Wilson Polygons in N=4 SYM: Systematics and New Results

Valya Khoze

(IPPP Durham University)

arXiv:1003.4405, 0910.4898 & 0902.2245

with

Babis Anastasiou, Andi Brandhuber,
Paul Heslop, Bill Spence, Gabriele Travaglini

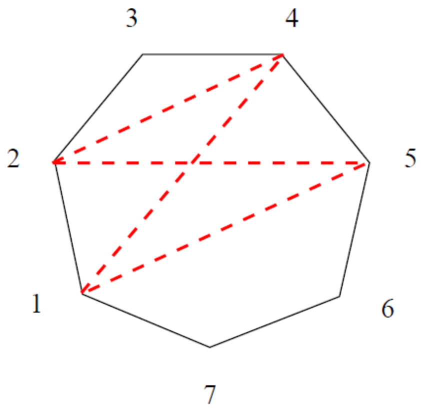
1. MHV Amplitudes and Wilson Loops

- At any loop order L, an MHV amplitude in $\mathcal{N}$=4 SYM can be expressed as the tree-level amplitude times a helicity-independent loop factor: $$\mathcal{A}_n^{(L)} = \mathcal{A}_n^{\text{tree}} \mathcal{M}_n^{(L)}$$
- At 1-loop: $$\mathcal{M}_n^{(1)} = \sum_{p,q} F^{\text{2me}}(p,q,P,Q)$$
- At 2-loops: Anastasiou-Bern-Dixon-Kosower'03 (ABDK) found an iterative structure relating the 2-loop to the 1-loop.
- To all-orders: Bern-Dixon-Smirnov'05 (BDS) proposed a re-summed exponentiated expression. (checked by BDS at 3 loops for n=4)

- Alday-Maldacena'07:

At strong coupling: amplitudes are related to the Wilson loops with the contour made out of external momenta

$$W[\mathcal{C}_n] \ := \ \mathrm{Tr}\,\mathcal{P}\exp\left[ig\oint_{\mathcal{C}_n} d\tau\ \dot{x}^{\mu}(\tau)A_{\mu}(x(\tau))\right]$$



- Drummond-Korchemsky-Sokatchev'07
  Brandhuber-Heslop-Travaglini'07
  Drummond-Henn-Korchemsky-Sokatchev'07-08

found that also at weak coupling there is a relation between planar MHV amplitudes and these light-like polygon Wilson loops:

Wilson-Loops/Amplitudes duality in perturbation theory

- The BDS expression in the exponent (for MHV ampls):

$$(BDS)_n = \sum_{L=1}^{\infty} a^L f^{(L)}(\epsilon)\, \mathcal{M}_n^{(1)}(L\epsilon) + C(a)$$

kinematic dependence governed by the 1-loop result.

- For Wilson Loops the BDS formula takes the same form with a substitution: 1-loop ampl. ⟶ 1-loop Wilson loop:

$$W_n^{(1)} = \mathcal{M}_n^{(1)} - n\frac{\pi^2}{12}$$

and the coefficient functions are different:

$$f^{(1)}(\epsilon) = 1 \qquad f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$$

$$f_{WL}^{(1)}(\epsilon) = 1 \qquad f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\epsilon - 5\zeta_4\epsilon^2$$

- At 1-loop the BDS formula is exact by construction.
- At higher-loops (dual conformal invariance indicates that) it must be correct for n=4 and n=5 points.

- Explicit calculations show that

at n=6 points and at 2 loops the BDS formula must be modified by an addition of the remainder function which is the same for the amplitudes and for the Wilson loops :

$$\mathcal{M}_n^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 = f^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{R}_n + \mathcal{O}(\epsilon)$$

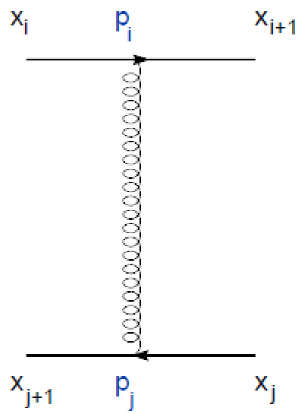
Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich;

Drummond-Henn-Korchemsky-Sokatchev'08

DHKS conformal Ward identity: BDS is one particular solution => the Remainder must be conformally invariant

At 1-loop: the Wilson Loop and the Amplitude contributions are the same up to a multiplicative factor:

