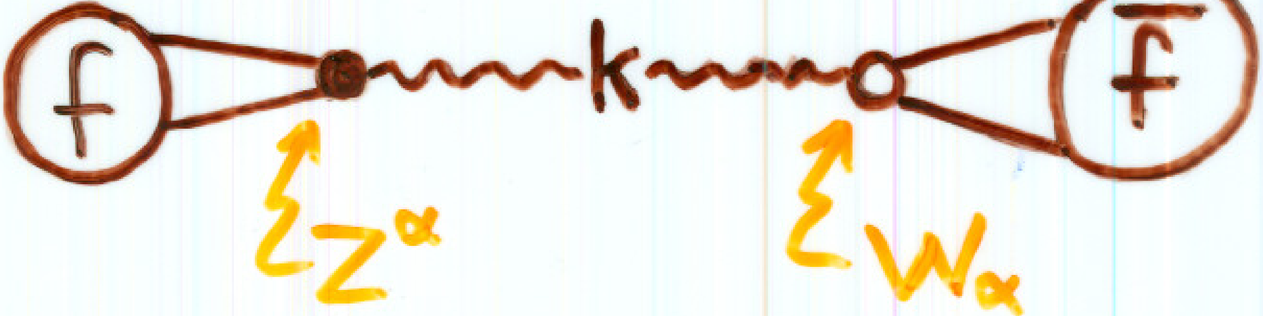
Manifest Twistor Unitarity?

Old problem: Can we make it manifest, from twistor formulae, that the scalar product between positive frequency massless fields (wavefunctions) is **Positive Definite**, in a way that is conformally invariant and uniform across helicities?

(Momentum method: not conformally invariant; space-time method not uniform across helicities .... fermions?)

I think: YES! Use Hodges boundary method:



Boundary of 8-contour on $W\cdot Z = k$. Then: $\int_0^\infty e^{-k} \times$ above dk

Twistor Notation

Unfortunate choices made in R.P. (1967) Twistor Algebra, J Math Phys 8, 345-

EW:	λ^{α}	$\mu_{\dot\alpha}$	$x_{\alpha\dot\alpha}$	$(++--)$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	
later RP:	$\pi_{A'}$	ω^{A}	$-ix^{AA'}$	$(+---)$

N.B. There are several good reasons for my changing my notation from that of 1967 (and 1968) to: $\boxed{Z^{\alpha}=(\omega^{A},\pi_{A'})}$

- The use of capital Latin indices for 2-spinors, together with l.c. latin indices for tensors greatly simplifies tensor-spinor correspondence when abstract indices are used, e.g.

$$C_{abcd}=\Psi_{ABCD}\,\varepsilon_{A'B'}\,\varepsilon_{C'D'}+\varepsilon_{AB}\,\varepsilon_{CD}\,\overline{\Psi}_{A'B'C'D'}$$

- The secondary part (or projection part) of a twistor is really a <u>momentum</u> thing (not a velocity), & momentum is <u>co</u>vector

$$p_a=\overline{\pi}_A\pi_{A'} \qquad \text{(explaining "}\pi_{A'}\text{")}$$

whereas the origin-dependent ω^A is an <u>angular</u> momentum thing:

$$M^{ab}=i\omega^{(A}\overline{\pi}^{B)}\varepsilon^{A'B'}-i\varepsilon^{AB}\overline{\omega}^{(A'}\pi^{B')}$$

2-spinor Notation

Vector/tensor indices: $a, b, c, \dots, a_0, \dots, a_1, \dots$

4-dimensional $0, 1, 2, 3$

$\angle$ time

2-spinor indices: $A, B, C, \dots, A_0, \dots, A_1, \dots$

$0, 1 \,;\, 0', 1'$ $A', B', C', \dots, A'_0, \dots, A'_1, \dots$

complex conjugate

$$\boxed{\text{Abstract indices: } a = AA', \quad b = BB', \quad c = CC', \dots, a_0 = A_0A'_0, \dots, a_1 = A_1A'_1, \dots}$$

Standard Coordinates:

$$V^{AA'}: \quad \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}$$

Raise/lower indices

$$g_{ab}, \, g^{ab} \; ; \qquad \varepsilon_{AB}, \varepsilon^{AB}, \varepsilon_{A'B'}, \varepsilon^{A'B'}$$

Symmetric — anti-symmetric

$$\boxed{g_{ab} = g_{AA'BB'} = \varepsilon_{AB} \, \varepsilon_{A'B'}}$$

Interpretation of η^A:

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} r^0 + r^3 & r^1 + i r^2 \\ r^1 - i r^2 & r^0 - r^3 \end{bmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

$$\omega^A = i\, r^{AA'}\, \pi_{A'}$$

$$Z^\alpha = (\omega^A, \pi_{A'})$$

$\boxed{\text{incidence: } \omega^A = i\, r^{AA'} \pi_{A'}}$

Shift of origin

$$\overset{Q}{\omega}{}^A = \overset{O}{\omega}{}^A - i\, q^{AA'} \overset{O}{\pi}_{A'}$$

$$\overset{Q}{\pi}_{A'} = \overset{O}{\pi}_{A'}$$

complex conjugate twistor

$$\boxed{\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})}$$

Null twistor

$$Z^\alpha \bar{Z}_\alpha = 0$$

equation of $\mathbb{PN}$

Momentum-angular mom. for massless ptcle.

p_a = 4-momentum $\quad \left.\begin{matrix} p_a p^a = 0 \\ p_0 > 0 \end{matrix}\right\}$

M^{ab} = 6-angular momentum

Pauli-Lubanski spin vector $\quad S_a = \frac{1}{2} e_{abcd} \underbrace{p^b M^{cd}}_{*(p \wedge M)} = s\, p_a$

helicity

$$p_a = \pi_{A'} \bar{\pi}_A, \quad M^{ab} = i \omega^{(A} \pi^{B)} \varepsilon^{A'B'} - i \varepsilon^{AB} \bar{\omega}^{(A'} \bar{\pi}^{B')}, \quad 2s = Z^\alpha \bar{Z}_\alpha$$

Note: (...) denotes symmetrization

Remark on terminology:

To a twistor theorist, the term "twistor transform" refers to the operation which takes us from a twistor function $f(Z^\alpha)$ to a corresponding (holomorphic) function on the dual twistor space $\tilde{f}(W_\alpha)$ (or back again), describing the same space-time field, by use of contour $\oint$s.

