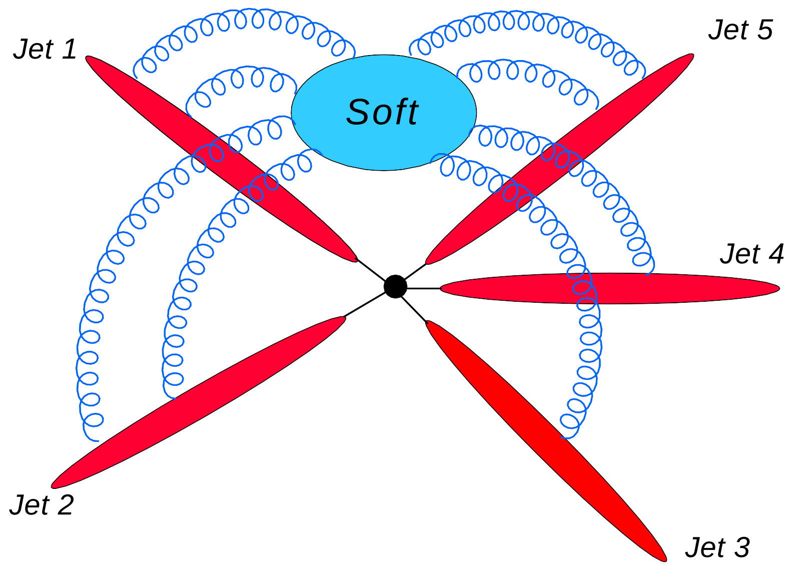
Amplitudes 2010

The infrared singularity structure of multi-leg gauge-theory amplitudes

Einan Gardi (Edinburgh)



In collaboration with Lorenzo Magnea and Lance J. Dixon

Prelude: the sum-over-dipoles formula

It is possible that all IR singularities in any massless gauge-theory amplitude are summarized by

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s, \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda, \alpha_s(\lambda^2, \epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s\right)$$

$$\Gamma(\lambda, \alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{s_{ij}}{\lambda^2}\right)\mathrm{T}_i\cdot\mathrm{T}_j - \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

A remarkably simple structure!

T. Becher & M. Neubert

E.G. & L. Magnea

IR singularities are highly constrained.

A window into the all-order structure of gauge theory amplitudes.

Plan of the talk

- The sum-over-dipoles formula — how does it arise?
  E.G. & L. Magnea, *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes*, [arXiv:0901.1091] JHEP 0903:079, 2009.
  - factorization of fixed-angle scattering amplitude
  - rescaling symmetry and the cusp anomaly
  - factorization constraints
- Beyond the sum-over-dipoles formula
  L.J. Dixon, E.G. & L. Magnea, *On soft singularities at three loops and beyond*, [arXiv:0910.3653] JHEP 1002:081, 2010.

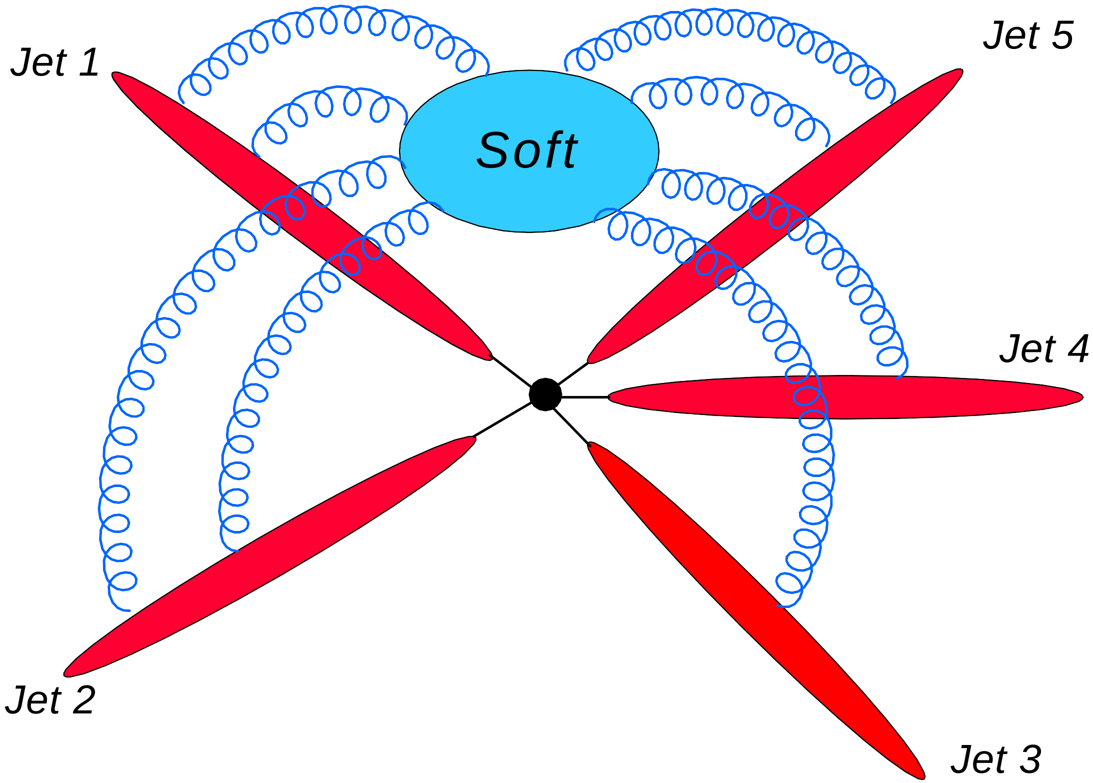
Factorization of a multi-leg amplitude

Fixed–angle scattering amplitude in a massless gauge theory.

We assume:

- All invariants are large: $|p_i \cdot p_j| \gg \Lambda^2_{\rm QCD}$
- All hard partons are massless $p_i^2 = 0$

Mueller (81)
Sen (83)
Botts Sterman (89)
Kidonakis Oderda Sterman (98)
Catani (98)
Tejeda-Yeomans Sterman (02)
Kosower (03)
Aybat Dixon Sterman (06)
Becher Neubert (09)
E.G. Magnea (09)
Dixon E.G. Magnea (09)



Factorization of a multi-leg amplitude

Established results:

- Amplitudes factorise into Soft–Jet–Hard modes.
- All singularities are in Soft and Jet; they all exponentiate.

Recent progress: understanding the all–order structure of the exponent.

