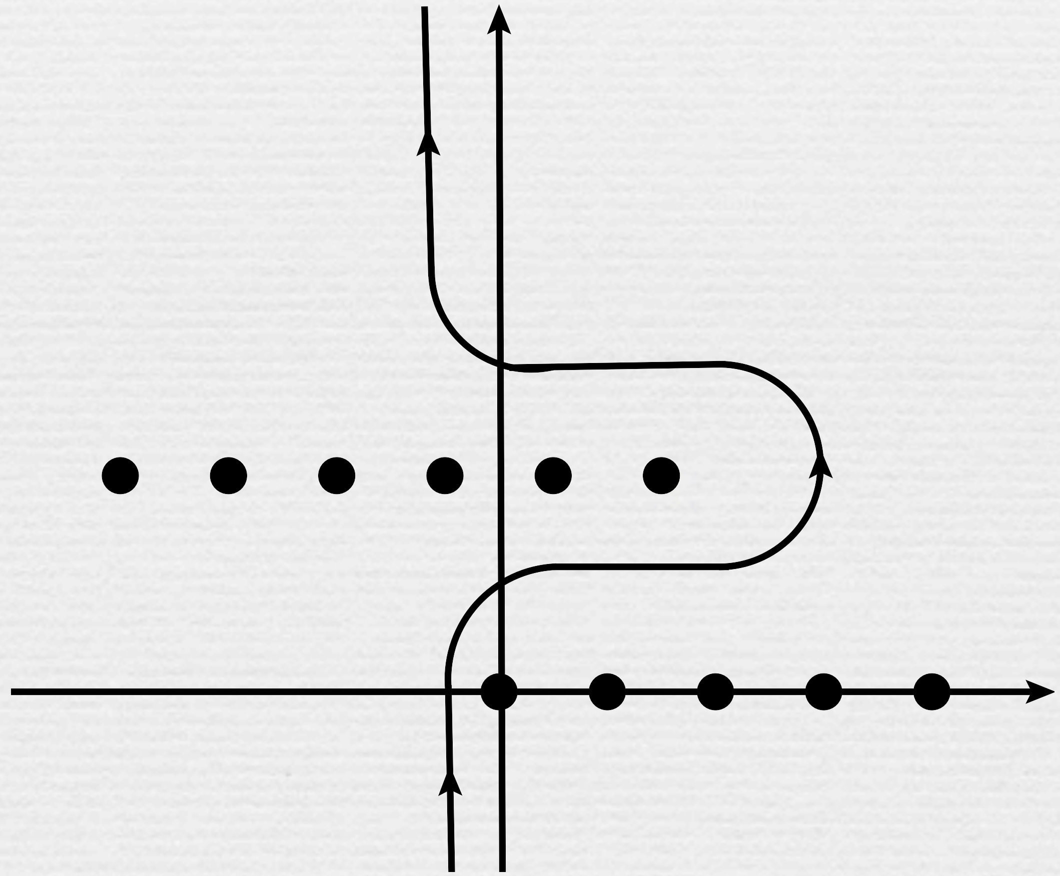
$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$$

A Recipe to compute Wilson loops

- Step 2:

  We exploit Regge exactness and we only compute the leading behavior of each integral in the quasi-multi-Regge limit

- The Mellin-Barnes approach is very suitable for this!



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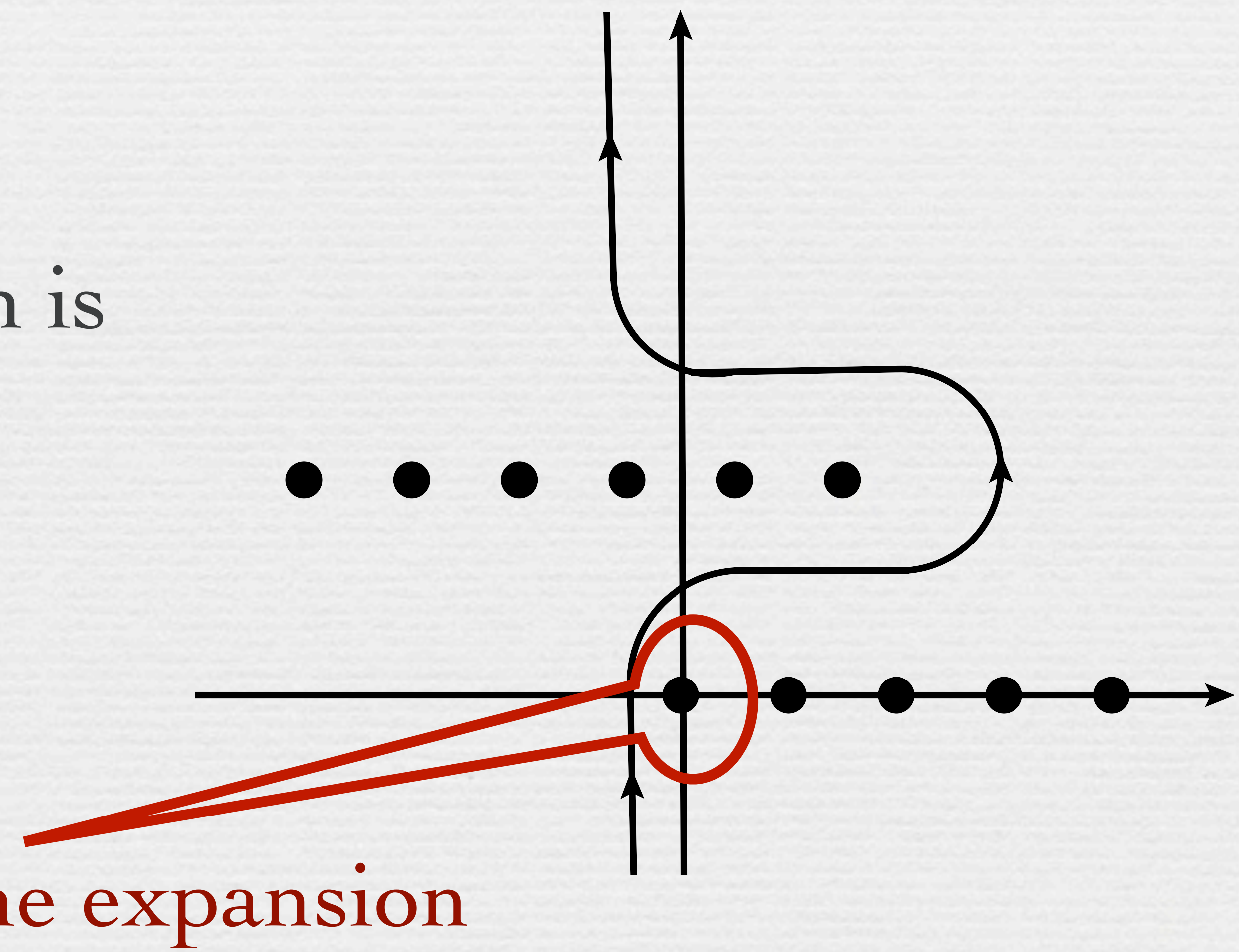
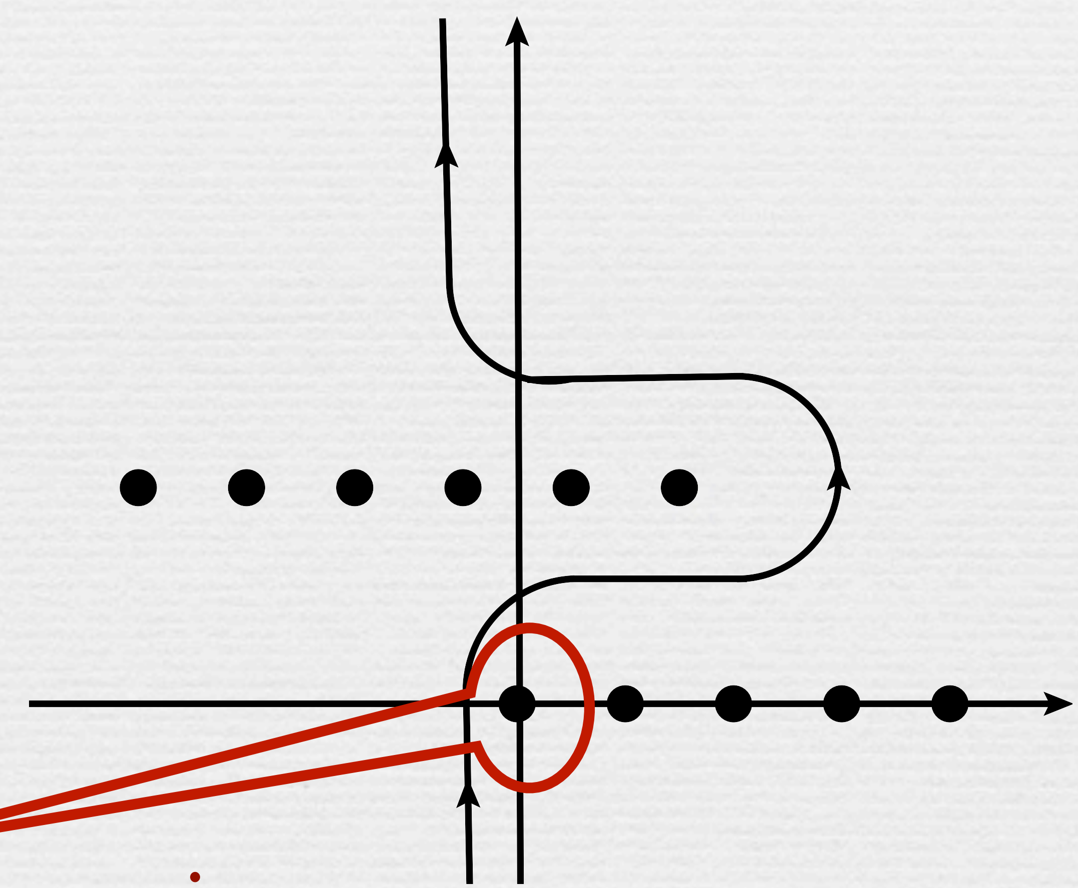