I recommend that the correspondence ("half Fourier transform") between a momentum representation $\psi(p_a)$ and a twistor function $\chi(Z^\alpha)$ given by

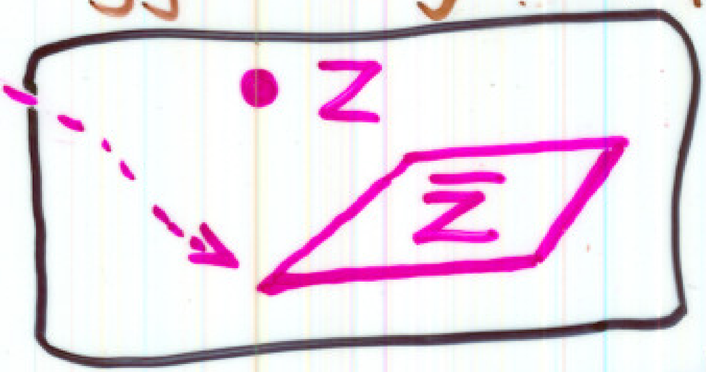
$$\chi(\omega^A, \pi_{A'}) \xleftrightarrow{\text{Fourier transform}} \psi(\pi_{A'}, \bar{\pi}_A)$$

be called the

$$\boxed{\text{Nair Transform}}$$

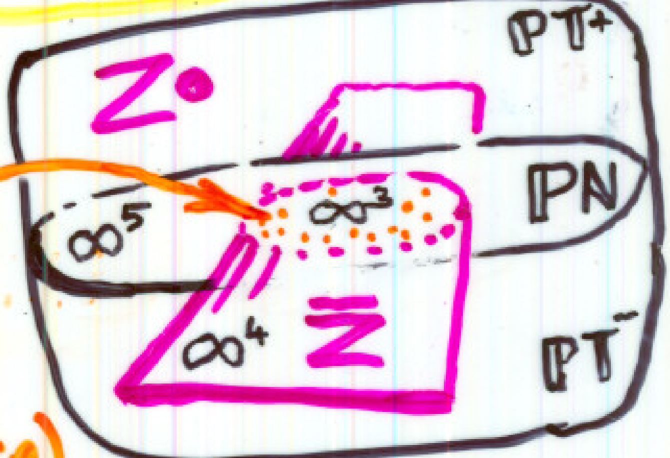
Note that, with the Lorentz space-time signature $\boxed{+---}$, the complex conjugate of a twistor (Z^α) is a dual twistor ($\bar{Z}_\alpha$), as is suggested by index placing

$\mathbb{PT} = \mathbb{CP}^3$



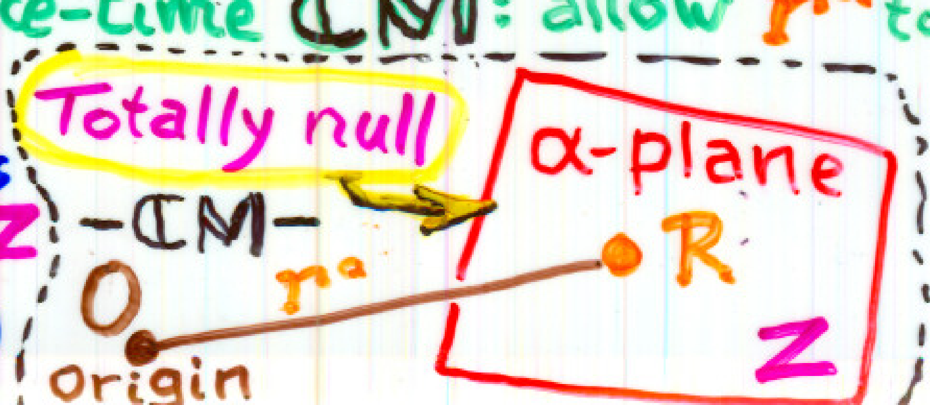
So $\bar{Z}$ represents a complex projective plane.

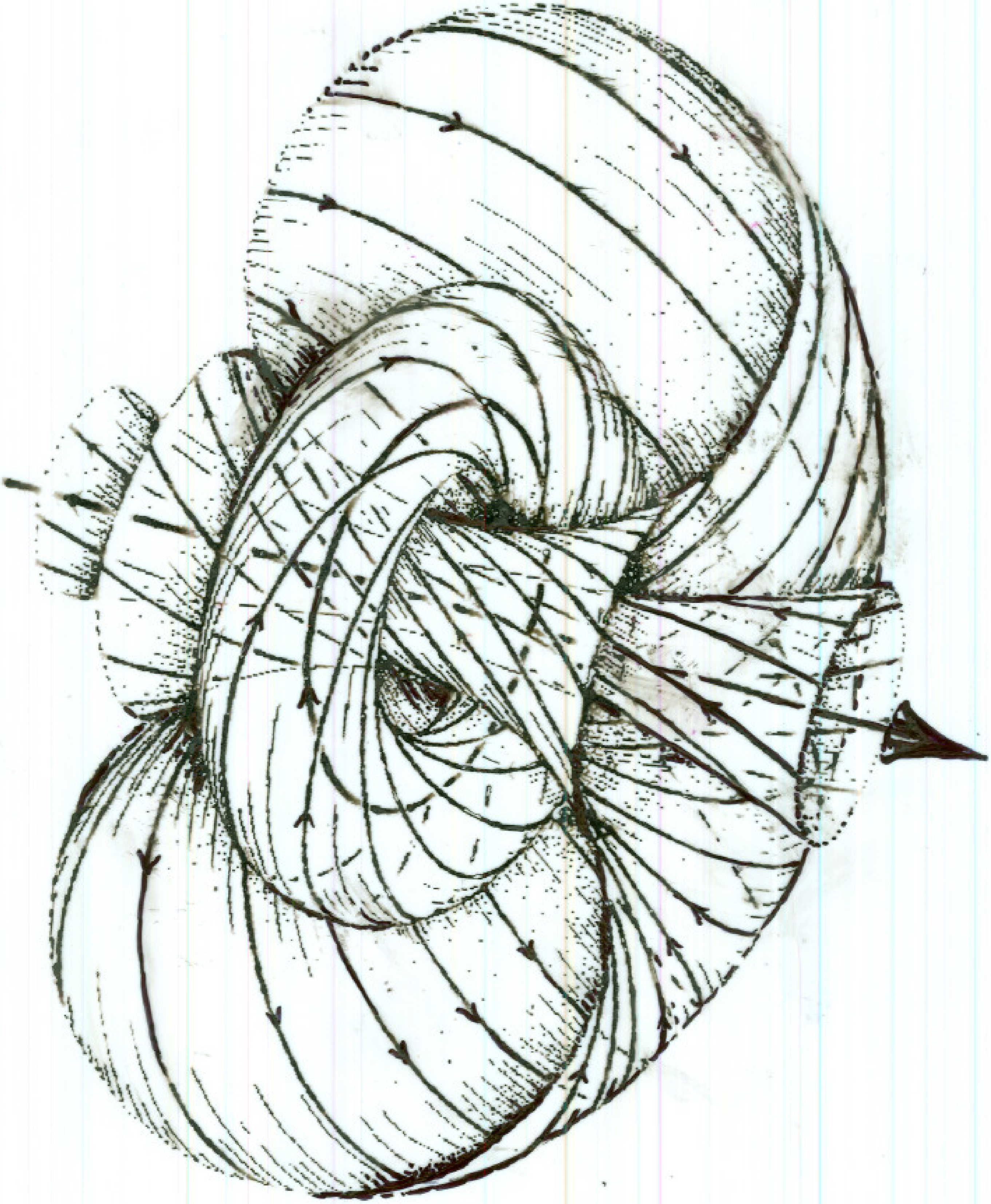
This leads to a real description as a twisting congruence of light rays (Robinson congruence)