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \mathcal{M}_n^{(1)} = (1+\zeta_2\epsilon^2)\mathcal{M}_n^{(1)} + \mathcal{O}(\epsilon)$$



$$p_i := x_i - x_{i+1}$$

Drummond-Korchemsky-Sokatchev'07 Brandhuber-Heslop-Travaglini'07

The relation between Amplitudes and Wilson loops at 2 loops and beyond is (traditionally) in terms of the Remainder fs.:

$$\mathcal{R}_n = \log(\mathcal{M}_n) - (BDS)_n$$

$$\mathcal{R}_n^{WL} = \log(W_n) - (BDS)_n^{WL}$$

$$\boxed{\mathcal{R}_n \; = \; \mathcal{R}_n^{\mathrm{WL}}}$$

with no additional (n-dependent) constant shifts.

Holds for n=4,5 since: $\mathcal{R}_4^{\mathrm{WL}} \; = \; \mathcal{R}_5^{\mathrm{WL}} \; = \; 0$

Collinear limits for amplitudes imply that Wilson loops:

$$\mathcal{R}_n^{\mathrm{WL}} \longrightarrow \mathcal{R}_{n-1}^{\mathrm{WL}}$$

We checked that this holds for n=6,7,8 (and no constant shifts allowed).

- Remainder functions are defined by subtracting BDS:

$$\mathcal{R}_n = \log(\mathcal{M}_n) - (BDS)_n$$
$$\mathcal{R}_n^{WL} = \log(W_n) - (BDS)_n^{WL}$$

- BDS formulae have a 1-loop origin and are not the most natural quantities to consider e.g. at strong coupling.

[At strong coupling, another solution of the conf anomaly equation makes a more prominent appearance: `BDS-like' expression.]

- In the second half of the talk instead consider a ratio of a Wilson loop and a `reference' Wilson loop.

  This ratio is finite, regularisation-independent and conformally-invariant

  (it does not refer to BDS)

  and can be used to:

$$\frac{W_n}{W_n^{\mathrm{ref}}} = \frac{\mathcal{M}_n}{\mathcal{M}_n^{\mathrm{ref}}} .$$

2. Wilson Loops: Systematics

$$W[\mathcal{C}_n] \ := \ \mathrm{Tr}\,\mathcal{P}\exp\left[ig\oint_{\mathcal{C}_n} d\tau\ \dot{x}^\mu(\tau)A_\mu(x(\tau))\right]$$

- The closed contour C_n is made out of the lightlike external momenta in the order dictated by the colour ordering of the amplitude.
- The non-Abelian exponentiation theorem (Gatheral'83; Frenkel-Taylor'84) allows to calculate directly the log of W:

$$\langle W[\mathcal{C}_n]\rangle \ := 1 + \sum_{l=1}^{\infty} a^l W_n^{(l)} \ := \ \exp\sum_{l=1}^{\infty} a^l w_n^{(l)}$$

$$w_n^{(2)} \ = \ W_n^{(2)} - \frac{1}{2}(W_n^{(1)})^2$$

At 2-loops:

- There are five main ingredients to the logarithm of the Wilson loop calculation at two lops for any number of edges, n.

We call them:

the ``hard diagram'' $f_H(p_i, p_j, p_k; Q_{jk}, Q_{ki}, Q_{ij})$

the ``curtain diagram'' $f_C(p_i, p_j, p_k; Q_{jk}, Q_{ki}, Q_{ij})$

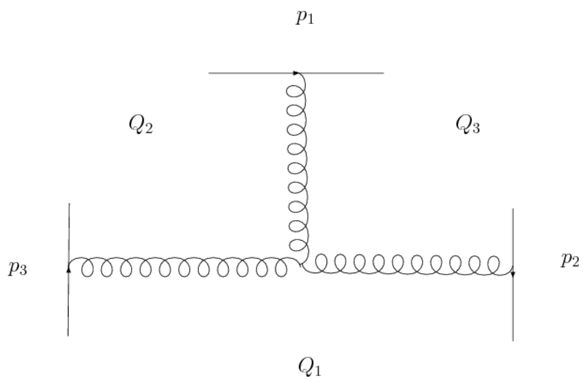
the ``cross diagram'' $f_X(p_i, p_j; Q_{ji}, Q_{ij})$