A Recipe to compute Wilson loops

- Step 3:

  Iterate the limits: There are six different ways to take the limits, corresponding to the six cyclic permutations of the external legs.

- Regge-exactness allows us to take all six limits at the same time!



Leading term in the expansion

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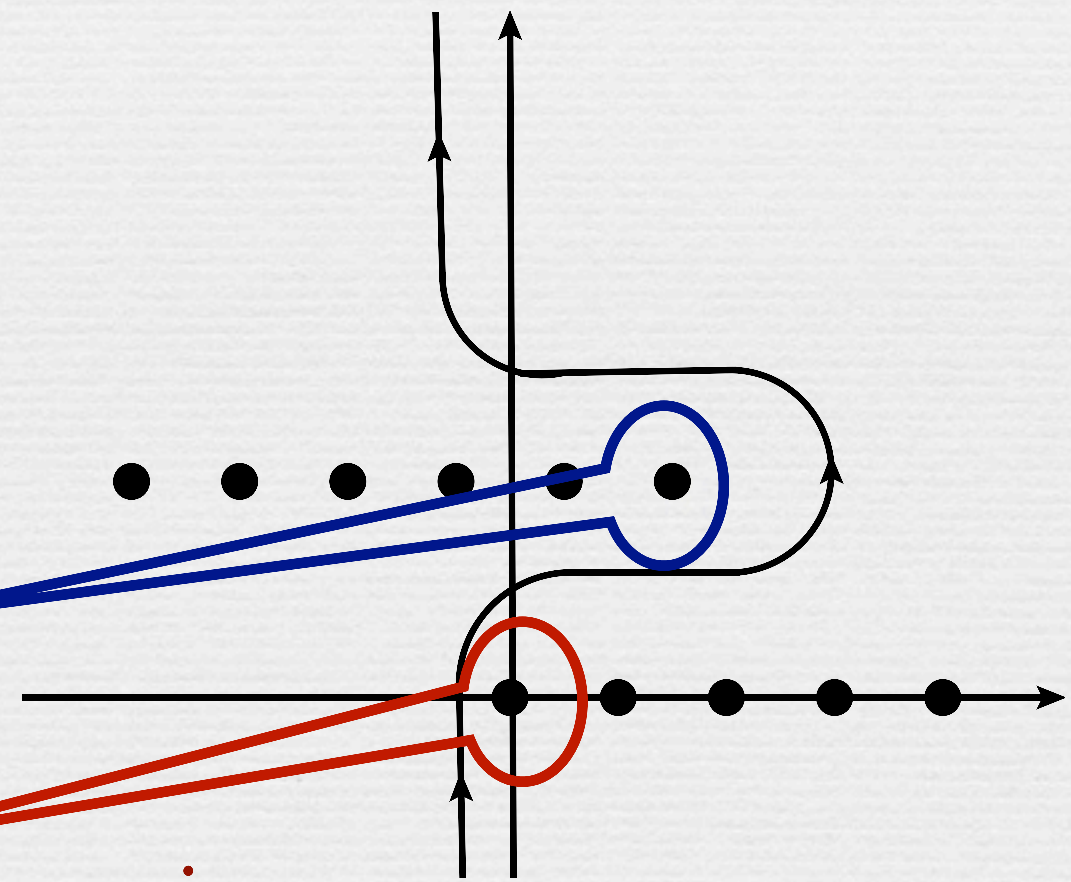


Leading term in the expansion in limit 2

Leading term in the expansion in limit1

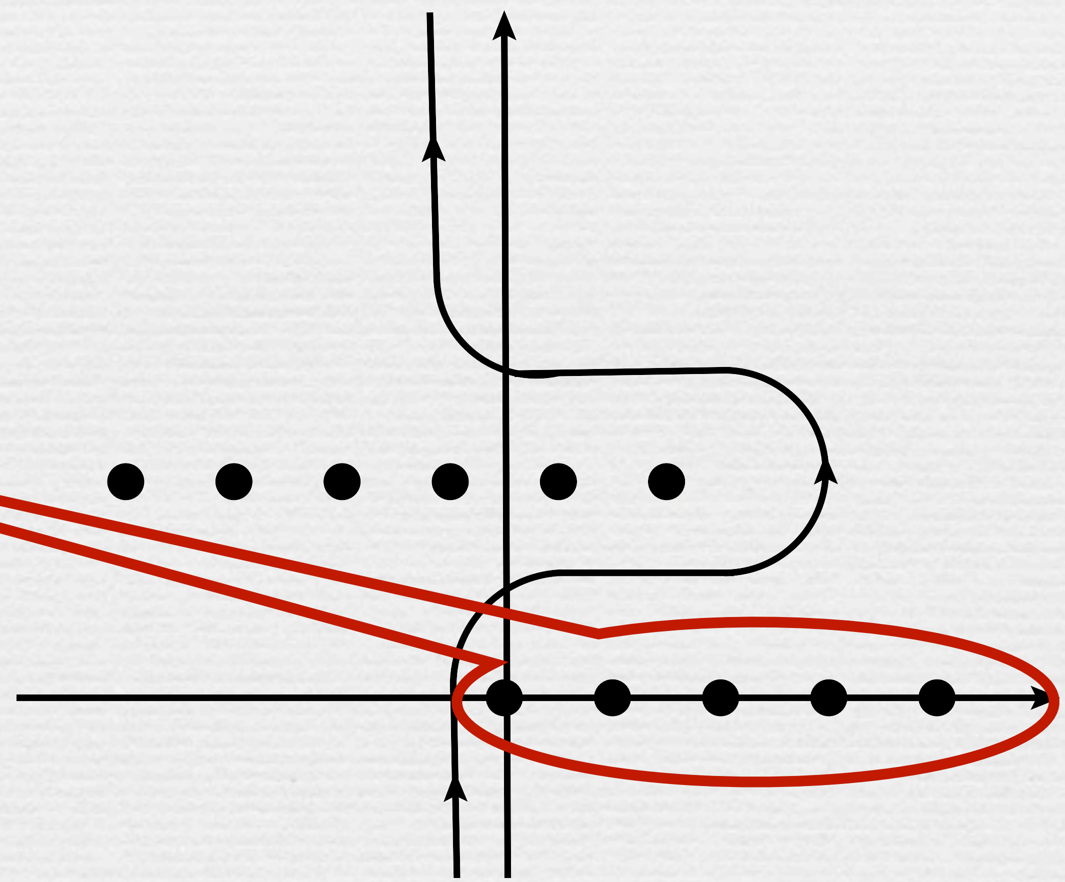
A Recipe to compute Wilson loops

- Step 4:

  Sum up the remaining towers of residues:

$$\sum_{n=1}^{\infty} \frac{u_i^n}{n} = -\ln(1-u_i)$$

$$\sum_{n=1}^{\infty} \frac{u_i^n}{n^k} = \mathrm{Li}_k(u_i)$$



The six-point remainder function

- We applied this recipe to the two-loop six-edged Wilson loop.
- In the limit, all integrals are
  - ➡ at most three-fold.
  - ➡ dependent on conformal cross ratios only.

$$
-\frac{1}{4}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}z_1}{2\pi i}\frac{\mathrm{d}z_2}{2\pi i}\frac{\mathrm{d}z_3}{2\pi i}\left(z_1\,z_2+z_2\,z_3+z_3\,z_1\right)u_1^{z_1}\,u_2^{z_2}\,u_3^{z_3}
$$
$$
\times\,\Gamma\left(-z_1\right)^2\,\Gamma\left(-z_2\right)^2\,\Gamma\left(-z_3\right)^2\,\Gamma\left(z_1+z_2\right)\,\Gamma\left(z_2+z_3\right)\,\Gamma\left(z_3+z_1\right)\,,
$$

The six-point remainder function

$$-\frac{1}{4}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}z_1}{2\pi i}\frac{\mathrm{d}z_2}{2\pi i}\frac{\mathrm{d}z_3}{2\pi i}\left(z_1\,z_2+z_2\,z_3+z_3\,z_1\right)u_1^{z_1}\,u_2^{z_2}\,u_3^{z_3}$$
$$\times\,\Gamma\left(-z_1\right)^2\,\Gamma\left(-z_2\right)^2\,\Gamma\left(-z_3\right)^2\,\Gamma\left(z_1+z_2\right)\,\Gamma\left(z_2+z_3\right)\,\Gamma\left(z_3+z_1\right)\,,$$

- We still need to compute this threefold integral.
- Naive solution: Close the contours and sum up the residues.