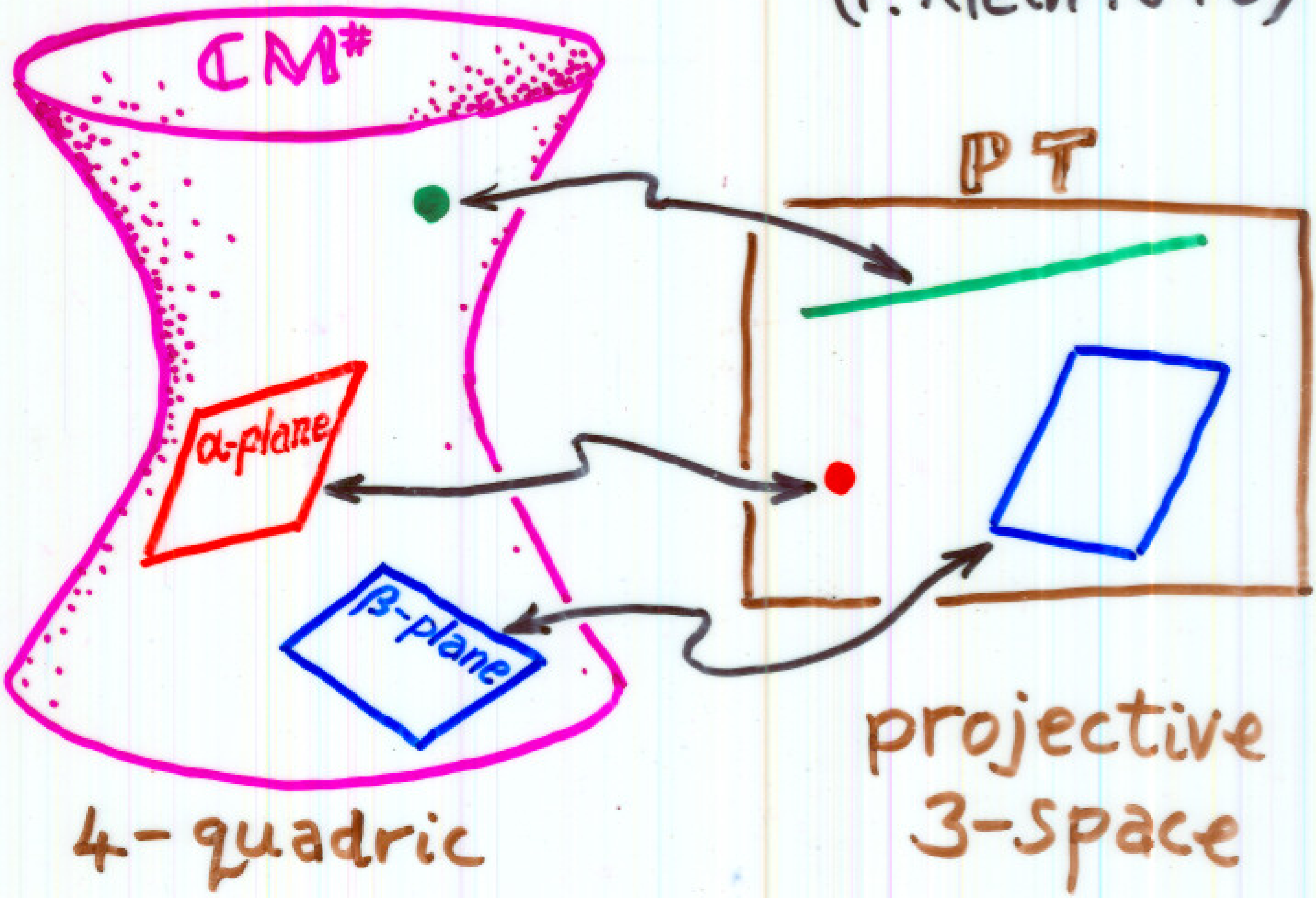
Complexified space-time $\mathbb{CM}$: allow r^a to be complex.

Fix Z. Set of points of $\mathbb{CM}$ incident with Z constitutes an α-plane. (For a dual twistor, get β-plane.)

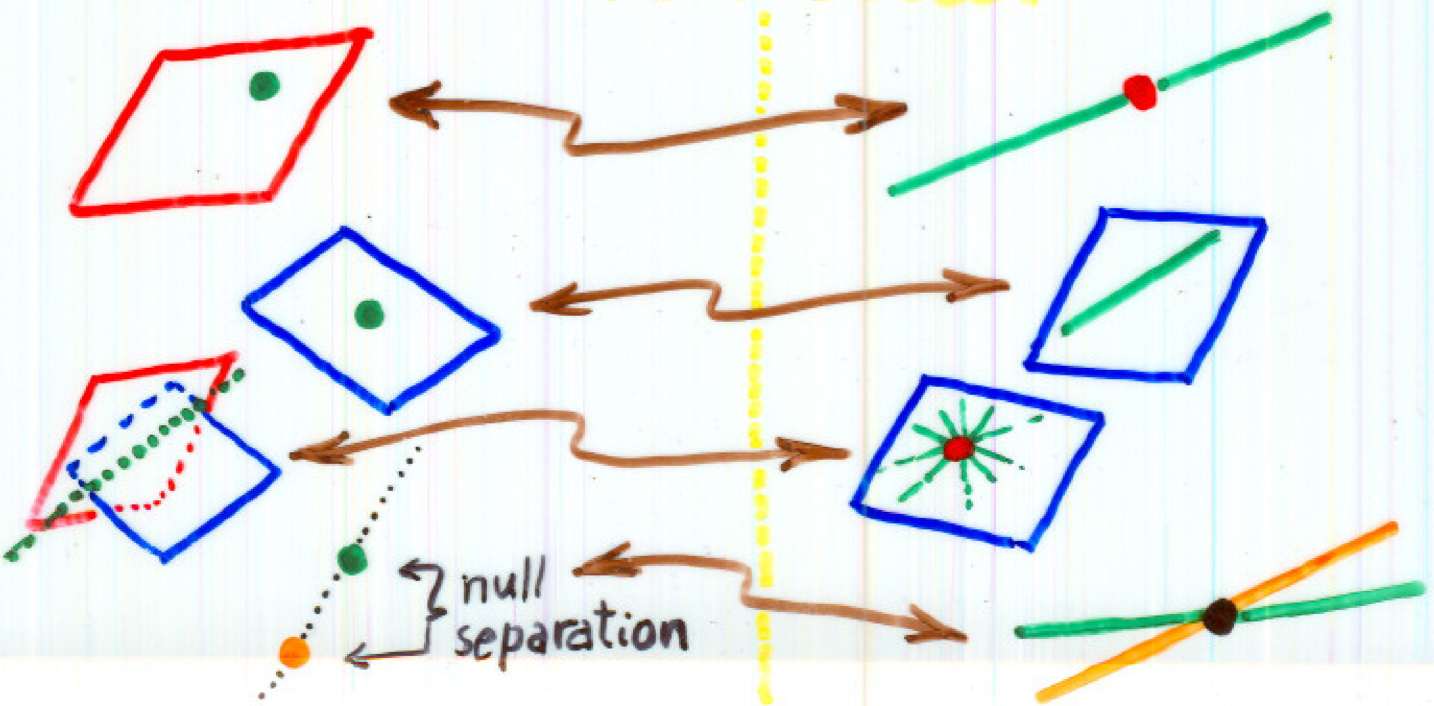




Klein Correspondence (F. Klein 1870)