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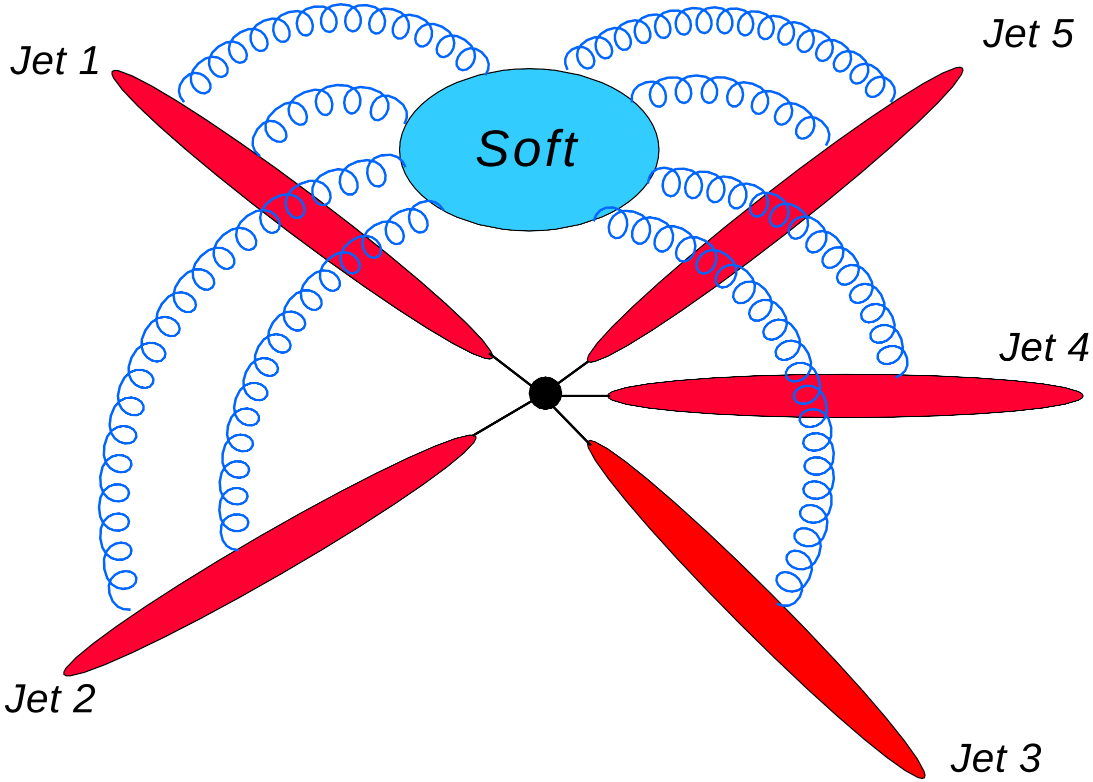
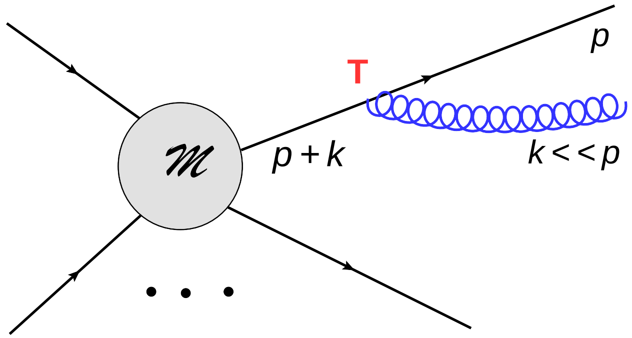


Eikonal approximation and rescaling invariance

Eikonal Feynman rules



gluon emission in the limit $k \to 0$:

$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{\mathrm{i}(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \longrightarrow \bar{u}(p)g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

- Valid when all momentum components of k are small (not valid when k is collinear to p but hard)
- Only the direction and the colour charge of the emitter are important. Rescaling invariance: $\beta \propto p$

$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon} = g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+\mathrm{i}\varepsilon}$$

- Equivalent to radiation off a Wilson line along the quark trajectory:

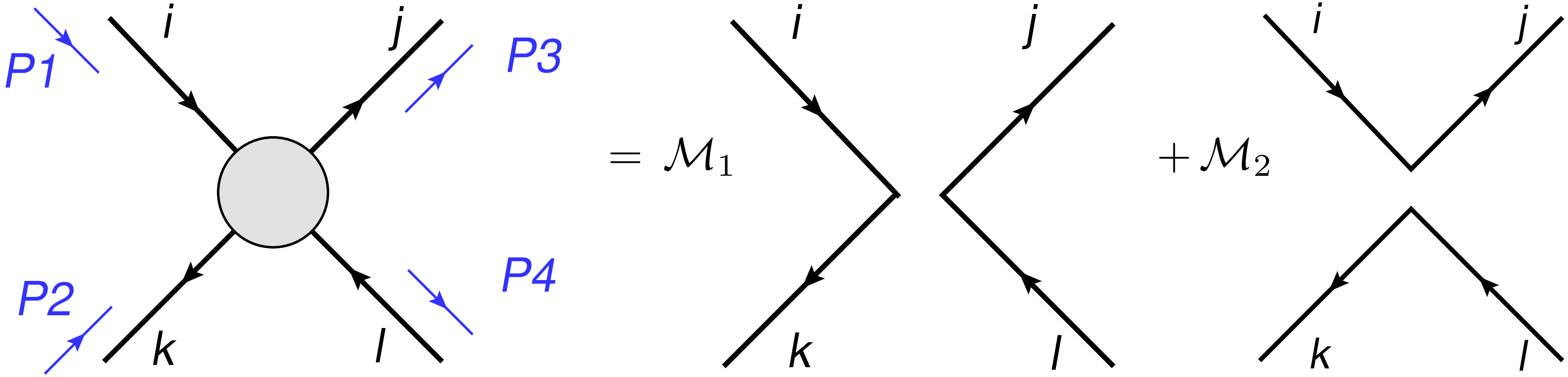
$$P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$$

Colour flow

Decompose the amplitude in a colour basis (independent colour tensors with the index structure of the external partons):

Example:

$q(p_1)\bar{q}(p_2) \to q(p_3)\bar{q}(p_4)$ $\qquad$ $c_1 = \delta_{ik}\delta_{jl}$ $\qquad$ $c_2 = \delta_{ij}\delta_{kl}$



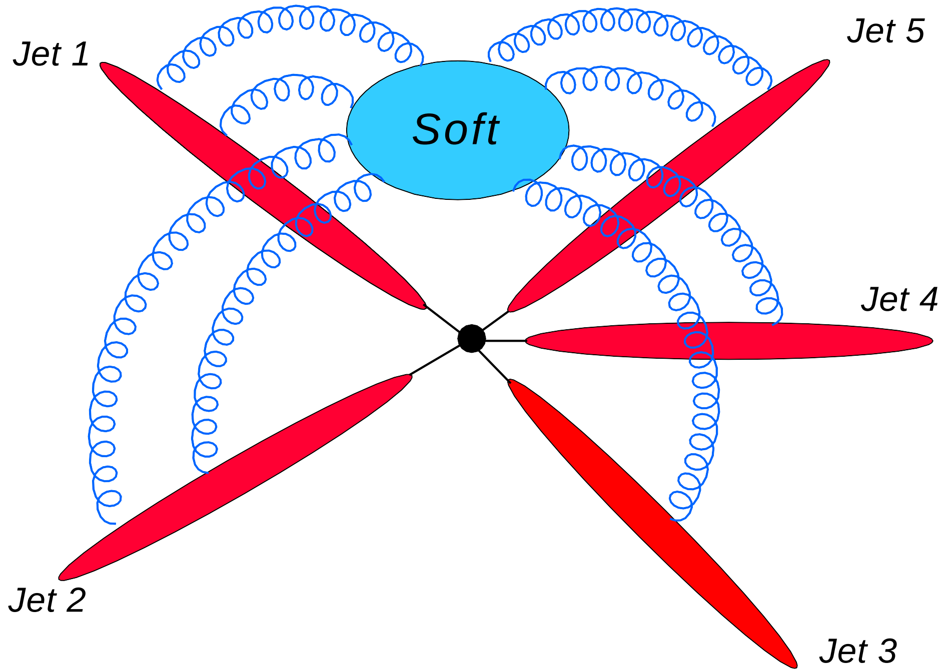
In general:

$$\mathcal{M}_{\{\alpha_i\}}\left(p_i/\mu, \epsilon\right) = \sum_{L=1}^{n_{\rm rep}} \mathcal{M}_L\left(p_i/\mu, \epsilon\right)\,\left(c_L\right)_{\{\alpha_i\}}$$

$n_{\rm rep}$ is the number of elements in the basis (number of irreducible representations that can be constructed with the given external particles).