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\binom{n_1+n_2}{n_1}\binom{n_2+n_3}{n_2}\binom{n_3+n_1}{n_3}u_1^{n_1}\,u_2^{n_2}\,u_3^{n_3}\;\times\text{(harmonic numbers)}$$

- Sums of this type are in general unknown. They are related to the *Srivastava HB function.*

$$H_B(a,b,c;d,e,f;x,y,z)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(a)_{n_1+n_2}(b)_{n_2+n_3}(c)_{n_3+n_1}}{(d)_{n_1}(e)_{n_2}(f)_{n_3}}\frac{x^{n_1}}{n_1!}\frac{y^{n_2}}{n_2!}\frac{z^{n_3}}{n_3!}$$

The six-point remainder function

$$-\frac{1}{4}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}z_1}{2\pi i}\frac{\mathrm{d}z_2}{2\pi i}\frac{\mathrm{d}z_3}{2\pi i}\left(z_1\,z_2+z_2\,z_3+z_3\,z_1\right)u_1^{z_1}\,u_2^{z_2}\,u_3^{z_3}$$
$$\times\,\Gamma\left(-z_1\right)^2\Gamma\left(-z_2\right)^2\Gamma\left(-z_3\right)^2\Gamma\left(z_1+z_2\right)\Gamma\left(z_2+z_3\right)\Gamma\left(z_3+z_1\right)\,,$$

- We can turn the MB integrals into Euler integrals

$$\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(-z_1)\,\Gamma(c-z_1)\,\Gamma(b+z_1)\,\Gamma(c+z_1)\,X^{z_1}$$
$$=\Gamma(a)\,\Gamma(b+c)\,\int_0^1\mathrm{d}v\,v^{b-1}\,(1-v)^{a+c-1}\,(1-(1-X)v)^{-a}$$

- The hard diagram then becomes

$$-\int_0^1\mathrm{d}v_1\int_0^1\mathrm{d}v_2\,v_1^{-1}\,v_2^{-1}\left(\frac{\pi^2}{6}-\mathrm{Li}_2\left(1-v_1v_2u_3\right)\right)$$
$$\times\left(-u_2v_2+v_1\left(u_1(v_2-1)+(u_2-1)v_2+1\right)+v_2-1\right)^{-1}$$

Multiple polylogarithms

- The result is completely expressed in terms Goncharov's multiple polylogarithm,

$$G(\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a}\, G(\vec{w}';t) \qquad \Big| \qquad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(t)$$

- Multiple polylogarithms form both a shuffle and a quasi-shuffle algebra, and hence also a Hopf algebra.

- Numerical evaluation is very fast and easy (Mathematica or GiNaC).

Multiple polylogarithms

- In some cases, Goncharov's polylogarithm can be reduced to simpler functions

$$G(a,b;z) = \mathrm{Li}_2\left(\frac{b-z}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \ln\left(1-\frac{z}{b}\right)\ln\left(\frac{z-a}{b-a}\right)$$

$$\begin{aligned}
G\left(0,a^2,a,1;1\right) = &-\frac{1}{6}\pi^2 H\left(0,-1;\frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0,1;\frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1,1;\frac{1}{a}\right) \\
&- 2H\left(0,-1,0,-1;\frac{1}{a}\right) + 4H\left(0,-1,0,1;\frac{1}{a}\right) + 2H\left(0,-1,1,1;\frac{1}{a}\right) \\
&- 4H\left(0,0,-1,-1;\frac{1}{a}\right) + 8H\left(0,0,-1,1;\frac{1}{a}\right) + 18H\left(0,0,0,-1;\frac{1}{a}\right) \\
&- 18H\left(0,0,0,1;\frac{1}{a}\right) + 8H\left(0,0,1,-1;\frac{1}{a}\right) - 8H\left(0,0,1,1;\frac{1}{a}\right) + 2H\left(0,1,-1,1;\frac{1}{a}\right) \\
&+ 6H\left(0,1,0,-1;\frac{1}{a}\right) - 5H\left(0,1,0,1;\frac{1}{a}\right) + 2H\left(0,1,1,-1;\frac{1}{a}\right) - 2H\left(0,1,1,1;\frac{1}{a}\right) \\
&+ 2H\left(1,0,-1,1;\frac{1}{a}\right) + 4H\left(1,0,0,-1;\frac{1}{a}\right) - 2H\left(1,0,0,1;\frac{1}{a}\right) + 2H\left(1,0,1,-1;\frac{1}{a}\right) \\
&- H\left(1,0,1,1;\frac{1}{a}\right) + 2H\left(1,1,0,-1;\frac{1}{a}\right)
\end{aligned}$$

The six-point remainder function

- Some of polylogarithms depend on complicated arguments:

$$u_{jkl}^{(\pm)} = \frac{1 - u_j - u_k + u_l \pm \sqrt{(u_j + u_k - u_l - 1)^2 - 4\,(1 - u_j)\,(1 - u_k)\,u_l}}{2\,(1 - u_j)\,u_l},$$

$$v_{jkl}^{(\pm)} = \frac{u_k - u_l \pm \sqrt{-4u_j u_k u_l + 2u_k u_l + u_k^2 + u_l^2}}{2\,(1 - u_j)\,u_k}.$$

- For some values of the u's, the square roots can become complex.
- They however always come in pairs such that the full result is real.

$$\mathcal{G}(\ldots, u_{ijk}, \ldots; z) = G\left(\ldots, u_{ijk}^{(+)}, \ldots; z\right) + G\left(\ldots, u_{ijk}^{(-)}, \ldots; z\right)$$

The six-point remainder function

$$
\begin{aligned}
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+\\
&\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-
\end{aligned}
$$

The six-point remainder function

- We checked that our result has all the properties required for the remainder function:

  - ✓ the result is of uniform transcendental weight 4.
  - ✓ no new transcendental numbers appear (only $\zeta_2, \zeta_3, \zeta_4$).
  - ✓ explicitly dependent on conformal cross-ratios.
  - ✓ symmetric in all its arguments.
  - ✓ vanishes in all collinear and multi-Regge limits.
  - ✓ we checked numerically several points.

The six-point remainder function

- We can derive easily the asymptotic behavior of the remainder function

  ➡ all cross ratios small (strongly ordered)

$$\lim_{u_1 \ll u_2 \ll u_3 \ll 1} R^{(2)}_{6,WL}(u_1, u_2, u_3)$$
$$= \frac{\pi^2}{24}(\ln u_1 \ln u_2 + \ln u_2 \ln u_3 + \ln u_3 \ln u_1) + \frac{17\pi^2}{1440} + \mathcal{O}(u_i)\,.$$

  ➡ all cross ratios large (strongly ordered)

$$\lim_{u_1 \gg u_2 \gg u_3 \gg 1} R^{(2)}_{6,WL}(u_1, u_2, u_3) =$$
$$-\frac{1}{96}\ln^4 \frac{u_1}{u_2 u_3} - \frac{5}{48}\pi^2 \ln^2 \frac{u_1}{u_2 u_3} - \frac{1}{2}\zeta_3 \ln \frac{u_1}{u_2 u_3} - \frac{157\pi^4}{1440} + \mathcal{O}(1/u_i)\,.$$

Regular hexagons

- Let us concentrate now on $\mathbf{Z}_n$ symmetric regular hexagons,

$$s_{12} = s_{23} = s_{34} = s_{45} = s_{56} = s_{61} \qquad s_{123} = s_{234} = s_{345}$$