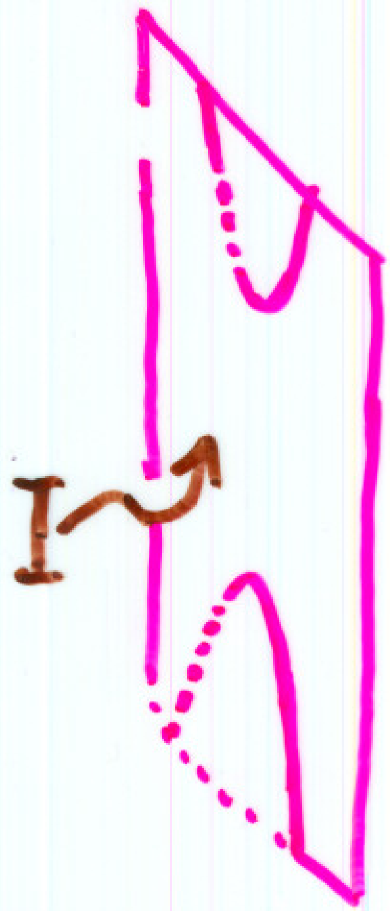


$\mathbb{C}\mathbb{M}^{\#}$ Incidence $\mathbb{P}\mathbb{T}$



$I^{\alpha\beta}$ (or dual $I_{\alpha\beta}$) is the (simple) $\boxed{\text{INFINITY TWISTOR}}$

$$I^{\alpha\beta} I_{\beta\gamma} = 0$$



$I^{\alpha\beta}$ (or dual $I_{\alpha\beta}$) is the (non-simple) $\boxed{\text{INFINITY TWISTOR}}$

$I^{\alpha\beta} I_{\alpha\beta} > 0$ for de Sitter

$I^{\alpha\beta} I_{\alpha\beta} < 0$ for Anti-de Sitter

Quantum Twistor Theory

Z^α and $\bar{Z}_\alpha$ become non-commuting:

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0$$

$$\bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$

$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar\, \delta^\alpha_\beta$$

Z^α and $\bar{Z}_\alpha$ are canonical conjugate variables (as well as complex conjugate)

Choose $\hbar = 1$, for convenience. We find

$p_a = \pi_A \bar{\pi}_{A'}$, $M^{ab} = i\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - i\varepsilon^{AB}\bar{\omega}^{(A'}\pi^{B')}$

undisturbed by factor ordering, but

$$s = \frac{1}{4}\left(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha\right)$$

The standard commutators for p_a and M^{ab} follow

$p_a p_b - p_b p_a = 0$, $p_a M^{bc} - M^{bc} p_a = i(g_a{}^b p^c - g_a{}^c p^b)$,

$M^{ab}M^{cd} - M^{cd}M^{ab} = i(g^{bc}M^{ad} - g^{bd}M^{ac} + g^{ad}M^{bc} - g^{ac}M^{bd})$.

Lie algebra generators for the Poincaré group.

Massless field equations:

$\phi_{AB\ldots L}$ and $\phi_{A'B'\ldots L'}$

$\square\phi = 0$, $\nabla^{AA'}\phi_{AB\ldots L} = 0$ helicity $-\frac{n}{2}$

$\nabla^{AA'}\phi_{A'B'\ldots L'} = 0$ helicity $+\frac{n}{2}$

(assuming these are positive-frequency wave functions)

		Helicity	Twistor function hom. deg.
Scalar wave	$\square\phi = 0$	0	-2
Dirac-Weyl neutrino	$\nabla^{AA'}\nu_A = 0$	$-1/2$	-1
Dirac-Weyl anti-neutrino	$\nabla^{AA'}\bar{\nu}_{A'} = 0$	$+1/2$	-3
Maxwell photon $F_{ab} \rightsquigarrow \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\tilde{\varphi}_{A'B'}$			
left-handed (anti-s.-d.)	$\nabla^{AA'}\varphi_{AB} = 0$	-1	0
right-handed (self-dual)	$\nabla^{AA'}\tilde{\varphi}_{A'B'} = 0$	$+1$	-4

Linearized Einstein graviton

$K_{abcd} \rightsquigarrow \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\tilde{\Psi}_{A'B'C'D'}$

Twistor Wavefunctions

Since Z^α and (i)$\bar{Z}_\alpha$ are conjugate variables, a twistor wavefunction f ought to depend on *either* Z^α or $\bar{Z}_\alpha$, but not both. But what does it mean to say that $f(Z^\alpha)$ does not depend on $\bar{Z}_\alpha$? The condition is $\frac{\partial f}{\partial \bar{Z}_\alpha} = 0$, the *Cauchy-Riemann* eqns, asserting that f is *holomorphic* in Z^α.

Helicity eigenstates

These are eigenstates of the helicity operator $s = \frac{\hbar}{2}\left(-2 - Z^\alpha \frac{\partial}{\partial Z^\alpha}\right)$.

But $Z^\alpha \frac{\partial}{\partial Z^\alpha}$ is the Euler homogeneity operator, whose eigenstates are homogeneous functions, with eigenvalue = degree of homogeneity.

Take the integer $n = 2s/\hbar$ to represent the helicity; then f is homogeneous (holomorphic) in Z^α of $\boxed{\text{degree } -2-n}$. Alternatively use $\tilde{f}(W_\alpha)$, with $W_\alpha = \bar{Z}_\alpha$. Then $\tilde{f}$ is hol. hom. $\boxed{\text{deg. } -2+n}$

Contour Integral Expressions

(Whittaker[1903], Bateman[1904, 1944], RP[1968,9,75], Hughston[1973,9])

zero Helicity:

$$\phi(x^a) = \text{con.} \times \oint_{\omega = ix\pi} f(\omega^A, \pi_{A'})\, \delta\pi$$

incidence

Homogeneous version (1-dim $\oint$) $\delta\pi = \pi_{A'}\, d\pi^{A'}$

Inhomogeneous version (2-dim $\oint$) $\delta\pi = d\pi_{0'} \wedge d\pi_{1'}$

Positive Helicity:

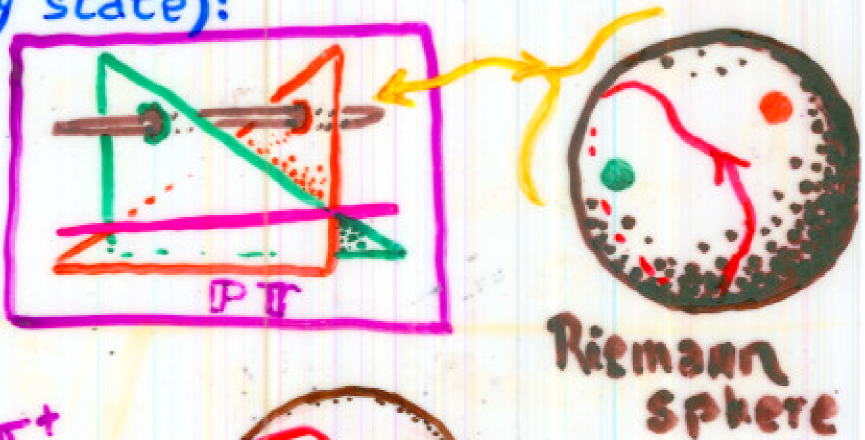
$$\varphi_{A'B'\ldots L'}(x^a) = \text{con.} \times \oint_{\omega = ix\pi} \pi_{A'}\pi_{B'}\ldots\pi_{L'}\, f(\omega, \pi)\, \delta\pi$$

Negative Helicity:

$$\psi_{AB\ldots L}(x^a) = \text{con.} \times \oint_{\omega = ix\pi} \frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\cdots\frac{\partial}{\partial\omega^L} f(\omega, \pi)\, \delta\pi$$

Canonical case (elementary state):