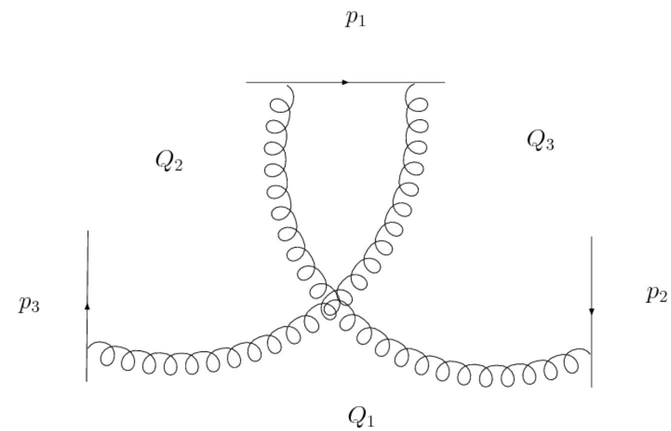
the ``Y diagram'', and $f_Y(p_i, p_j; Q_{ji}, Q_{ij})$

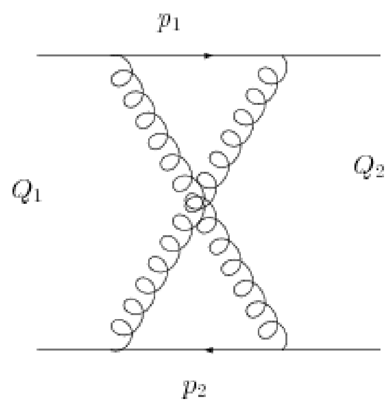
the ``factorised cross diagram"

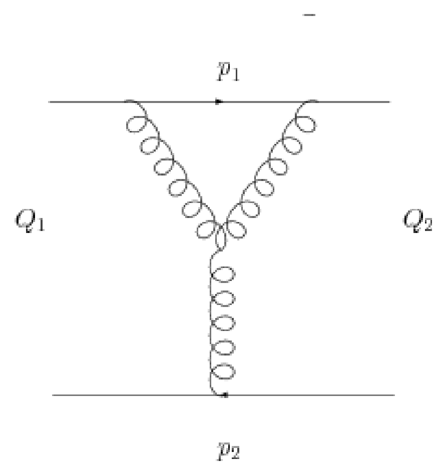
$$(-1/2) f_P(p_i, p_j; Q_{ji}, Q_{ij}) f_P(p_k, p_l; Q_{lk}, Q_{kl})$$

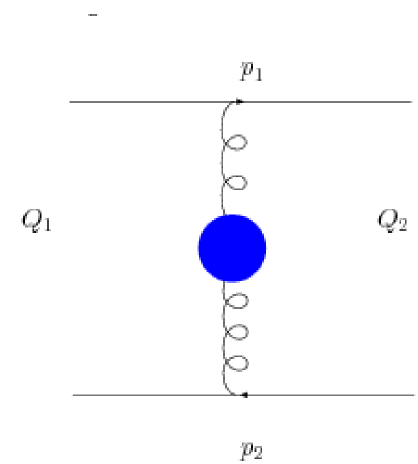
(these are Feynman diagrams arising from the use of the non-Abelian exponentiation theorem)

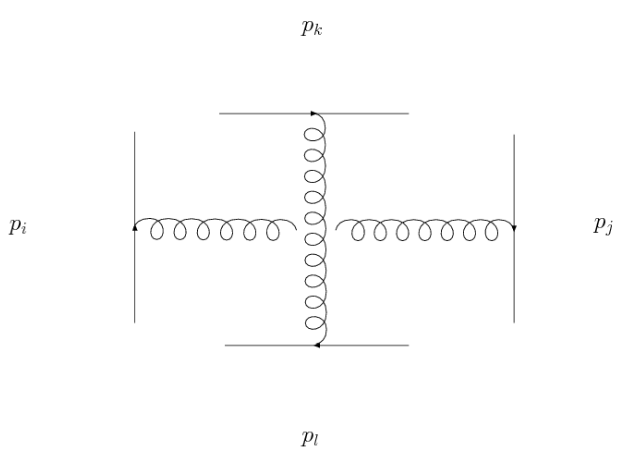

p_1
Q_2
Q_3
p_3
p_2
Q_1


p_1
Q_2
Q_3
p_3
p_2
Q_1


p_1
Q_1
Q_2
p_2


p_1
Q_1
Q_2
p_2


p_1
Q_1
Q_2
p_2


p_k
p_i
p_j
p_l

- The logarithm of the complete n-sided Wilson loop (at 2 loops) is given by the sum over these 5 types of diagrams: [all symmetry factors automatically = 1]