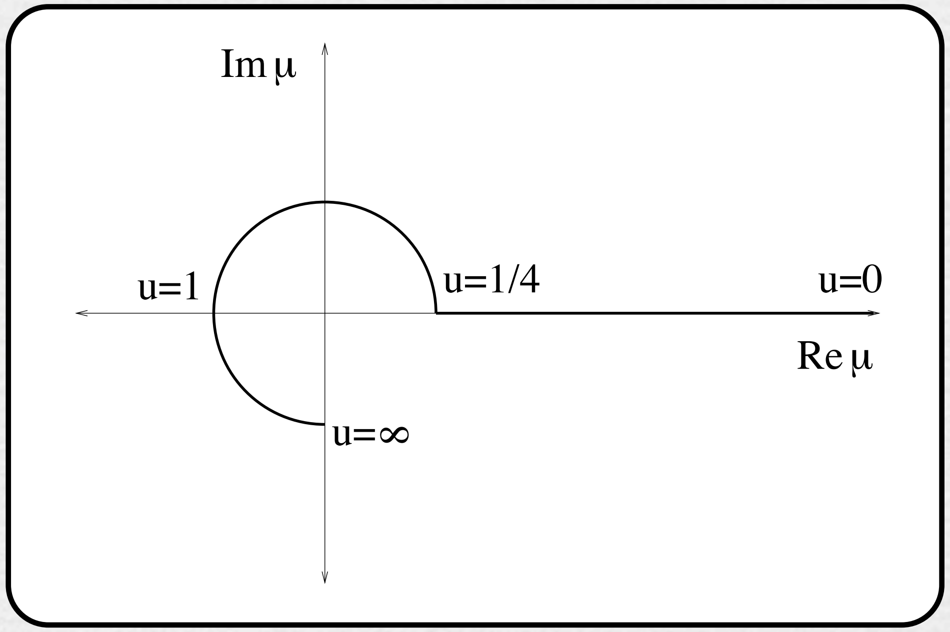
$$u_1 = u_2 = u_3 \quad (= u)$$

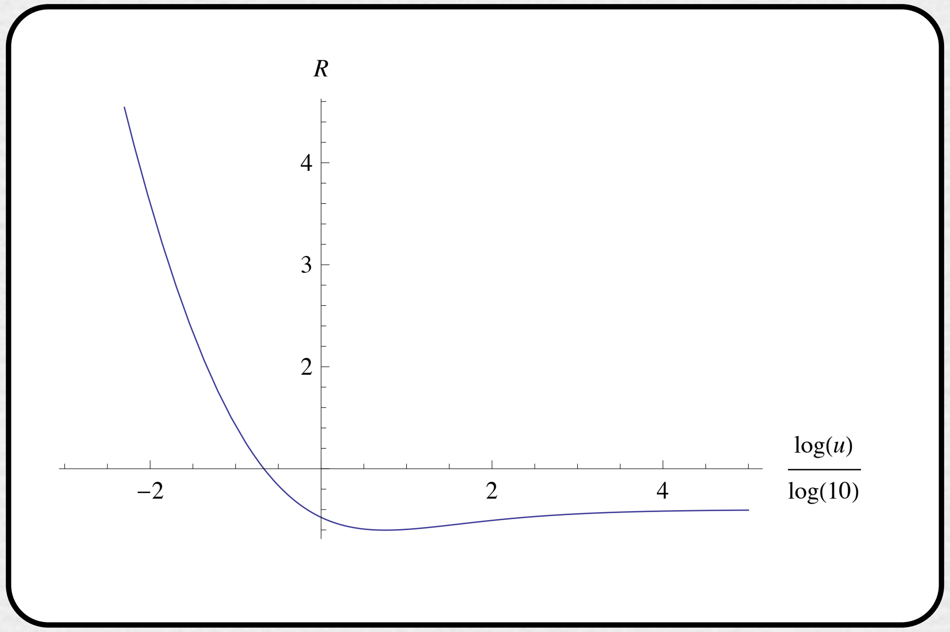
- These hexagons are interesting because
    1. the remainder function is a function of one single variable u.
    2. The corresponding result at strong coupling is also known analytically. [Alday, Maldacena, Gaiotto]

Regular hexagons at strong coupling

$$R(u,u,u) = -\frac{\pi}{6} + \frac{1}{3\pi}\phi^2 + \frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right)$$

$$\mu + \mu^{-1} = 2\cos\phi = \frac{1-3u}{u^{3/2}} \qquad\qquad \mu = e^{i\phi}$$





[Alday, Maldacena, Gaiotto]

Regular hexagons at strong coupling

- The observation is that numerically the strong and weak coupling results look extremely similar, and AGM suggested that the weak coupling result could be written in the form

$$R_{c_1,c_2,c_3} = c_1 \left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2 \right) + c_2 \frac{3}{8} \left(\log^2 u + 2Li_2(1-u) \right) + c_3$$

$$c_1 \approx 12.2, \;\; c_2 \approx 11.4, \;\; c_3 \approx -9.1$$

- Does this hint to some deeper structure..?

Regular hexagons at weak coupling

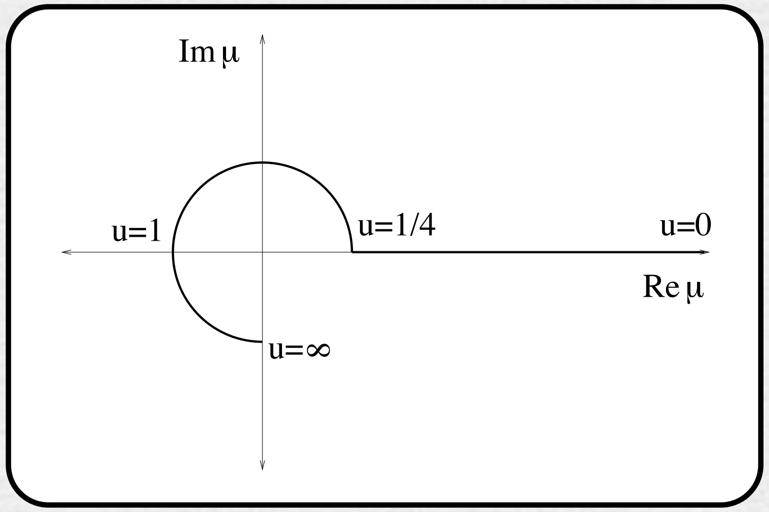
- Putting all u's equal, the square roots reduce to

$$u_{ijk}^{(\pm)} \to \mu^{(\pm)} = \frac{1 \pm \sqrt{1-4u}}{2u}\,, \qquad v_{ijk}^{(\pm)} \to \nu^{(\pm)} = \pm\frac{1}{\sqrt{1-u}}$$

- The answer reduces to ~ 100 terms.
- Two interesting observations:
  - ➡ The point u=1/4 is special.
  - ➡ We find back the arguments we had at strong coupling:

$$(1-u)\,\mu^{(+)} = 1 + \frac{\mu}{x_\epsilon}, \qquad (1-u)\,\mu^{(-)} = 1 + \frac{1}{\mu x_\epsilon}$$