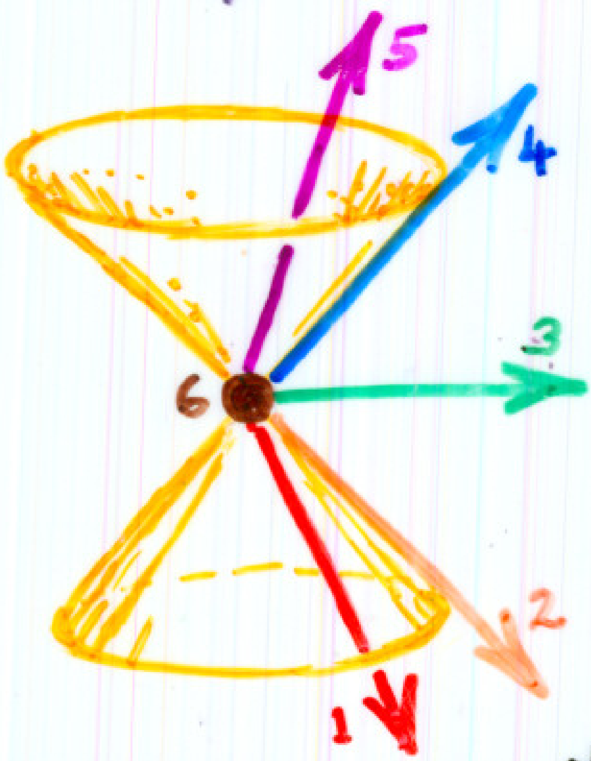
$$f = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}$$

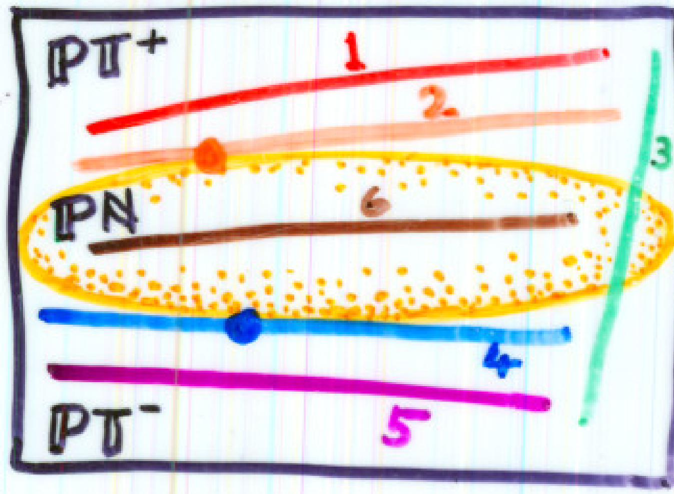


General positive frequency:

Lines in $\mathbb{PT}^+$ corr. pts. in the forward tube

Complex Minkowski Points

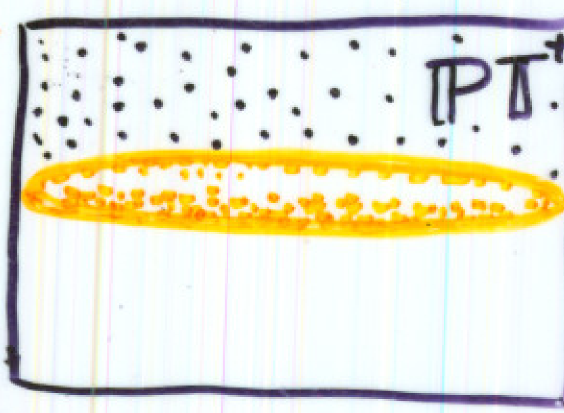




Projective twistor space $\mathbb{PT}$

Imaginary part of complex position vector

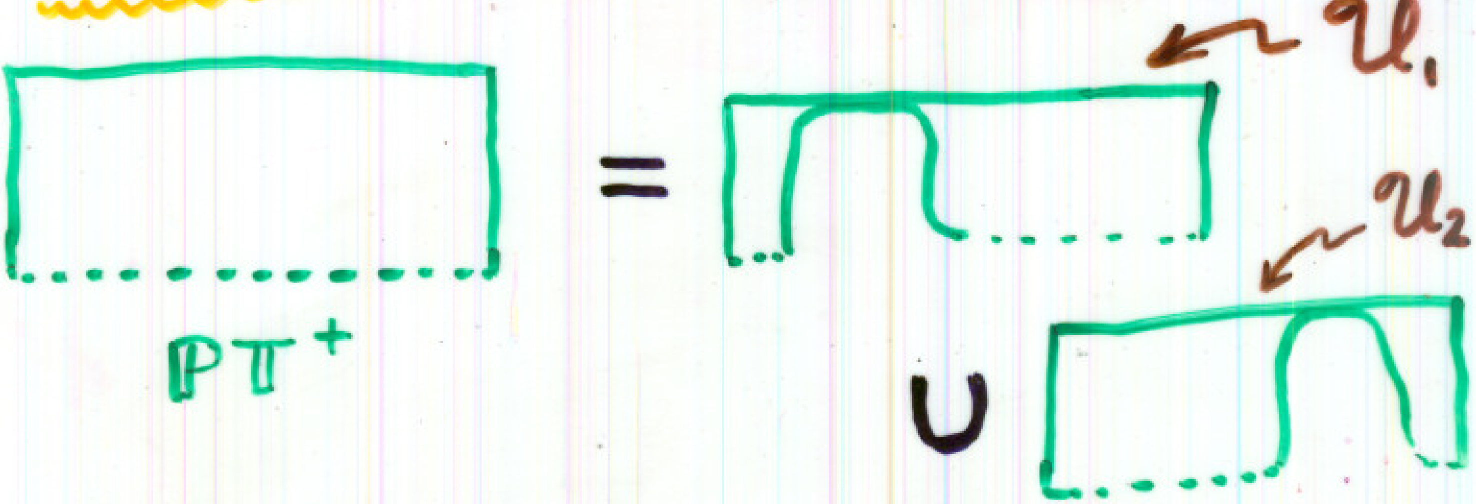
Corresponds to *forward tube* of complex Minkowski space: *past-timelike* imaginary part



Positive-frequency fields: extend holomorphically to the forward tube

composed of $e^{P_a x^a / i\hbar}$ with  : tails off in forward tube

Twistor sheaf cohomology



$\mathbb{PT}^+ = \mathcal{U}_1 \cup \mathcal{U}_2$

f defined (holomorphic) on $\mathcal{U}_1 \cap \mathcal{U}_2$

More generally: space $\mathcal{X}$ $\left(\overset{\text{here}}{=}\mathbb{PT}^+\right)$