Factorization of a multi-leg amplitude

- All singularities are in $\mathcal{S}$, $J_i/\mathcal{J}_i$.
- colour:
  $\mathcal{S}$ is a matrix acting on H
- kinematics:
  $\mathcal{S}$ depends on all velocities;
  $J_i/\mathcal{J}_i$ depends on a single p_i

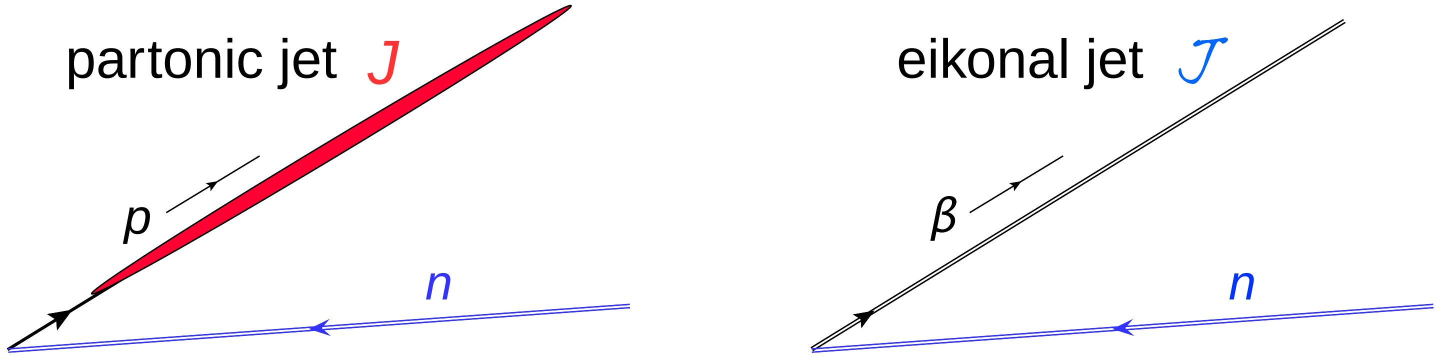


$$\mathcal{M}_N\left(p_i/\mu,\epsilon\right)=\sum_L \mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\epsilon\right)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$
$$\times\prod_{i=1}^{n} J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)\Big/\,\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)$$

To avoid double counting of the soft-collinear region: $\mathcal{J}_i$ removes from J_i its eikonal part, which is already taken into account in $\mathcal{S}$.

The jet function: definition

- Introduce auxiliary vectors n_i ($n_i^2 \neq 0$) to separate collinear regions. Intuitive picture: jet i contains gluons (k) such that: $k \cdot p_i < n_i \cdot p_i$
- Define a gauge–invariant jet using a Wilson line along a ray n_i.

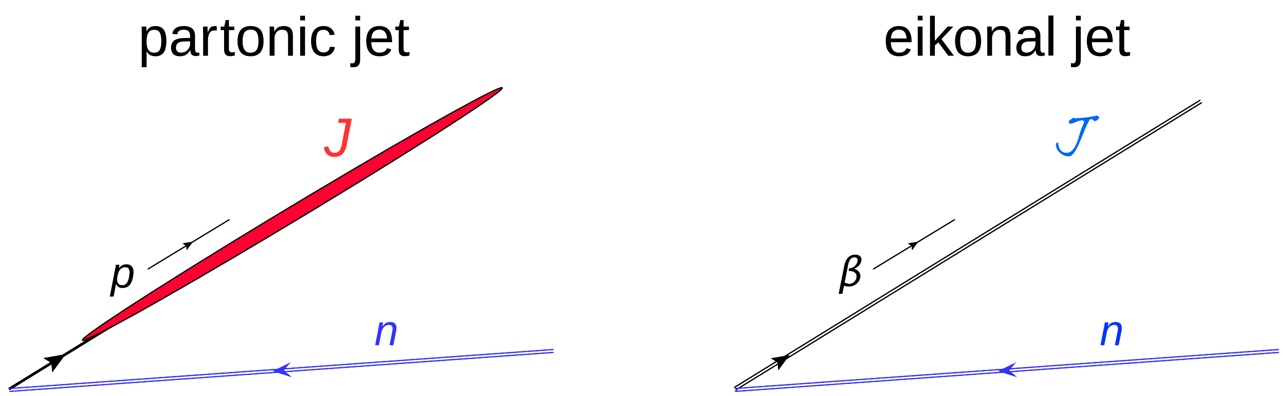


partonic jet: $$\overline{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \epsilon\right) = \langle p\,|\,\overline{\psi}(0)\,\Phi_n(0,-\infty)\,|0\rangle$$

where $$\Phi_n(\lambda_2,\lambda_1) = P\exp\left\{ ig\int_{\lambda_1}^{\lambda_2} d\lambda\, n\cdot A(\lambda n)\right\}$$

eikonal jet: $$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \epsilon\right) = \langle 0|\,\Phi_\beta(\infty,0)\,\Phi_n(0,-\infty)\,|0\rangle$$

Jet functions: evolution equations



partonic jet: $$\overline{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \langle p\,|\,\overline{\psi}(0)\,\Phi_n(0,-\infty)\,|0\rangle$$

eikonal jet: $$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0|\,\Phi_\beta(\infty,0)\,\Phi_n(0,-\infty)\,|0\rangle\,.$$

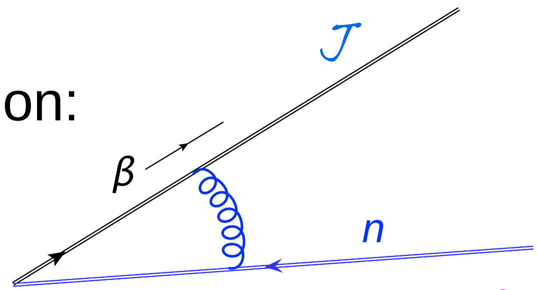
Renormalization $\Longrightarrow$ evolution equations: [based on Korchemsky (89),...]

$$\mu\frac{d}{d\mu}\ln J_i\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) = -\,\gamma_{J_i}(\alpha_s(\mu^2))$$

$$\mu\frac{d}{d\mu}\ln \mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right) \equiv -\gamma_{\mathcal{J}_i} = \frac{1}{2}G_{\mathcal{J}_i}\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s\right) - \underbrace{\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K^{(i)}\left(\alpha_s(\lambda^2,\epsilon)\right)}_{\mathcal{O}(1/\epsilon)}$$

The eikonal jet and the cusp anomaly

- $\mathcal{J}$ doesn't depend on any scale.
  Radiative corrections: only due to renormalization:
  In dimensional regularization: $UV + IR = 0$.

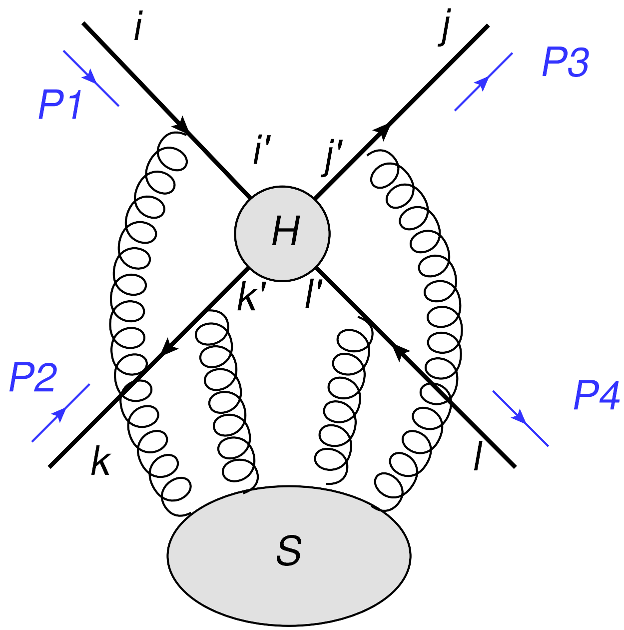


- if $\beta^2 \neq 0$: finite anomalous dimension, rescaling invariance: $\dfrac{(\beta \cdot n)^2}{\beta^2 n^2}$
- if $\beta^2 = 0$: overlapping ultraviolet and collinear singularities $\Longrightarrow$
  - double poles
  - single poles that carry $(\beta \cdot n)^2 / n^2$ dependence, violating classical rescaling symmetry wrt β. This is the cusp anomaly!