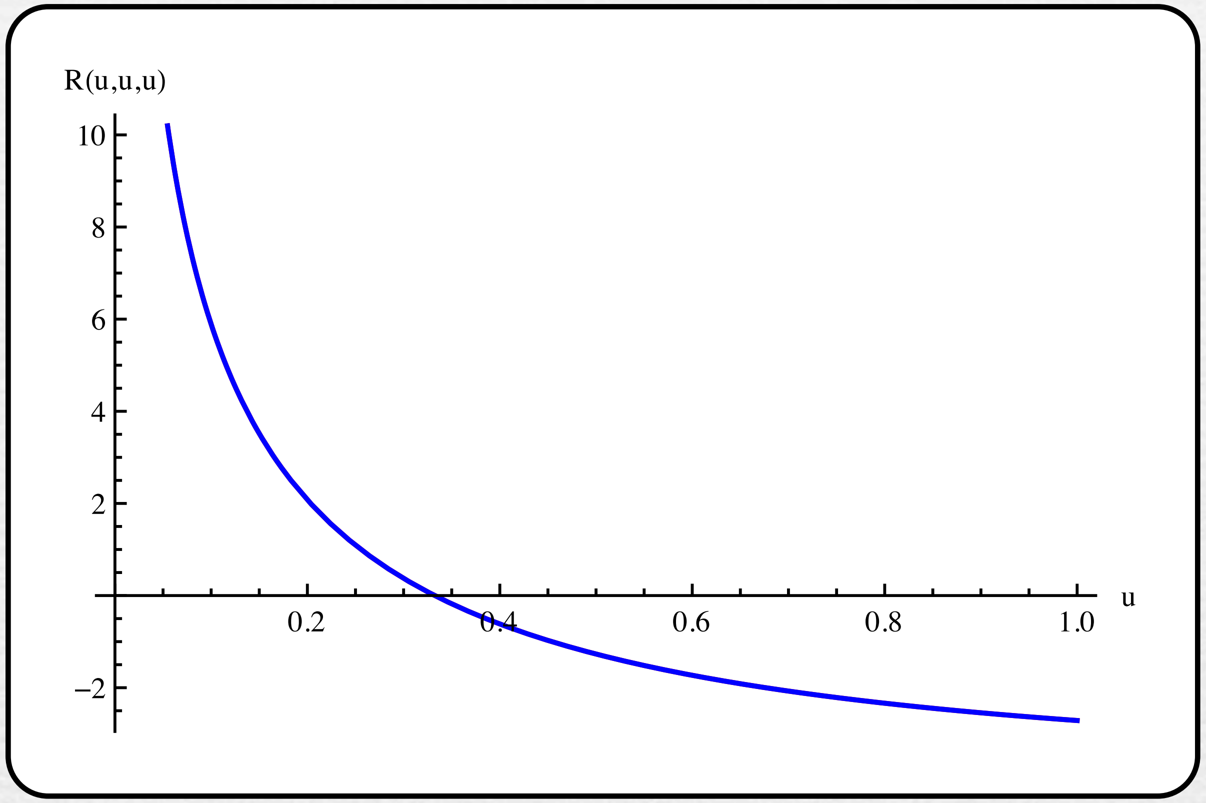
$$x_\epsilon = \mu^{1/3} + \mu^{-1/3}$$



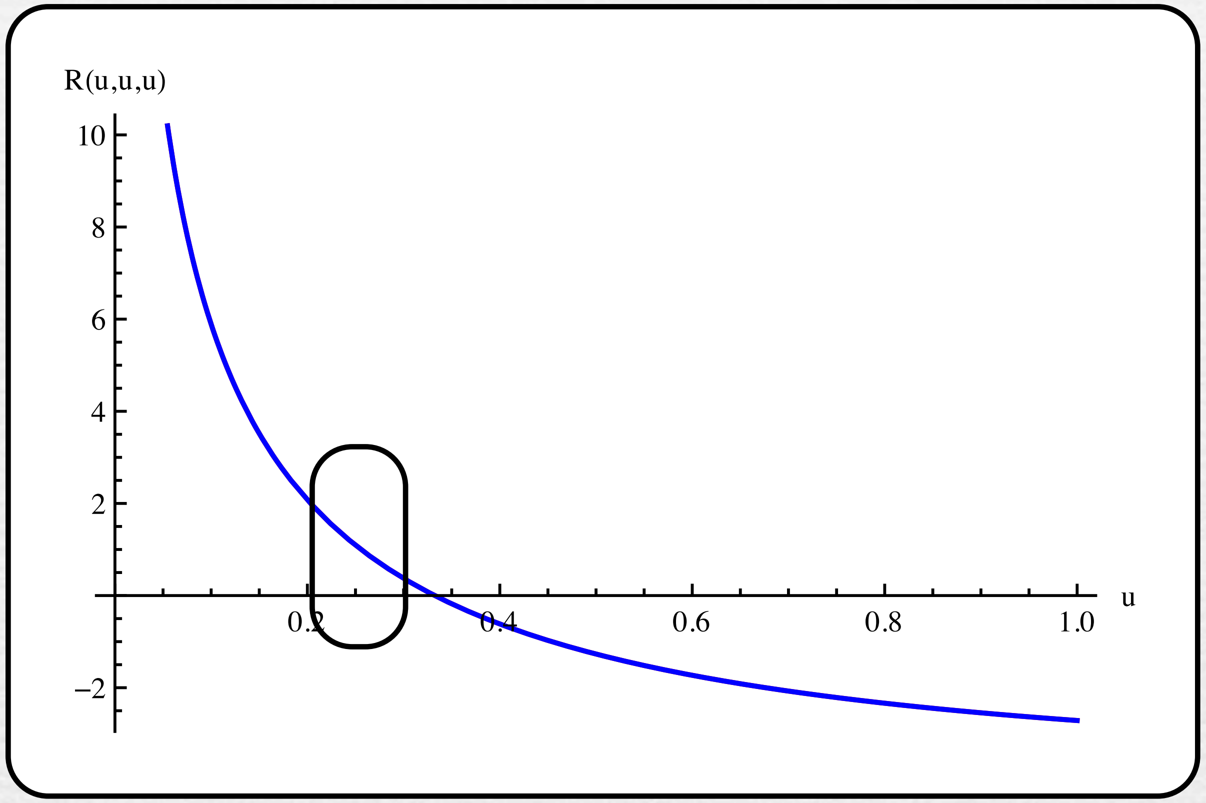
Regular hexagons at weak coupling

$$
\begin{aligned}
&R^{(2)}_{6,WL}(u,u,u) = \\
&\frac{1}{4}\pi^2 G\left(1,\frac{1}{2};u\right)+\frac{1}{8}\pi^2 G\left(\frac{1}{1-u},\frac{u-1}{2u-1};1\right)-3G\left(0,1,0,\frac{1}{2};u\right)-3G\left(0,1,\frac{1}{2},0;u\right)+ \\
&\frac{3}{4}G\left(0,\frac{u-1}{2u-1},0,\frac{1}{1-u};1\right)+\frac{3}{4}G\left(0,\frac{u-1}{2u-1},\frac{1}{1-u},0;1\right)-\frac{3}{4}G\left(0,\frac{u-1}{2u-1},\frac{1}{1-u},1;1\right)+ \\
&\frac{3}{4}G\left(0,\frac{u-1}{2u-1},\frac{1}{1-u},\frac{1}{1-u};1\right)-\frac{3}{4}G\left(0,\frac{u-1}{2u-1},\frac{u-1}{2u-1},\frac{1}{1-u};1\right)-6G\left(1,0,0,\frac{1}{2};u\right)- \\
&3G\left(1,0,\frac{1}{2},0;u\right)+3G\left(1,\frac{1}{2},0,0;u\right)-3G\left(1,\frac{1}{2},1,0;u\right)-\frac{3}{4}G\left(\frac{1}{1-u},1,\frac{1}{u},0;1\right)+ \\
&\frac{3}{2}G\left(\frac{1}{1-u},\frac{1}{1-u},1,\frac{1}{1-u};1\right)+\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},0,1;1\right)- \\
&\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},0,\frac{1}{1-u};1\right)+\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},1,0;1\right)- \\
&\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},\frac{1}{1-u},0;1\right)+\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},\frac{1}{1-u},1;1\right)- \\
&\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},\frac{1}{1-u},\frac{1}{1-u};1\right)-\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},\frac{u-1}{2u-1},1;1\right)+ \\
&\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},\frac{u-1}{2u-1},\frac{1}{1-u};1\right)-\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u},\mu;1\right)-\frac{3}{4}\mathcal{G}\left(0,\mu,0,\frac{1}{1-u};1\right)- \\
&\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{1}{1-u},0;1\right)+\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{1}{1-u},1;1\right)-\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{1}{1-u},\frac{1}{1-u};1\right)- \\
&\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{1}{u},0;1\right)-\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{u-1}{2u-1},1;1\right)+\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{u-1}{2u-1},\frac{1}{1-u};1\right)- \\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,0,1;1\right)+\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,0,\frac{1}{1-u};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,1,0;1\right)+ \\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{1}{1-u},0;1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{1}{1-u},1;1\right)+\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{1}{1-u},\frac{1}{1-u};1\right)+ \\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{1}{u},0;1\right)+\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{u-1}{2u-1},1;1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{u-1}{2u-1},\frac{1}{1-u};1\right)- \\
&\frac{3}{4}G\left(\frac{1}{1-u},1,\frac{1}{u};1\right)H(0;u)+\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},0;1\right)H(0;u)+ \\
&\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},1;1\right)H(0;u)-\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1},\frac{u-1}{2u-1};1\right)H(0;u)- \\
&\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{1}{u};1\right)H(0;u)-\frac{3}{4}\mathcal{G}\left(0,\mu,\frac{u-1}{2u-1};1\right)H(0;u)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,0;1\right)H(0;u)- \\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,1;1\right)H(0;u)+\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{1}{u};1\right)H(0;u)+ \\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu,\frac{u-1}{2u-1};1\right)H(0;u)+\frac{3}{2}G\left(\frac{1}{1-u},\frac{u-1}{2u-1};1\right)H(0,0;u)- \\
&\frac{3}{2}\mathcal{G}\left(\frac{1}{1-u},\mu;1\right)H(0,0;u)-\frac{1}{8}\pi^2 H(0,0;u)+3H(0,0;u)H(0,1;(2u))+\frac{1}{4}\pi^2 H(0,1;(2u))+ \\
&\frac{3}{2}H(0,0;u)H\left(0,1;\frac{2u-1}{u-1}\right)-\frac{1}{8}\pi^2 H\left(0,1;\frac{2u-1}{u-1}\right)+\frac{3}{4}G\left(\frac{1}{1-u},\frac{u-1}{2u-1};1\right)H(1,0;u)- \\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u},\mu;1\right)H(1,0;u)+3H(0,1;(2u))H(1,0;u)+\frac{3}{4}H\left(0,1;\frac{2u-1}{u-1}\right)H(1,0;u)+ \\
&\frac{1}{8}\pi^2 H(1,0;u)+\frac{1}{8}\pi^2 H(1,1;u)-6H(0;u)H(0,0,1;(2u))-3H(0;u)H\left(0,0,1;\frac{2u-1}{u-1}\right)- \\
&\frac{3}{4}H(0;u)H\left(1,0,1;\frac{2u-1}{u-1}\right)-\frac{3}{2}H(0,0,0,0;u)+9H(0,0,0,1;(2u))+ \\
&\frac{9}{2}H\left(0,0,0,1;\frac{2u-1}{u-1}\right)+\frac{15}{4}H(0,0,1,0;u)-\frac{9}{2}H(0,1,0,0;u)+\frac{3}{2}H\left(0,1,0,1;\frac{2u-1}{u-1}\right)+ \\
&\frac{3}{2}H(0,1,1,0;u)-\frac{3}{4}H\left(0,1,1,1;\frac{2u-1}{u-1}\right)+3H\left(1,0,0,1;\frac{2u-1}{u-1}\right)+\frac{15}{4}H(1,0,1,0;u)- \\
&3H(1,1,0,0;u)+\frac{3}{4}H\left(1,1,0,1;\frac{2u-1}{u-1}\right)+\frac{3}{2}H(1,1,1,0;u)-\frac{1}{8}\pi^2 H(0;u)\mathcal{H}\left(1;\frac{1}{\mu}\right)- \\
&\frac{3}{2}H(0,0;u)\mathcal{H}\left(0,1;\frac{1}{\mu}\right)-\frac{3}{4}H(1,0;u)\mathcal{H}\left(0,1;\frac{1}{\mu}\right)+\frac{1}{8}\pi^2\mathcal{H}\left(0,1;\frac{1}{\mu}\right)+ \\
&3H(0;u)\mathcal{H}\left(0,0,1;\frac{1}{\mu}\right)+\frac{3}{4}H(0;u)\mathcal{H}\left(0,1,1;\frac{1}{\mu}\right)+\frac{3}{4}H(0;u)\mathcal{H}\left(1,0,1;\frac{1}{\mu}\right)- \\
&\frac{9}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{\mu}\right)-\frac{3}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{\mu}\right)-\frac{3}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{\mu}\right)-\frac{3}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{\mu}\right)+ \\
&3H(0;u)\zeta_3+\frac{3}{2}\mathcal{H}\left(1;\frac{1}{\mu}\right)\zeta_3+\frac{89\pi^4}{1440}\,.
\end{aligned}
\tag{5.8}
$$