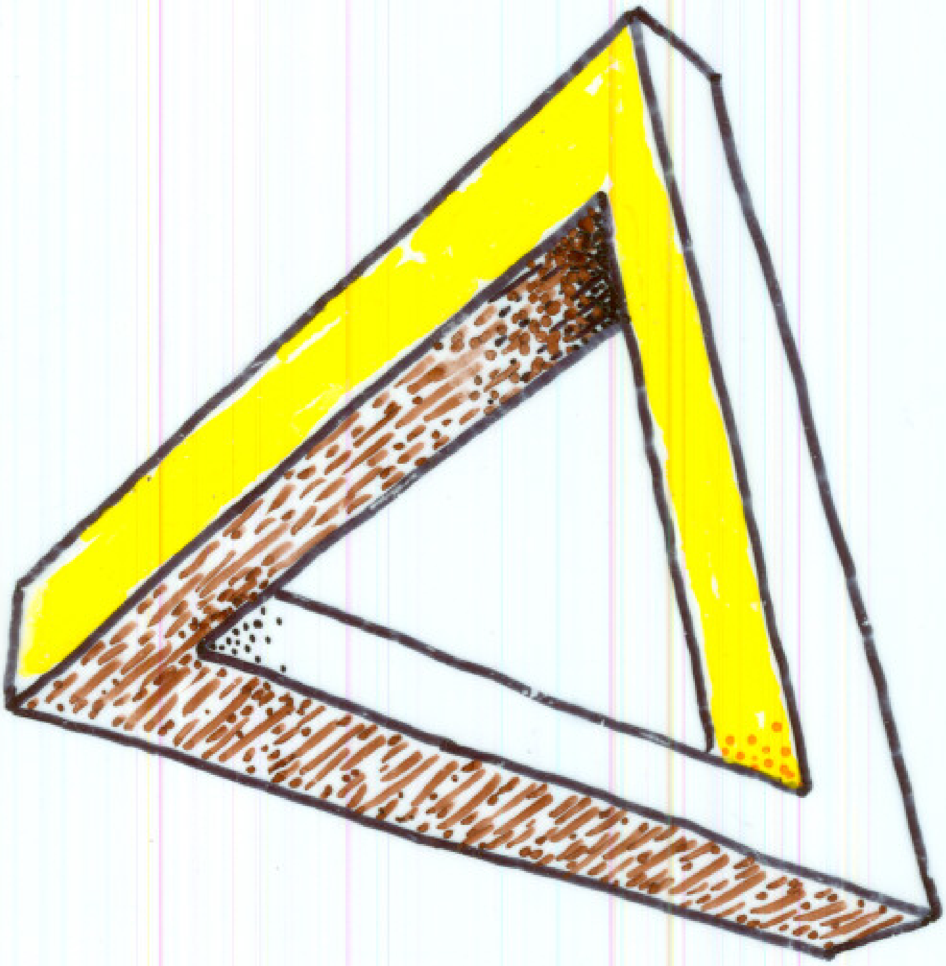
$\mathcal{X} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_n$ ($\{\mathcal{U}_i\}$ open cover)

collection $\{f_{ij}\}$, where $f_{ij}\,(=-f_{ji})$ hol. on $\mathcal{U}_i \cap \mathcal{U}_j$

We require $f_{ij} - f_{ik} + f_{jk} = 0$ on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$

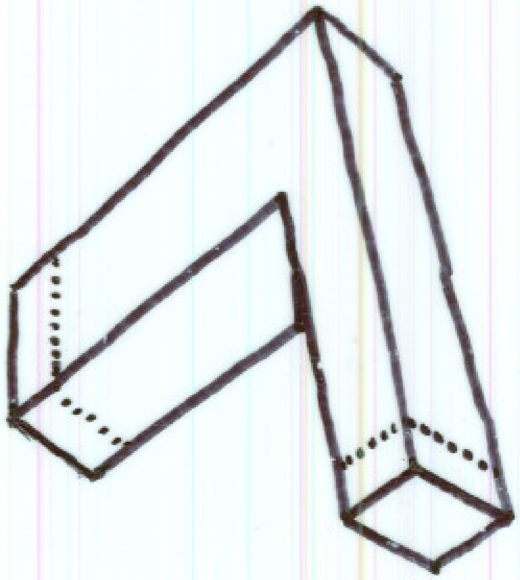
and $\{f_{ij}\} \equiv \{g_{ij}\}$ if each $f_{ij} - g_{ij} = h_i - h_j$ with h_k hol. on $\mathcal{U}_k$

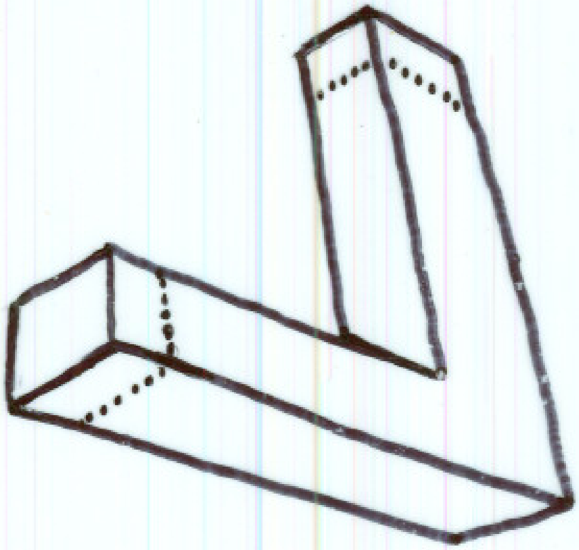
Branched contour integral:

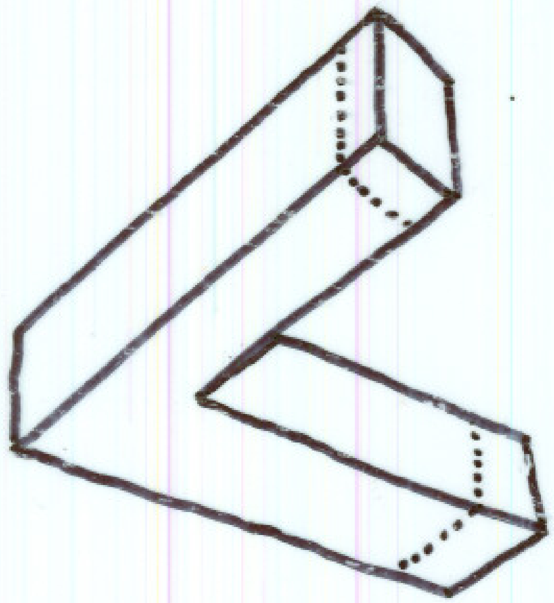


Cohomology:

a precise non-local measure — here of the degree of IMPOSSIBILITY







Non-Locality in the Wavefunction of a Single Particle

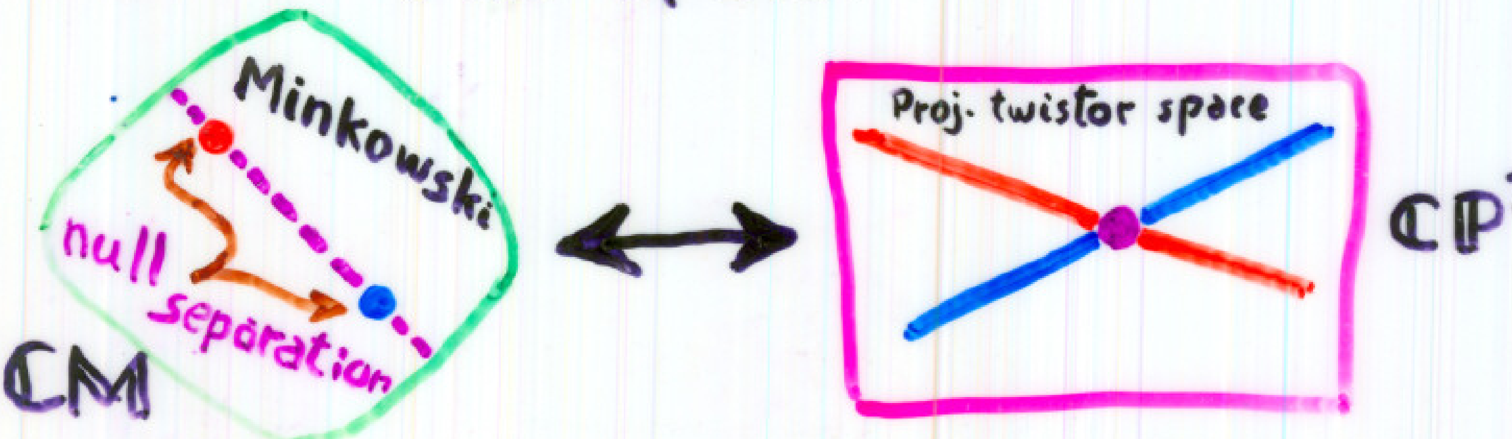
If a wavefunction were a LOCAL disturbance, then detections at both places (or nowhere) could occur — would seem to imply superluminal communication

General Relativity

Numerous special applications
(e.g. Woodhouse-Mason: stationary axi-symm.)