$$\mathcal{J}_i\left(\frac{2(\beta \cdot n)^2}{n^2}, \epsilon\right) = \exp\left\{\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[\frac{1}{4}\delta_{\mathcal{J}_i}\left(\alpha_s(\lambda^2, \epsilon)\right) - \frac{1}{8}\gamma_K^{(i)}\left(\alpha_s(\lambda^2, \epsilon)\right) \ln\left(\frac{2(\beta \cdot n)^2 \mu^2}{n^2 \lambda^2}\right)\right]\right\}$$

The double poles as well as the entire kinematic dependence of the simple poles are governed by $\gamma_K^{(i)}$! [EG & Magnea (09)]

The soft function $\mathcal{S}$



Definition:

$$(c_N)_{ijkl}\,\mathcal{S}_{NL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right)\;=$$

$$\sum_{i'j'k'l'}\langle 0|\;\Phi^{k,k'}_{-\beta_2}(0,\infty)\,\Phi^{i,i'}_{\beta_1}(\infty,0)\Phi^{j,j'}_{\beta_3}(0,\infty)\Phi^{l,l'}_{-\beta_4}(\infty,0)\;|0\rangle\;(c_L)_{i'j'k'l'}$$

matrix evolution equation:

$$\mu\frac{d}{d\mu}\mathcal{S}_{JL}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)=-\sum_N\,[\mathbf{\Gamma}_{\mathcal{S}}]_{JN}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)\,\mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)$$

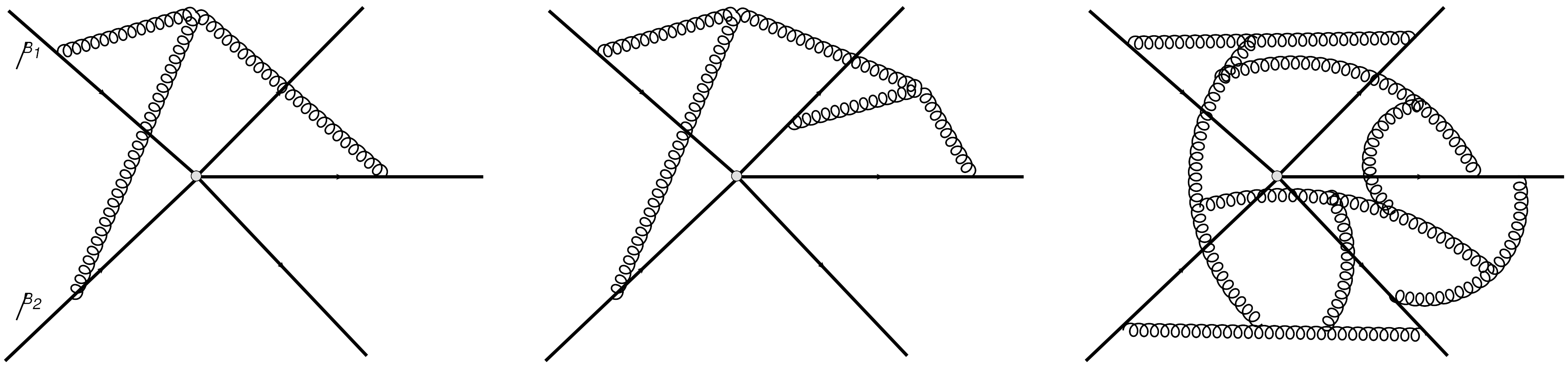
The soft function $\mathcal{S}$

Evolution $\Longrightarrow$ Exponentiation:

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp\left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \mathbf{\Gamma}_{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\lambda^2, \epsilon), \epsilon\right) \right\}$$

$\mathbf{\Gamma}_{\mathcal{S}}$ is a matrix of anomalous dimensions.

- A priori, $\mathbf{\Gamma}_{\mathcal{S}}$ can be very complicated: at each order in α_s it may contain new colour structures and kinematic dependence corresponding to sums of webs:



- In fact $\mathbf{\Gamma}^{\mathcal{S}}$ is (much?) simpler.

The soft anomalous dimension $\mathbf{\Gamma}_{\mathcal{S}}$ at two loops

Remarkable discovery: [Aybat Dixon Sterman (06)]
For any multi-leg amplitude:

$$\mathbf{\Gamma}_{\mathcal{S}}^{(2)} = \frac{K}{2}\mathbf{\Gamma}_{\mathcal{S}}^{(1)}$$

where $\mathbf{\Gamma}_{\mathcal{S}} = \sum_{n=1}^{\infty} \mathbf{\Gamma}_{\mathcal{S}}^{(n)} \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$ and $K = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F N_f$.

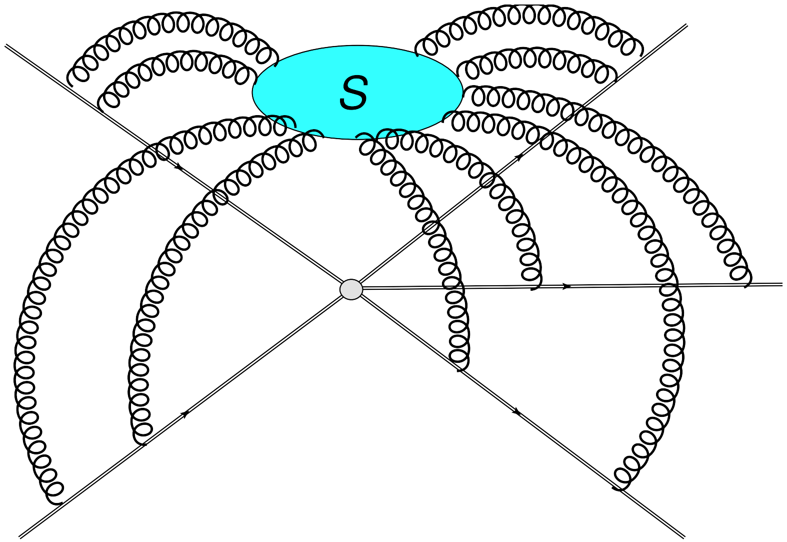
so at two loops: no new colour matrices, no new kinematic dependence...

- why?
- where is K coming from?
  This is the famous coefficient of the cusp anomalous dimension $\gamma_K^{(i)}$
  [Korchemsky Radyushkin (87), Kodaira Trentadue (82),...] :

$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s}{\pi} + K C_i \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$

- very suggestive... – is $\mathbf{\Gamma}_{\mathcal{S}}$ prop. to the 1-loop matrix to all orders?