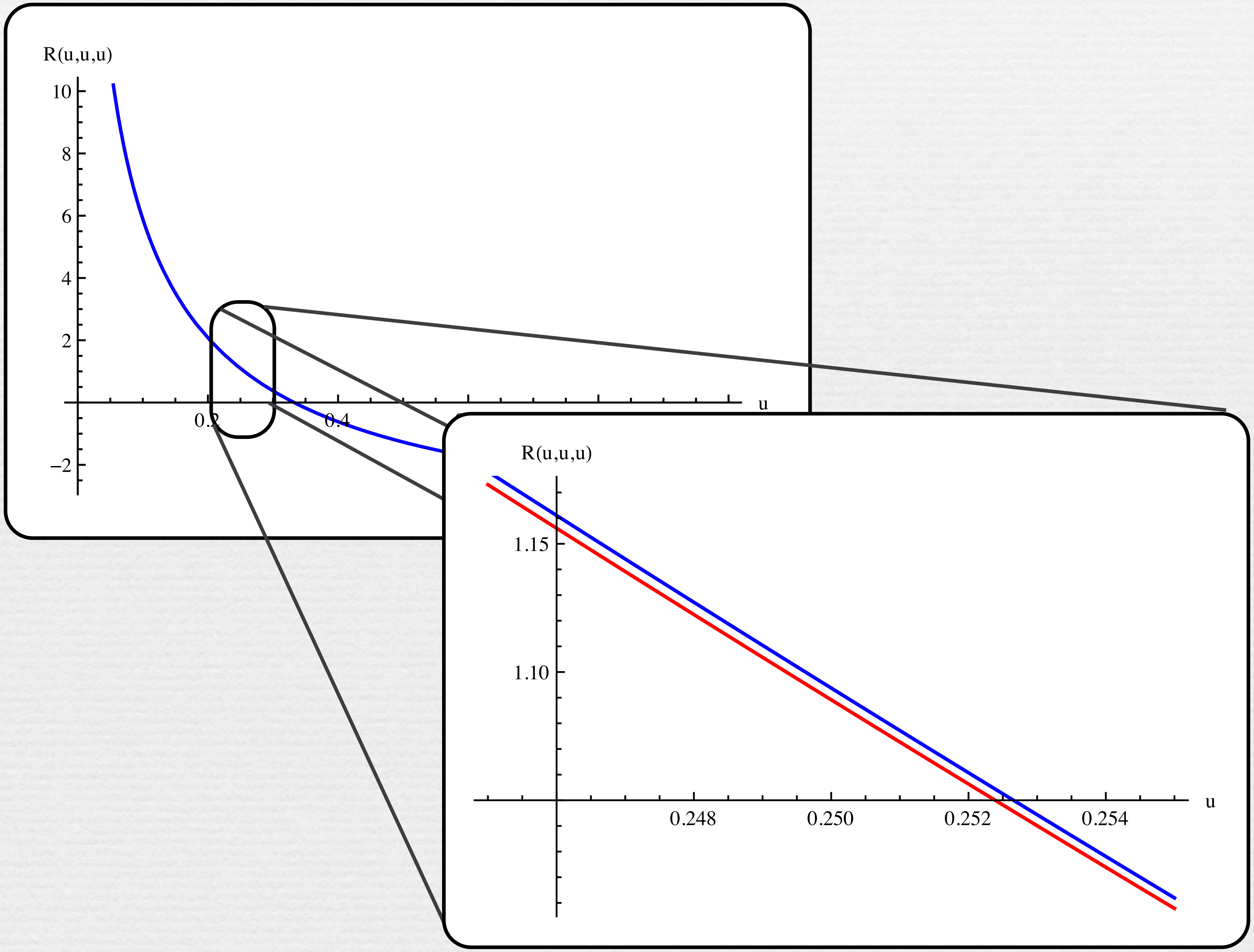
Regular hexagons



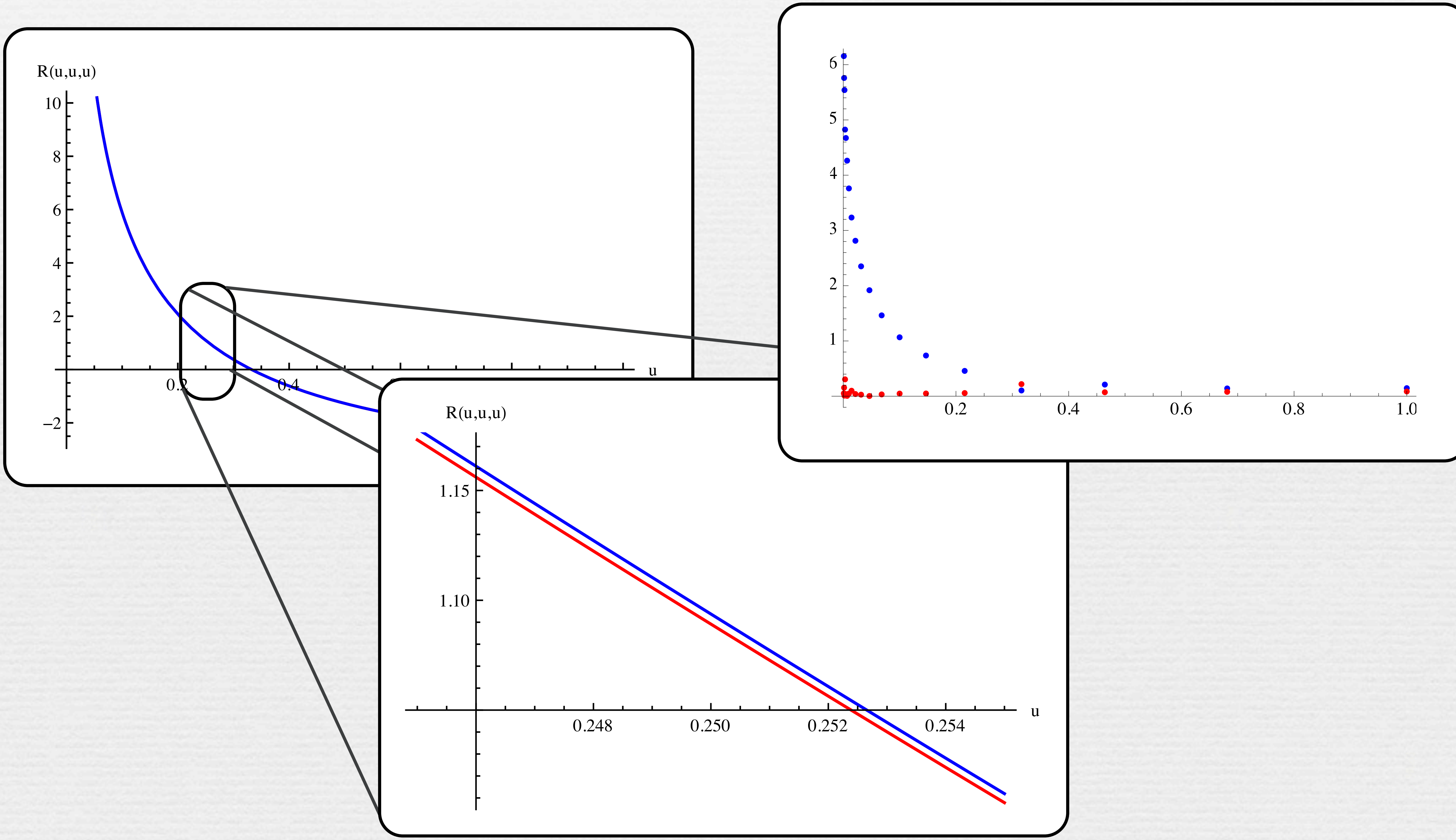
Regular hexagons



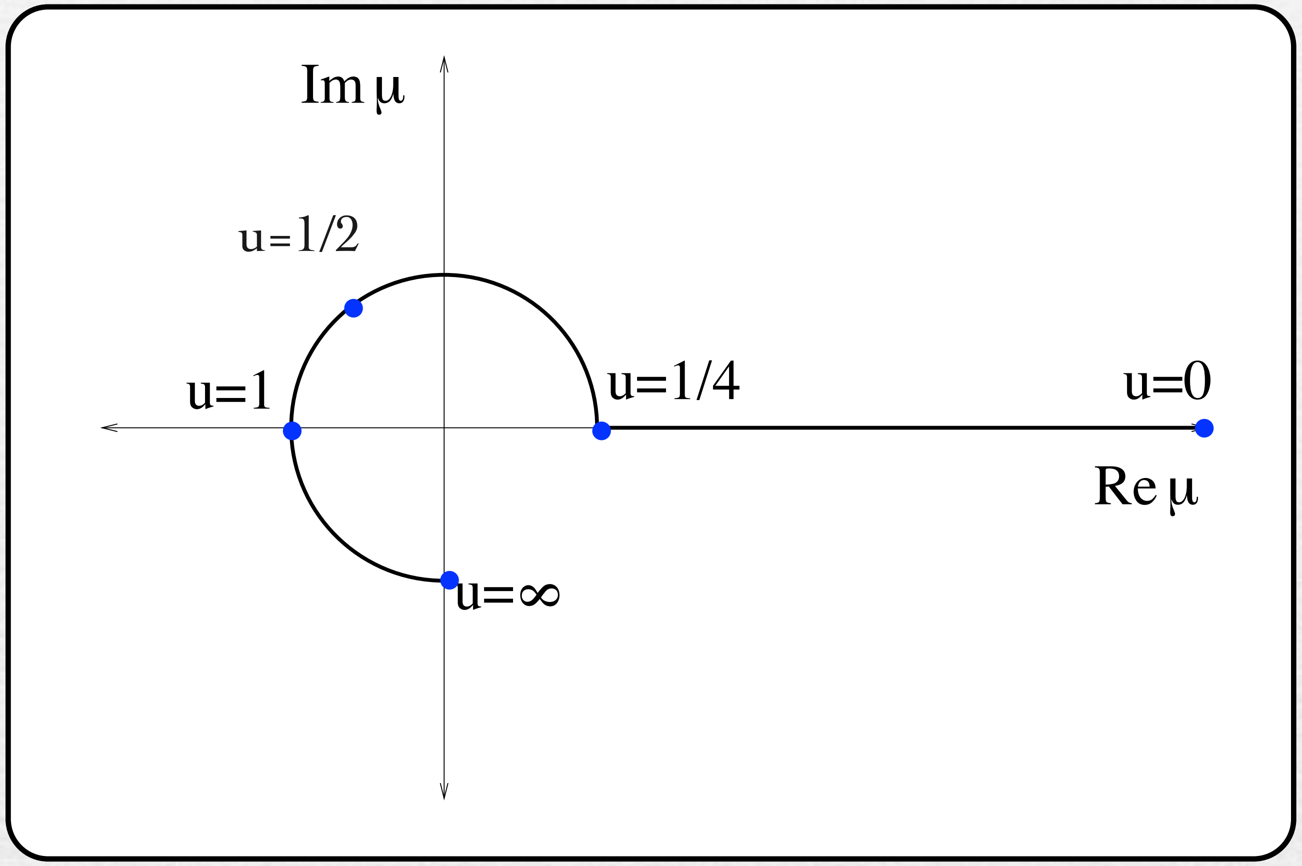
Regular hexagons



Regular hexagons