As part of general programme:
"non-linear graviton construction" [R.P. '76]

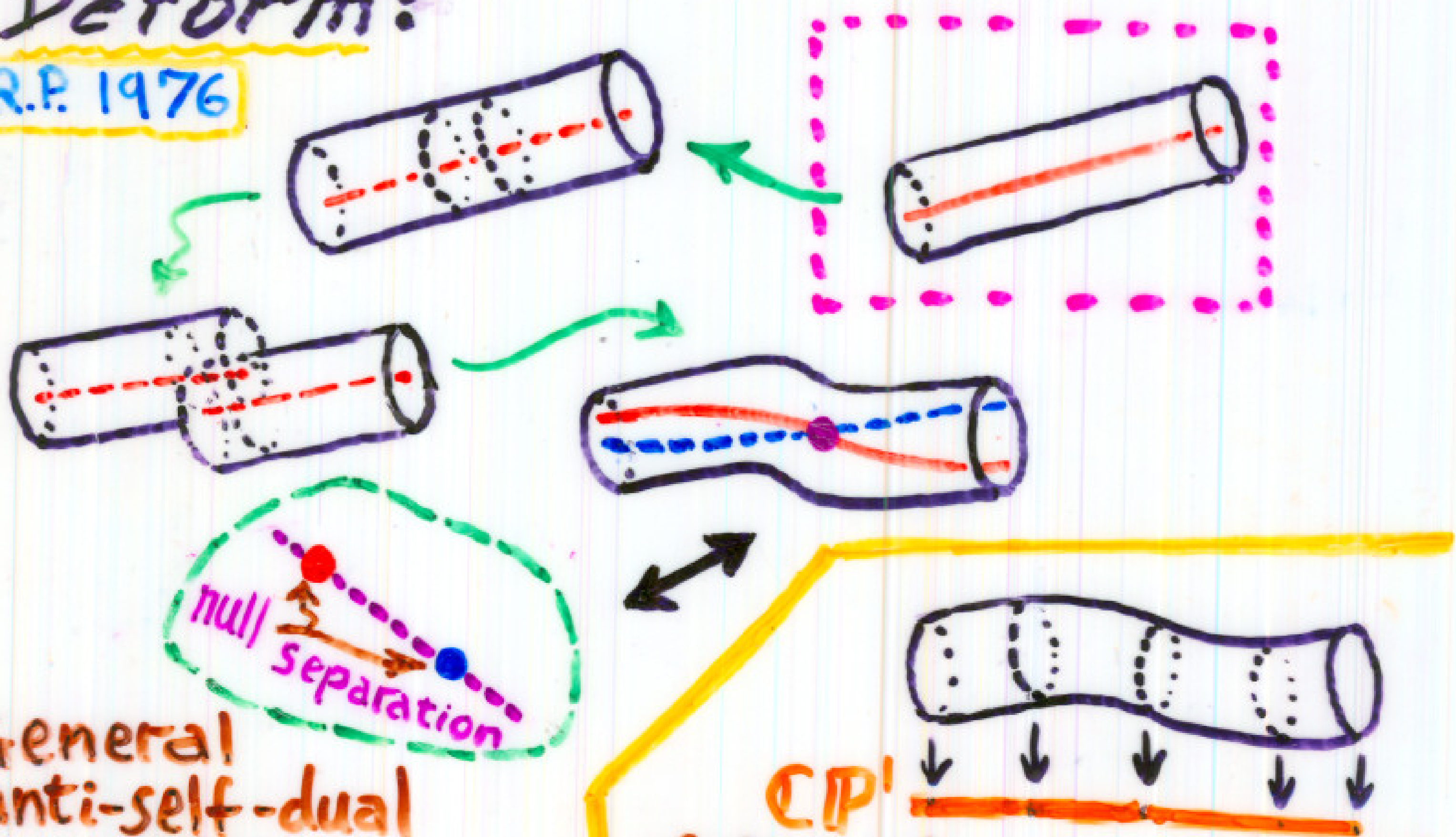
N.B. for flat space:



null separation ⟷ meeting lines

Deform:

R.P. 1976



General anti-self-dual complex "space-time"

General anti-self-dual Ricci-flat complex "space-time"

Deformation to obtain a general left-handed (-helicity) ("leg break") non-linear graviton — from a homogeneity +2 fn.

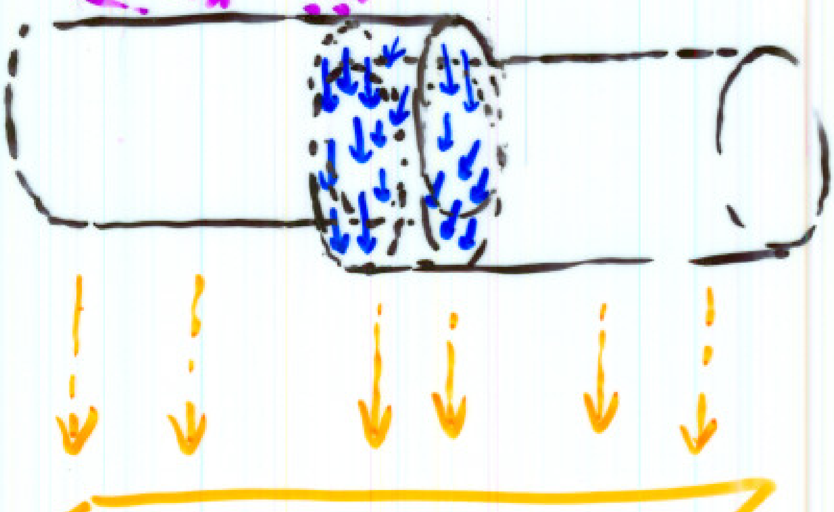
Vector field:

twistor function (1st cohomology)

$$I^{\alpha\beta}\frac{\partial f_{+2}}{\partial Z^\alpha}\frac{\partial}{\partial Z^\beta}$$

$$=\varepsilon^{AB}\frac{\partial f_{+2}(\omega,\pi)}{\partial\omega^A}\frac{\partial}{\partial\omega^B}$$