The soft function $\mathcal{S}$



$$\mu \frac{d}{d\mu} \mathcal{S}_{JL} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = - \sum_N \left[\mathbf{\Gamma}_{\mathcal{S}}\right]_{JN} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \, \mathcal{S}_{NL} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)$$

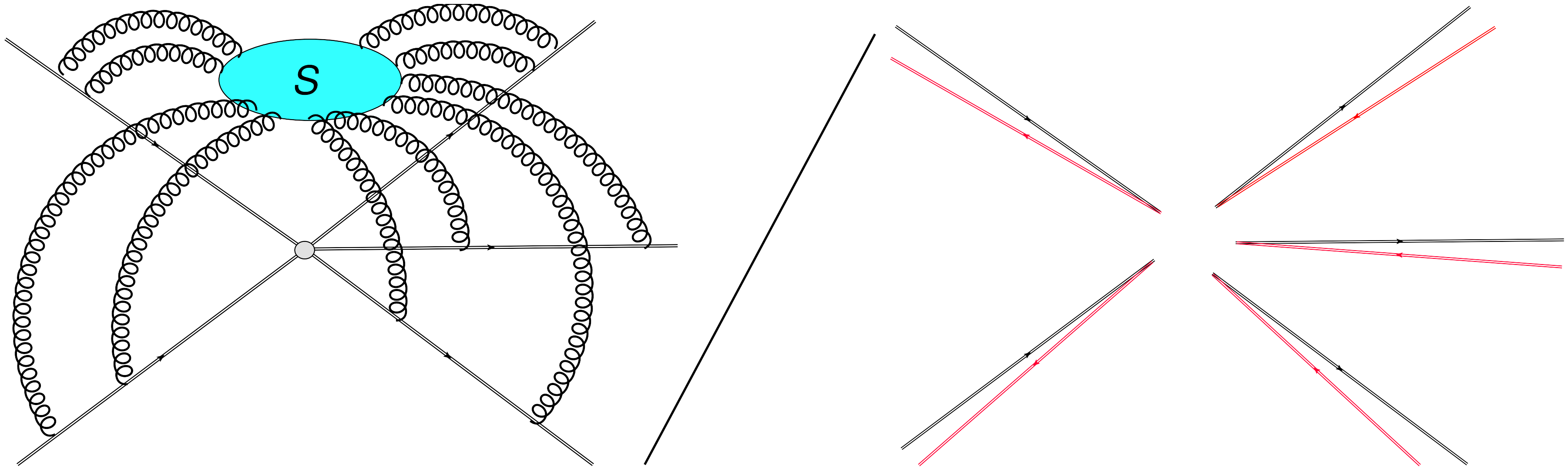
$\mathbf{\Gamma}_{\mathcal{S}}$ has cusp singularities, and therefore, similarly to $\gamma_{\mathcal{J}}$

- it has poles in ϵ ($\mathcal{S}$ itself has double poles).
- it is not invariant with respect to $\beta_i \longrightarrow \kappa_i \beta_i$

Both these issues can be ‘fixed’ by dividing by appropriate eikonal jets...

The reduced soft function $\overline{\mathcal{S}}$

$\overline{\mathcal{S}} \quad \equiv$



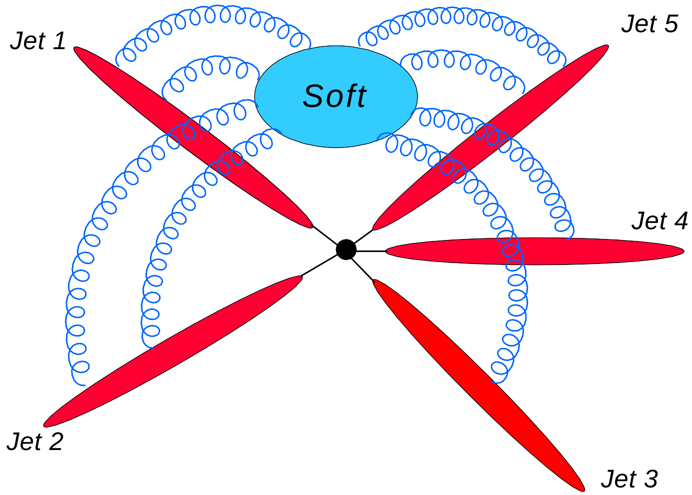
$$\overline{\mathcal{S}}_{JL}\left(\rho_{ij}, \epsilon\right) = \frac{\mathcal{S}_{JL}\left(\beta_i \cdot \beta_j, \epsilon\right)}{\prod_{i=1}^{n} \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)}$$

Having removed the collinear regions, $\overline{\mathcal{S}}$ does not suffer from the cusp anomaly, and must therefore respect rescaling $\beta_i \longrightarrow \kappa_i \beta_i$:

$\Longrightarrow$ $\overline{\mathcal{S}}$ depends only on $\rho_{ij} \equiv \frac{(\beta_i \cdot \beta_j)^2}{\left[2(\beta_i \cdot n_i)^2/n_i^2\right]\left[2(\beta_j \cdot n_j)^2/n_j^2\right]}$

Factorization in terms of the reduced soft function $\overline{\mathcal{S}}$

$$\mathcal{M}_N\left(p_i/\mu,\epsilon\right) =$$



$$=$$

$$= \sum_L \mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\epsilon\right) H_L \prod_{i=1}^{n} \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$$= \sum_L \overline{\mathcal{S}}_{NL}\left(\rho_{ij},\epsilon\right) H_L \prod_{i=1}^{n} J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)$$

- $\overline{\mathcal{S}}$ has only single poles due to large-angle soft gluons.
- $\overline{\mathcal{S}}$, like $\mathcal{M}$, cannot depend on the normalization of the velocities! Enforcing this requirement leads to new all-order constraints on $\Gamma^{\overline{\mathcal{S}}}$.

The equations for $\Gamma^{\overline{\mathcal{S}}}$

Factorization $+$ rescaling invariance imply:
$\Gamma^{\overline{\mathcal{S}}}$ for any multi-leg amplitude, in any colour basis, obeys:

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \left[\mathbf{\Gamma}^{\overline{\mathcal{S}}} \right]_{JN} \left(\rho_{ij}, \alpha_s \right) = \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s \right) \; \delta_{JN} , \qquad \forall i$$

[EG & Magnea (09)]

This is true to all orders (probably also at strong coupling).

- We have related the soft anomalous dimension of a general multi-leg amplitude to the cusp anomalous dimension.
- Intriguing relation between kinematics and colour.

Solving for $\Gamma^{\overline{\mathcal{S}}}$

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \mathbf{\Gamma}^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right) = \frac{1}{4}\, \gamma_K^{(i)}\left(\alpha_s\right) \,, \qquad \forall i$$

Does this set of differential equations have a unique solution?

- For two or three legs - yes! Then $\Gamma^{\overline{\mathcal{S}}}$ can be written in terms of γ_K, with explicitly determined kinematic dependence.
- For four or more legs - no: functions of conformal cross ratios

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left(\frac{\rho_{ij}\,\rho_{kl}}{\rho_{ik}\,\rho_{jl}}\right)^{1/2}$$

satisfy the homogeneous equation.

Yet, it has a simple all-order solution (minimal solution)

The sum-over-dipoles formula

$\gamma_K^{(i)}$ admits quadratic Casimir scaling ($C_i \equiv \mathrm{T}_i \cdot \mathrm{T}_i$) (at least to 3 loops):

$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s}{\pi} + K C_i \left(\frac{\alpha_s}{\pi}\right)^2 + K^{(2)} C_i \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots = C_i \, \widehat{\gamma}_K(\alpha_s) + \underbrace{\widetilde{\gamma}_K^{(i)}(\alpha_s)}_{\text{Higher Casimirs}}$$

The equations:

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma_{\text{dip.}}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \mathrm{T}_i \cdot \mathrm{T}_i \, \widehat{\gamma}_K(\alpha_s) \, , \qquad \forall i$$

are solved by the sum-over-dipoles formula [EG & Magnea (09)]:

$$\Gamma_{\text{dip.}}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = -\frac{1}{8} \widehat{\gamma}_K(\alpha_s) \sum_{i \neq j} \ln(\rho_{ij}) \, \mathrm{T}_i \cdot \mathrm{T}_j \; + \; \frac{1}{2} \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_{i=1}^{n} \mathrm{T}_i \cdot \mathrm{T}_i \, ,$$

- Generalises the two loop result to all orders (minimal solution!)
- Kinematics and colour are directly correlated.