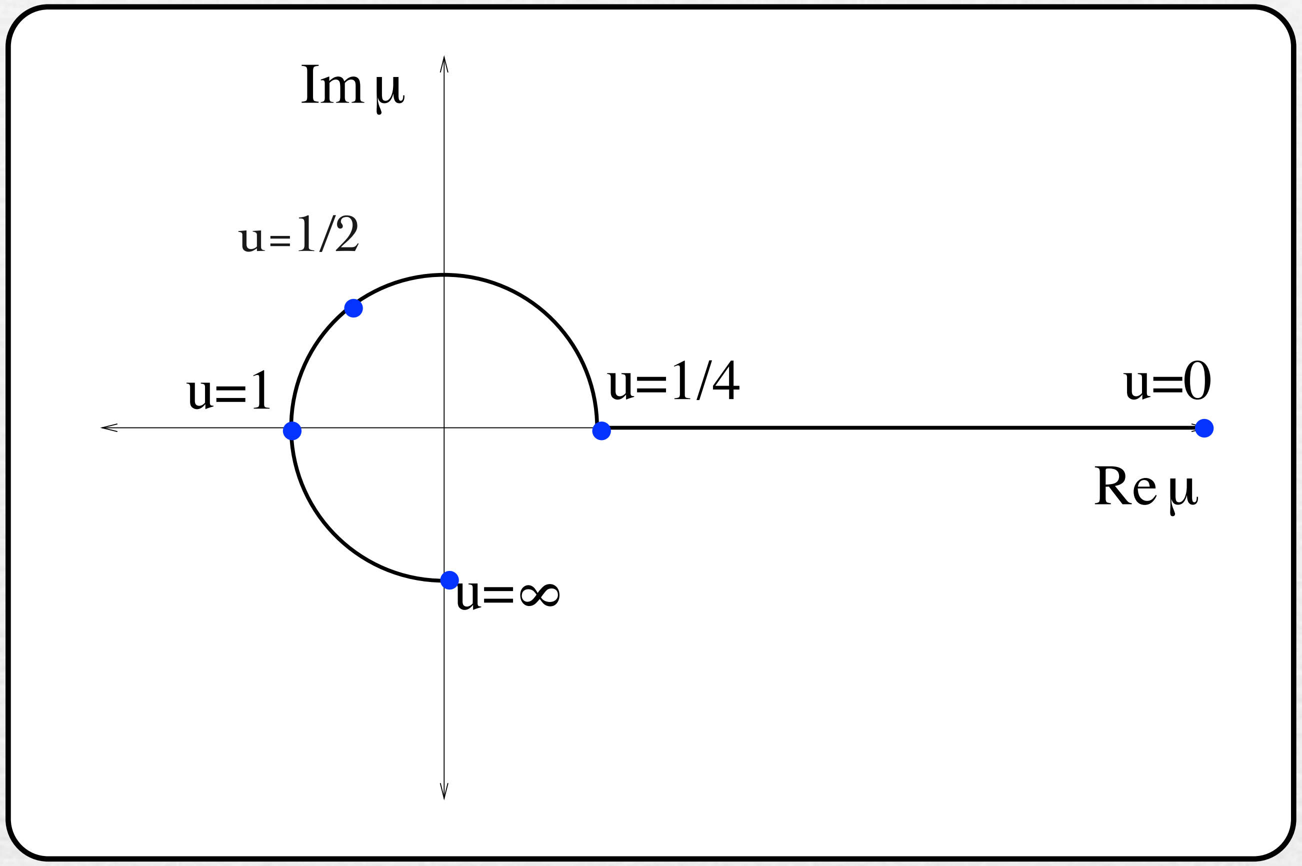


Regular hexagons



$$R_{c_1,c_2,c_3} = c_1\left(-\frac{\pi}{6}+\frac{1}{3\pi}\phi^2\right)+c_2\frac{3}{8}\left(\log^2 u+2Li_2(1-u)\right)+c_3$$