Vector field gives infinitesimal deformation

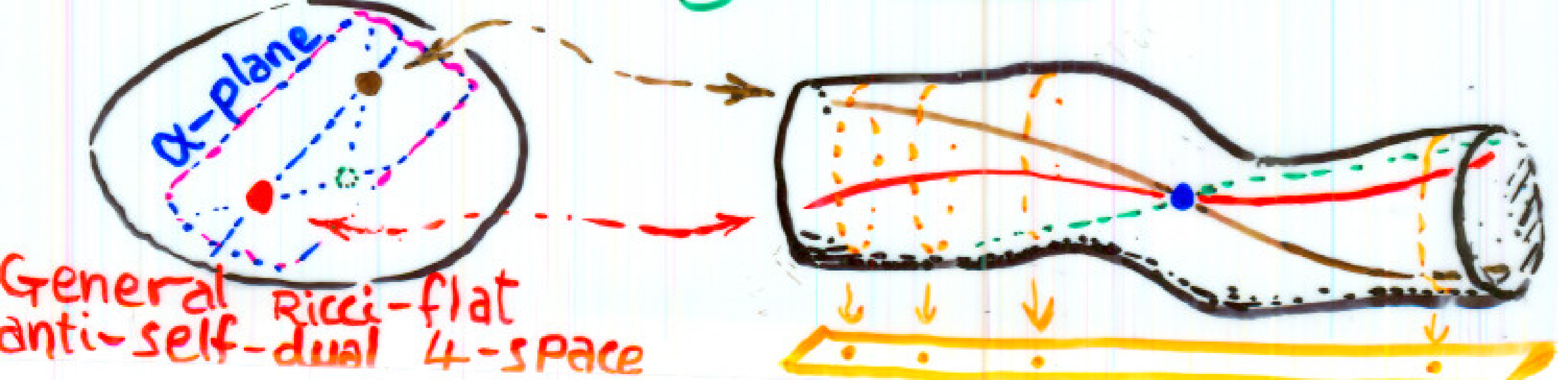


π-space

deformation preserves $\pi_{A'}$

∴ projection to π-space is preserved

"Exponentiate" to give finite deformation



Googly (graviton)

A cricket ball bowled so as to look as though it spins left-handed whereas it actually spins right-handed

Left-handed gravitons are described by anti-self-dual Weyl curvature, whereas right-handed gravitons are described by self-dual Weyl curvature.

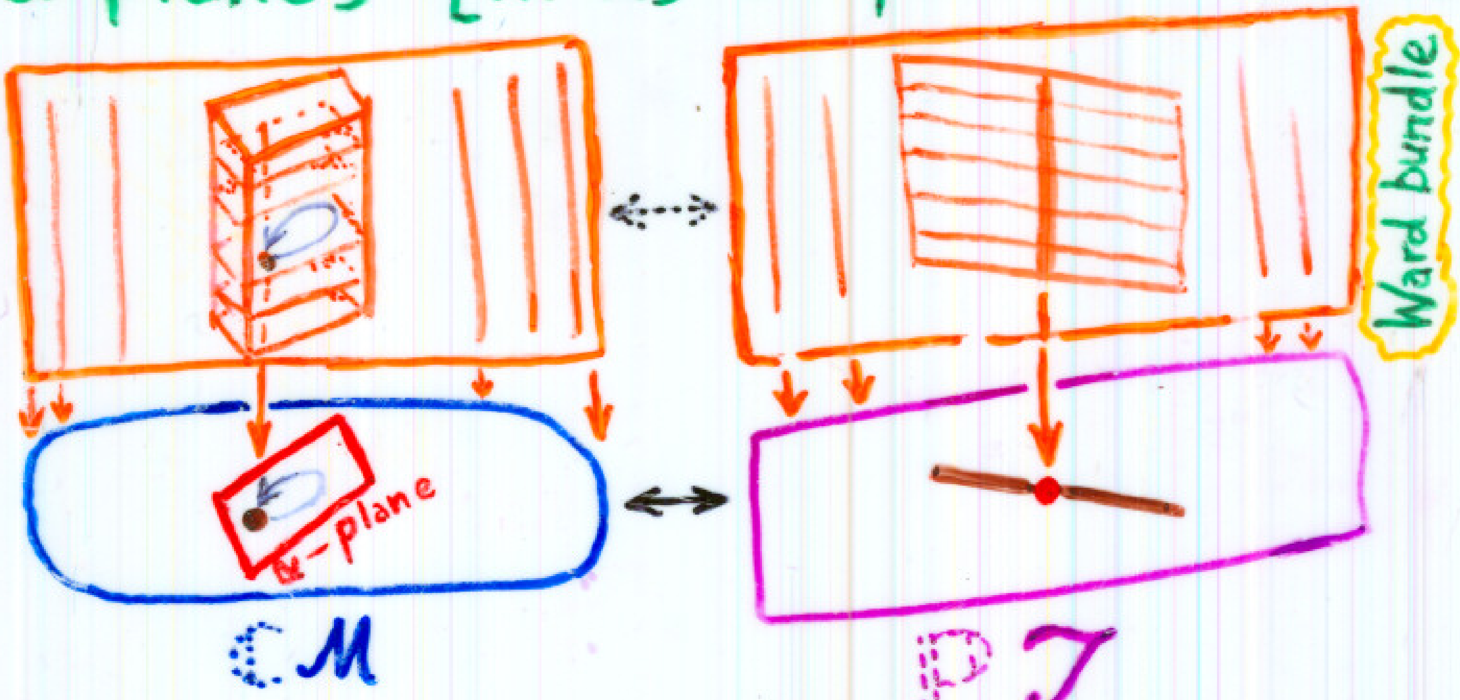
Compare:
photons,
linear spin 2 massless quanta

[positive-frequency wave function]

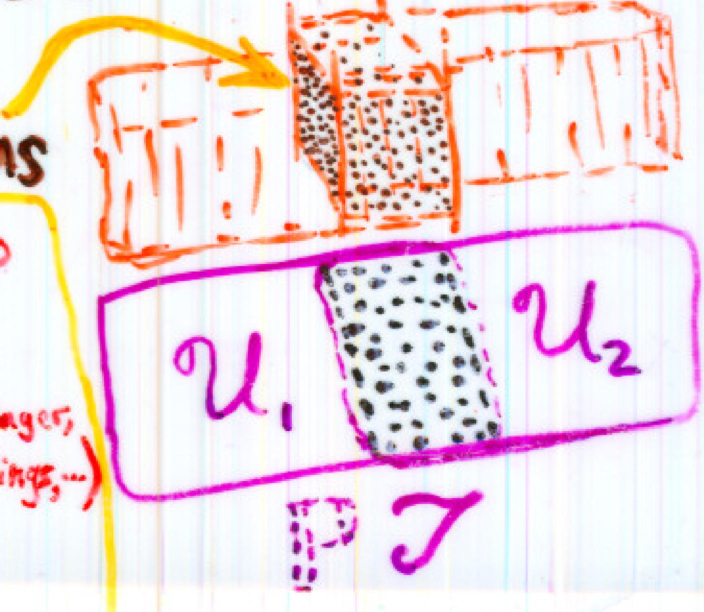
Twistors, with usual conventions, had seemed to describe anti-self-dual interactions so far!

Ward construction for (anti-)self-dual Yang-Mills fields

Y-M connection on an (analytic) space $\mathcal{M}$, with ASD conformal curvature, is ASD iff it is integrable on α-planes [in its complexification $\mathbb{C}\mathcal{M}$]



Ward bundle can be constructed in terms of free transition functions



Many applications to integrable systems (KdV, sine-Gordon, non-lin Schrödinger, Toda lattice, Einstein with 2 Killings, ...)

Ward, Sparling, Mason, Woodhouse, ...

The idea would be that suggestions for scattering amplitudes would have to be expressed as twistor integrals exhibiting such requirements of unitarity in a manifest way, & this would be a powerful selection principle.