The same formula was simultaneously proposed by Becher & Neubert.

Beyond the minimal solution

Potential corrections to the sum-over-dipoles formula are of two kinds

$$\Gamma^{\overline{\mathcal{S}}} = \Gamma^{\overline{\mathcal{S}}}_{\rm dip.} + \Gamma^{\overline{\mathcal{S}}}_{\textsf{H.C.}} + \Delta^{\overline{\mathcal{S}}}$$

- terms that are induced by higher Casimir contributions to γ_K — they may appear starting at four loops and must satisfy the equations

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\textsf{H.C.}} (\rho_{ij}, \alpha_s) = \frac{1}{4} \widetilde{\gamma}_K^{(i)} (\alpha_s) \,, \qquad \forall i,$$

- solutions of the homogeneous equations:

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Delta^{\overline{\mathcal{S}}} (\rho_{ij}, \alpha_s) = 0 \qquad \forall i$$

namely, functions of conformal cross ratios, $\rho_{ijkl} \equiv \dfrac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$

These may appear starting at three loops.

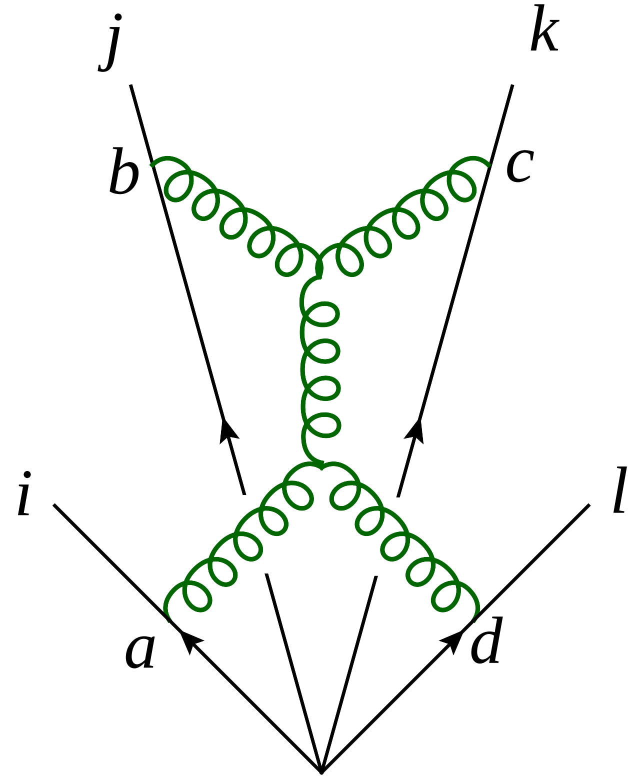
Beyond the minimal solution: 4–parton correlations

On soft singularities at three loops and beyond, Dixon, EG, Magnea

- Factorization & momentum-rescaling constraints require dependence through conformally-invariant cross ratios

$$\Delta^{\overline{\mathcal{S}}}\left(\rho_{ij},\alpha_s\right) = \Delta(\rho_{ijkl})\,, \qquad \rho_{ijkl} \equiv \frac{(\beta_i\cdot\beta_j)(\beta_k\cdot\beta_l)}{(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)}$$

- Δ directly correlate kinematic and colour d.o.f. of four partons.
- Non-Abelian exponentiation points to 4-leg colour-connected webs appearing first at 3-loops
- This is beyond the state-of-the-art... Can we exclude such contributions based on general considerations?



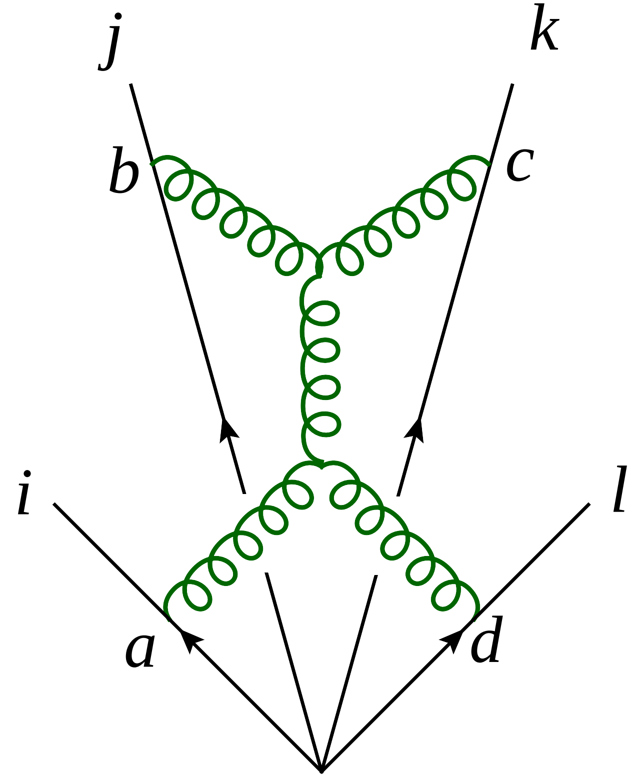
Beyond the minimal solution: 4–parton correlations

The soft function depends only on colour and kinematics.

The simplest structure Δ can have is

$$\Delta_4 = h_{abcd} \mathrm{T}_i^a \mathrm{T}_j^b \mathrm{T}_k^c \mathrm{T}_l^d \; \Delta_4^{\mathrm{kin.}}(\rho_{ijkl})$$

- at 3-loops:
  h_{abcd} is $f_{ade} f_{bce}$ + permutations
- one expect kinematic dependence through $\ln(\rho_{ijkl})$ (or polylogarithms) — not powers of ρ_{ijkl}.



Additional constraints:

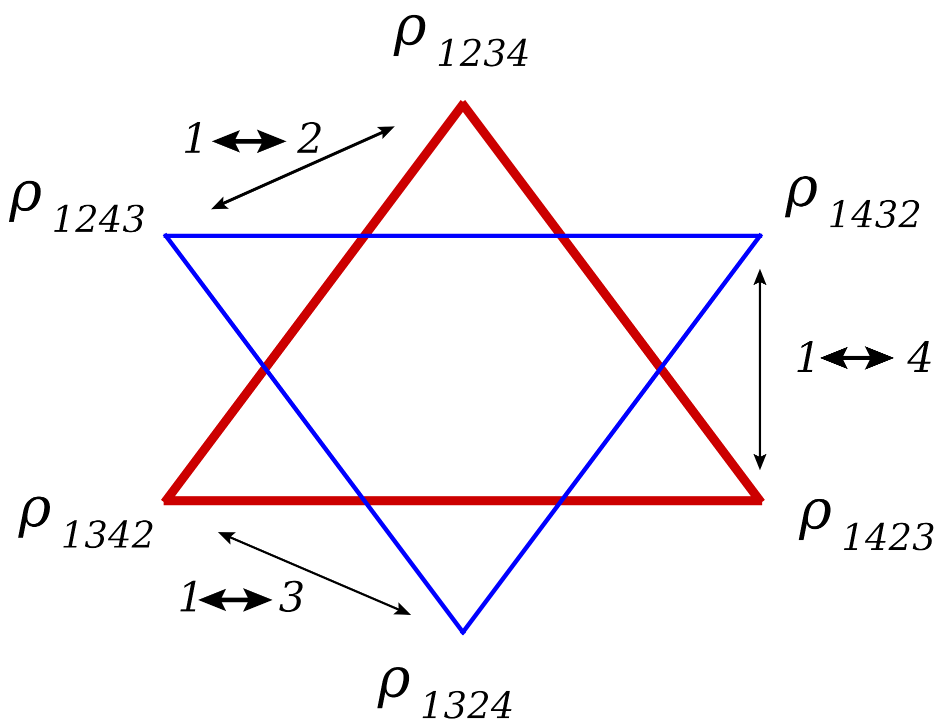
- Bose symmetry: colour and kinematic dependence are correlated
- Collinear limit
- Transcendentality

Identifying independent kinematic variables

For 4 legs there are 24 conformal cross ratios $\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$

but since $\rho_{ijkl} = \rho_{jilk} = \rho_{klij} = \rho_{lkji}$, only 6 are different.