Regular hexagons



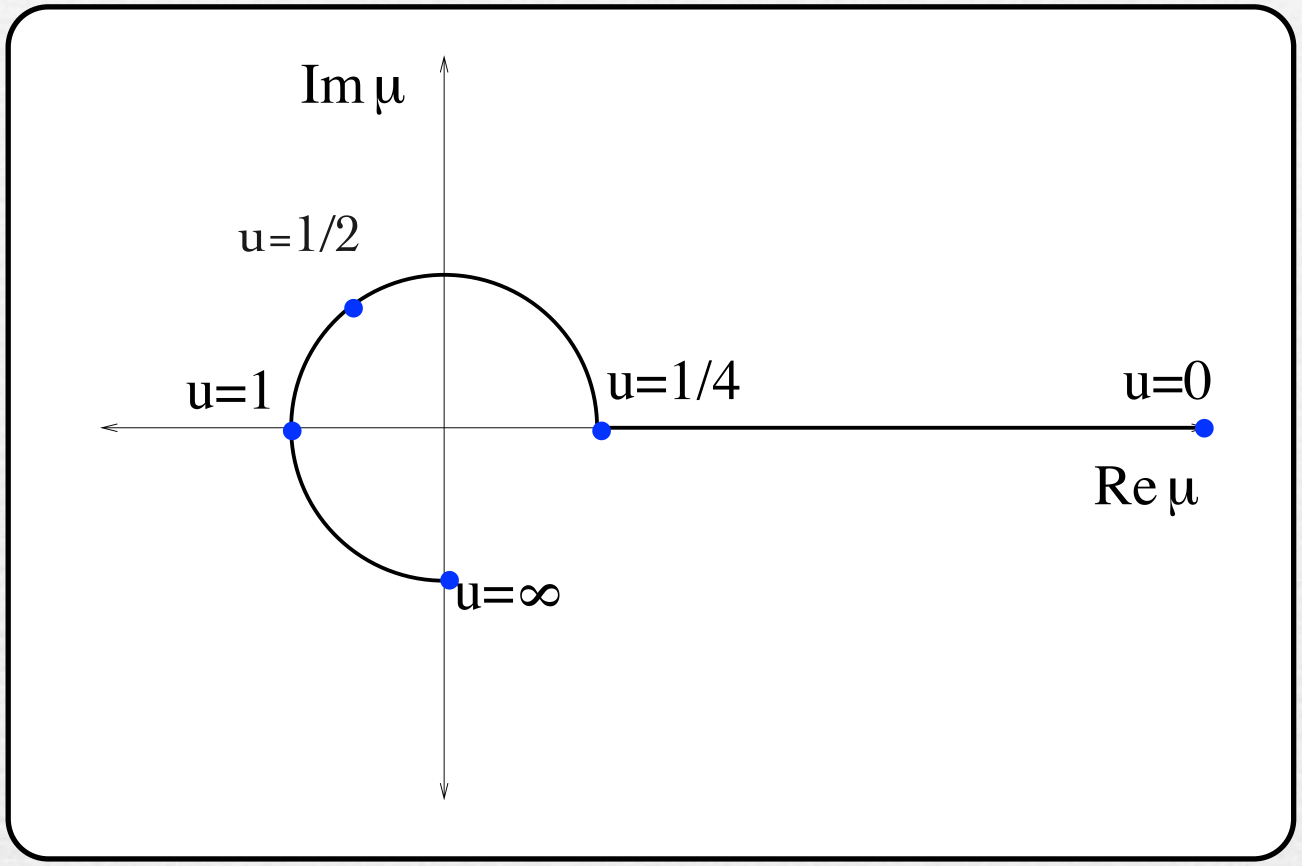
$$\lim_{u\to 0} R^{(2)}_{\mathrm{WL},6}(u,u,u) = \frac{\pi^2}{8}\ln^2 u + \frac{17\pi^4}{1440}$$

$$R^{(2)}_{\mathrm{WL},6}(1,1,1) = -\frac{\pi^4}{36}$$

$$\lim_{u\to \infty} R^{(2)}_{\mathrm{WL},6}(u,u,u) = -\frac{\pi^4}{144}$$

$$R_{c_1,c_2,c_3} = c_1\left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2\right) + c_2\frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right) + c_3$$

Regular hexagons



$$\lim_{u\to 0} R^{(2)}_{\mathrm{WL},6}(u,u,u) = \frac{\pi^2}{8}\ln^2 u + \frac{17\pi^4}{1440}$$

$$R^{(2)}_{\mathrm{WL},6}(1,1,1) = -\frac{\pi^4}{36}$$

$$\lim_{u\to \infty} R^{(2)}_{\mathrm{WL},6}(u,u,u) = -\frac{\pi^4}{144}$$

Pure zeta values

$$R_{c_1,c_2,c_3} = c_1\left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2\right) + c_2\frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right) + c_3$$

Regular hexagons

$$R_{c_1,c_2,c_3} = c_1\left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2\right) + c_2\frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right) + c_3$$

$$R^{(2)}_{6,WL}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = -\frac{105}{64}\zeta_3 \ln 2 - \frac{5}{64}\ln^4 2 + \frac{5}{64}\pi^2\ln^2 2 - \frac{15}{8}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{17\pi^4}{2304}$$

Regular hexagons

$$R_{c_1,c_2,c_3} = c_1\left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2\right) + c_2\frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right) + c_3$$

Pure weight 4, non zeta valued.

$$R^{(2)}_{6,WL}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = -\frac{105}{64}\zeta_3 \ln 2 - \frac{5}{64}\boxed{\ln^4 2} + \frac{5}{64}\pi^2\ln^2 2 - \frac{15}{8}\boxed{\mathrm{Li}_4\left(\frac{1}{2}\right)} + \frac{17\pi^4}{2304}$$

Regular hexagons

$$R_{c_1,c_2,c_3} = c_1\left(-\frac{\pi}{6}+\frac{1}{3\pi}\phi^2\right) + c_2\frac{3}{8}\left(\log^2 u + 2Li_2(1-u)\right) + c_3$$

$$\begin{aligned}
R^{(2)}_{6,WL}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) &= 3\mathrm{Li}_2\left(\frac{1}{3}\right)\ln^2 2 - \frac{9}{2}\mathrm{Li}_2\left(\frac{1}{3}\right)\ln^2 3 - \frac{567}{4}\mathrm{Li}_3\left(\frac{1}{3}\right)\ln 2 \\
&+ \frac{543}{4}\mathrm{Li}_3\left(-\frac{1}{2}\right)\ln 2 + \frac{567}{8}\mathrm{Li}_3\left(\frac{1}{3}\right)\ln 3 - \frac{567}{4}\mathrm{Li}_3\left(-\frac{1}{2}\right)\ln 3 + \frac{1323}{16}\zeta_3\ln 2 \\
&+ \frac{945}{32}\zeta_3\ln 3 - \frac{39}{32}\ln^4 2 - \frac{257}{64}\ln^4 3 + \frac{173}{8}\ln 3\ln^3 2 + \frac{189}{8}\ln^3 3\ln 2 - \frac{543}{16}\ln^2 3\ln^2 2 \\
&- \frac{63}{16}\pi^2\ln^2 2 - \frac{181}{64}\pi^2\ln^2 3 + \frac{189}{2}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1701}{8}\mathrm{Li}_4\left(\frac{1}{3}\right) - \frac{543}{16}\mathrm{Li}_4\left(-\frac{1}{3}\right) \\
&+ \frac{555}{2}\mathrm{Li}_4\left(-\frac{1}{2}\right) - \frac{9}{2}\mathrm{Li}_2\left(\frac{1}{3}\right)^2 - \frac{567}{16}S_{2,2}\left(-\frac{1}{3}\right) - \frac{567}{4}S_{2,2}\left(-\frac{1}{2}\right) - \frac{2123\pi^4}{2880}
\end{aligned}$$

Conclusion

- Many new interesting mathematical structures appear beyond one-loop:
  - ➡ A deeper understanding of these new functions and the relations among them might be required.

- Regge exactness of the (logarithm of the) Wilson loop in N=4 SYM provides a powerful tool for the analytic computation of Wilson loops.
- We applied this technique to the hexagon, and extracted in this way an analytic result for the two-loop six-point remainder function.