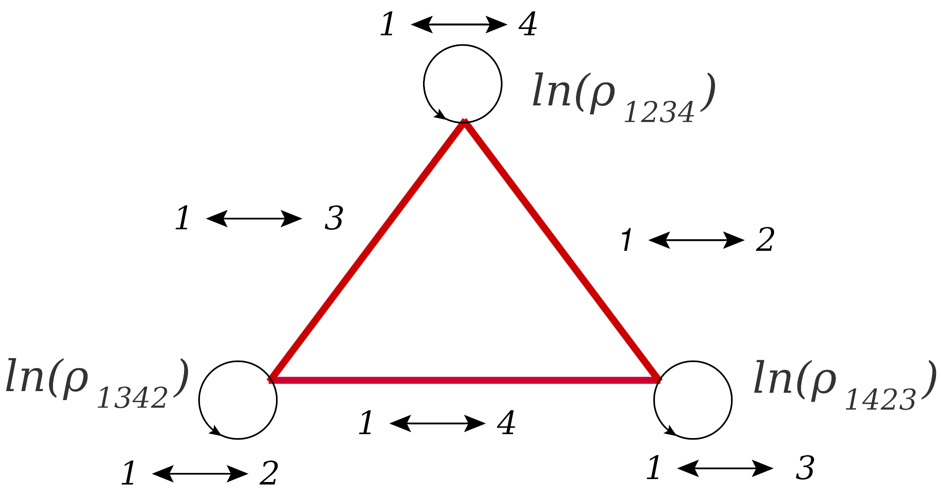
These $6 = 2 \times 3$ are related by permutations, and $\rho_{ijkl}\rho_{ikjl} = 1$:



Finally, $\ln(\rho_{ijkl}) + \ln(\rho_{iljk}) + \ln(\rho_{iklj}) = 0$, leaving two independent variables.

Bose symmetry

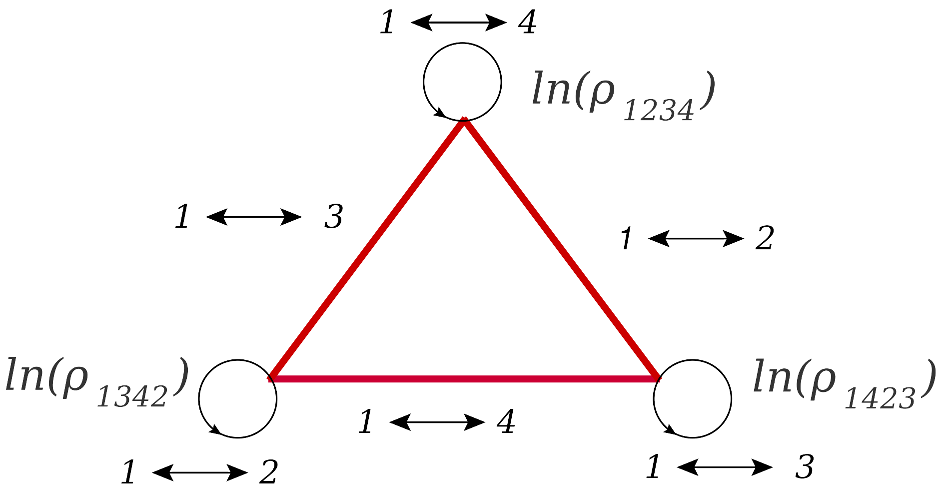
- Wilson lines are effectively Bosons (spin factored into jets): must be symmetric under all permutations.
- In $\Delta_4 = h_{abcd} \mathrm{T}_i^a \mathrm{T}_j^b \mathrm{T}_k^c \mathrm{T}_l^d \; \Delta_4^{\mathrm{kin.}}(\rho_{ijkl})$ the symmetry properties of $\Delta_4^{\mathrm{kin.}}(\rho_{ijkl})$ should mirror to those of h_{abcd}.
- We note: the antisymmtry of $\ln(\rho_{1234})$ under $1 \leftrightarrow 4$ (or $2 \leftrightarrow 3$) are mirrored by the antisymmetry of $h_{abcd} = f_{ade}\, f_{cbe}$.
- But under other permutations, the kinematic variables $\ln(\rho_{ijkl})$ transform into each other:



Bose symmetry: solution

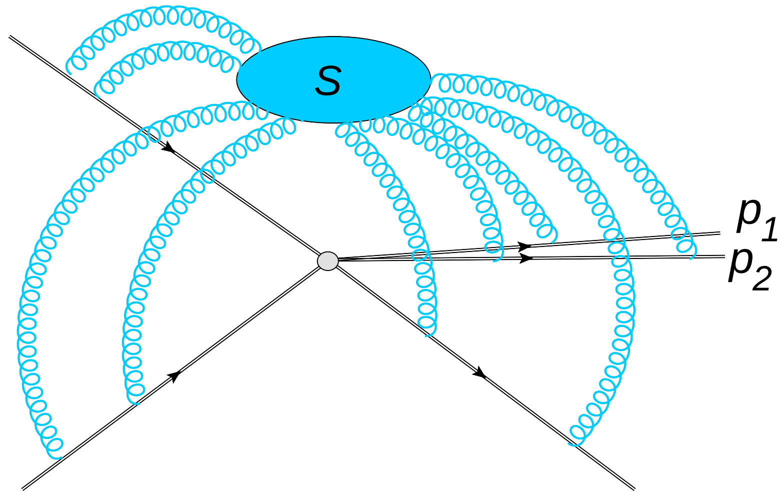
To satisfy the symmetry under all permutations with h_{abcd} built out of $f_{ade}\, f_{cbe}$, and polynomial dependence on $L_{ijkl} \equiv \ln(\rho_{ijkl})$ we must consider a sum:

$$\begin{aligned}
\Delta_4(\rho_{ijkl}) \;=\; & \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \\
\times \; & \Big[f^{ade} f^{cbe}\, L_{1234}^{h_1} \left(L_{1423}^{h_2}\, L_{1342}^{h_3} \;-\; (-1)^{h_1+h_2+h_3}\, L_{1342}^{h_2}\, L_{1423}^{h_3} \right) \\
& + f^{cae} f^{dbe}\, L_{1423}^{h_1} \left(L_{1234}^{h_2}\, L_{1342}^{h_3} \;-\; (-1)^{h_1+h_2+h_3}\, L_{1342}^{h_2}\, L_{1234}^{h_3} \right) \\
& + f^{bae} f^{cde}\, L_{1342}^{h_1} \left(L_{1423}^{h_2}\, L_{1234}^{h_3} \;-\; (-1)^{h_1+h_2+h_3}\, L_{1234}^{h_2}\, L_{1423}^{h_3} \right) \Big] \,,
\end{aligned}$$



Collinear limits

- We considered the ‘fixed angle’ amplitude, where all invariants $|p_i \cdot p_j|$ are simultaneously large.
- Consider now (minimally) violating this requirement, setting two of the hard partons collinear. In particular $|p_1 \cdot p_2| \ll |p_i \cdot p_j|$.



- This is a singular limit: additional singularities appear through $\ln(p_1 \cdot p_2)$.
- We define a Splitting Amplitude $\mathbf{Sp}$ to capture this singularity, relating the n parton amplitude with two collinear lines $1||2$ to the $n-1$ parton amplitude, where $P = p_1 + p_2$ and $T = T_1 + T2$:

$$\mathcal{M}_n\left(p_1, p_2, p_j; \mu, \epsilon\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu, \epsilon\right) \mathcal{M}_{n-1}\left(P, p_j; \mu, \epsilon\right) + \text{regular}.$$

- The property of $\mathbf{Sp}$: it depends only on d.o.f. of the collinear pair!

Collinear limits

Expressing the IR singularities of $\mathcal{M}_n$ and $\mathcal{M}_{n-1}$ by

$$\mathcal{M}_n\left(p_1, p_2, p_j; \mu, \epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu_f^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_n\left(p_1, p_2, p_j, \lambda, \epsilon\right)\right\}\mathcal{H}_n\left(p_1, p_2, p_j; \mu, \mu_f\right),$$

$$\mathcal{M}_{n-1}\left(P, p_j; \mu, \epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu_f^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_{n-1}\left(P, p_j, \lambda, \epsilon\right)\right\}\mathcal{H}_{n-1}\left(P, p_j; \mu, \mu_f\right).$$

the IR singularities in the Splitting Amplitude $\mathbf{Sp}$ are given by

$$\mathbf{Sp}(p_1, p_2; \mu, \epsilon) = \mathbf{Sp}_{\mathcal{H}}^{(0)}(p_1, p_2; \mu, \epsilon)\,\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda)\right\}.$$

where $\Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) = \Gamma_n\left(p_1, p_2, p_j, \lambda, \epsilon\right) - \Gamma_{n-1}\left(P, p_j, \lambda, \epsilon\right)$

$$= \Gamma_{\mathbf{Sp},\,\mathrm{dip.}}(p_1, p_2; \lambda) + \Delta_n\left(\rho_{ijkl}; \lambda\right) - \Delta_{n-1}\left(\rho_{ijkl}; \lambda\right)$$

So the requirement that $\mathbf{Sp}$ depends only on the collinear pair, amounts to having $\Delta_4\left(\rho_{ijkl}; \lambda\right)$ vanish in all collinear limits!

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Collinear limits: solution

- We usually consider $2p_i \cdot p_j = Q^2 \beta_i \beta_j$ where $\beta_i \cdot \beta_j = \mathcal{O}(1)$.
- Take collinear limit $2p_1 \cdot p_2 / Q^2 = P^2/Q^2 \to 0$ introducing: $p_1 = zP + k$ and $p_2 = (1-z)P - k$:

$$\ln\left(\frac{p_1 \cdot p_2 \, p_3 \cdot p_4}{p_1 \cdot p_3 \, p_2 \cdot p_4}\right) \simeq \underbrace{\ln\left(\frac{P^2 \, p_3 \cdot p_4}{2z(1-z)\, P \cdot p_3 \, P \cdot p_4}\right)}_{\mathcal{O}\left(\ln(P^2/Q^2)\right)} - \underbrace{\frac{k \cdot p_3}{z\, P \cdot p_3}}_{\mathcal{O}\left(\sqrt{P^2/Q^2}\right)} + \underbrace{\frac{k \cdot p_4}{(1-z)\, P \cdot p_4}}_{\mathcal{O}\left(\sqrt{P^2/Q^2}\right)} \to \infty\,,$$

$$\ln\left(\frac{p_1 \cdot p_4 \, p_2 \cdot p_3}{p_1 \cdot p_2 \, p_4 \cdot p_3}\right) \simeq \underbrace{\ln\left(\frac{2z(1-z)\, P \cdot p_4 \, P \cdot p_3}{P^2 \, p_4 \cdot p_3}\right)}_{\mathcal{O}\left(\ln(P^2/Q^2)\right)} - \underbrace{\frac{k \cdot p_3}{(1-z)\, P \cdot p_3}}_{\mathcal{O}\left(\sqrt{P^2/Q^2}\right)} + \underbrace{\frac{k \cdot p_4}{z\, P \cdot p_4}}_{\mathcal{O}\left(\sqrt{P^2/Q^2}\right)} \to -\infty\,,$$

$$\ln\left(\frac{p_1 \cdot p_3 \, p_4 \cdot p_2}{p_1 \cdot p_4 \, p_3 \cdot p_2}\right) = \frac{1}{z(1-z)}\left(\frac{k \cdot p_3}{P \cdot p_3} - \frac{k \cdot p_4}{P \cdot p_4}\right) = \mathcal{O}\left(\sqrt{P^2/Q^2}\right) \to 0\,,$$

- Conclusion: in a given collinear limit two of the cross ratios diverge logarithmically, while the third vanishes as a power!
- Any $\Delta_4 \sim \ln^{h_1}(\rho_{1234}) \ln^{h_2}(\rho_{1423}) \ln^{h_3}(\rho_{1342})$ with $h_i \geq 1, \forall i$ satisfies the Splitting Amplitude constraint! [Dixon, E.G. & L. Magnea]

Three-loop analysis

- Recall the solution to the Bose-symmetry constraint:

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} L_{1234}^{h_1} \left(L_{1423}^{h_2} L_{1342}^{h_3} - (-1)^{h_1+h_2+h_3} L_{1342}^{h_2} L_{1423}^{h_3} \right) + \text{perm.} \right]$$

- The collinear constraint is satisfied with $h_i \geq 1, \forall i$.

h_1	h_2	h_3	$h_{\rm tot}$	comment
1	1	1	3	vanishes identically by Jacobi identity
2	1	1	4	kinematic factor vanishes identically
1	1	2	4	allowed by symmetry, excluded by transcendentality
1	2	2	5	viable possibility
3	1	1	5	viable possibility
2	1	2	5	viable possibility
1	1	3	5	viable possibility

all coincide (Jacobi)

Beyond the minimal solution: 4–parton correlations

Within the space of polynomials in logs of conformal-invariant cross ratios

$$L_{ijkl} \equiv \ln \rho_{ijkl} = \ln\left(\frac{p_i \cdot p_j \; p_k \cdot p_l}{p_i \cdot p_k \; p_j \cdot p_l}\right) .$$

there is just one solution:

$$\Delta_4^{(122)}(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \Big[f^{ade} f^{cbe} L_{1234} \left(L_{1423} \, L_{1342}\right)^2$$
$$+ f^{cae} f^{dbe} L_{1423} \left(L_{1234} \, L_{1342}\right)^2 + f^{bae} f^{cde} L_{1342} \left(L_{1423} \, L_{1234}\right)^2 \Big] .$$

statisfying

- Rescaling symmetry: depends on conformal-invariant cross ratios
- Bose symmetry: simultaneous permutation of colour and kinematics
- Collinear limit: vanishes in all collinear limits
- Transcendentality: maximal, as in $\mathcal{N} = 4$ SYM.

Conclusions

- A completely general constraint was derived based on factorization and rescaling symmetry.
  It relates soft singularities in any amplitude, and any loop order, to the cusp anomalous dimension.
- An all-loop sum-over-dipoles formula emerges as a minimal solution.
- There are two types of potential corrections:
  - via higher-Casimir contributions to γ_K (starting at four loops)
  - functions of conformally–invariant cross ratios (starting at three loops, four legs)
- Additional constraints drastically reduce the space of potential corrections, yet there are candidates at three loops.

The full beauty of gauge theory amplitudes is not yet revealed...