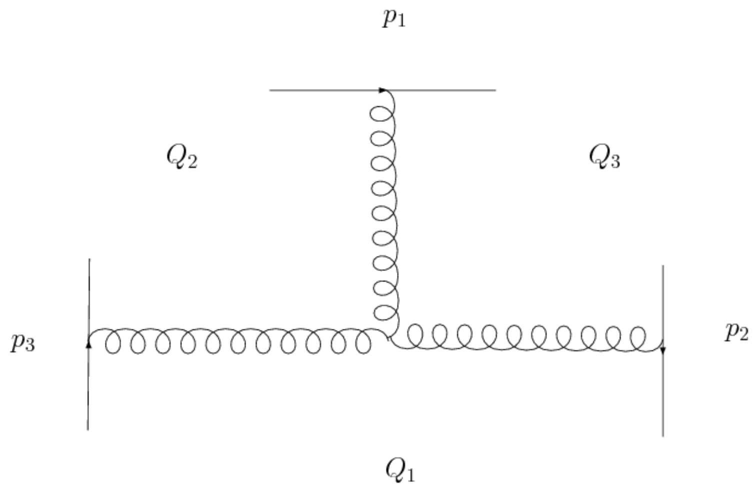
$$w_n^{(2)} = \mathcal{C} \Bigg\{ \sum_{1\le i<j<k\le n} \Big[f_H(p_i,p_j,p_k;Q_{jk},Q_{ki},Q_{ij}) + f_C(p_i,p_j,p_k;Q_{jk},Q_{ki},Q_{ij}) + f_C(p_j,p_k,p_i;Q_{ki},Q_{ij},Q_{jk}) + f_C(p_k,p_i,p_j;Q_{ij},Q_{jk},Q_{ki}) \Big] + \sum_{1\le i<j\le n} \Big[f_X(p_i,p_j;Q_{ji},Q_{ij}) + f_Y(p_i,p_j;Q_{ji},Q_{ij}) + f_Y(p_j,p_i;Q_{ij},Q_{ji}) \Big] + \sum_{1\le i<k<j<l\le n} (-1/2) f_P(p_i,p_j;Q_{ji},Q_{ij}) f_P(p_k,p_l;Q_{lk},Q_{kl}) \Bigg\}$$

$$\mathcal{C} := 2a^2\mu^{4\epsilon}\Big[\Gamma(1+\epsilon)e^{\gamma\epsilon}\Big]^2 \;=\; 2a^2\mu^{4\epsilon}\Big(1+\frac{\pi^2}{6}\epsilon^2\Big) \;+\; \mathcal{O}(\epsilon^3)$$

$$a \;:=\; \frac{g^2 N}{8\pi^2} \qquad \epsilon = -\epsilon_{\rm UV} \qquad \mu_{\rm WL}^2 \;:=\; \pi e^{\gamma}\mu^2 \qquad Q_{ij} = p_{i+1} + p_{i+2} + \dots p_{j-1}$$

Comment 1:

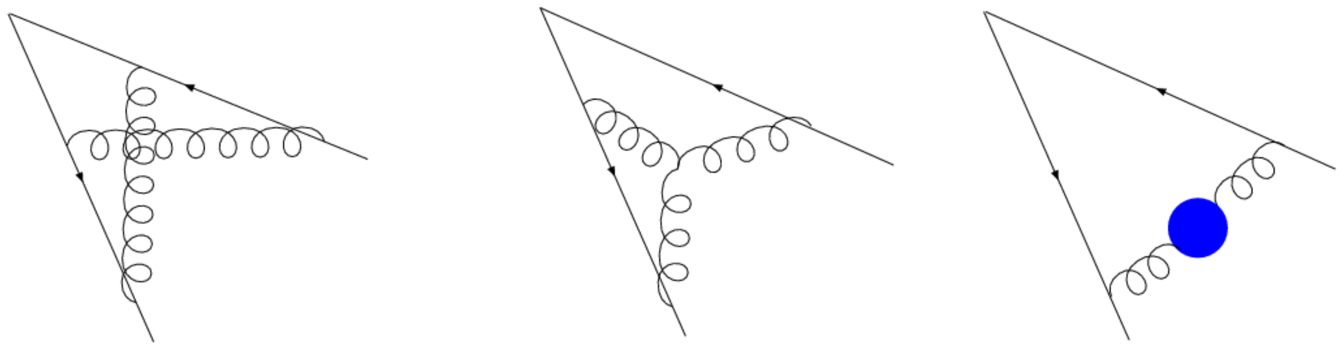
- UV singularities in these integrals depend on whether Q_i=0 or not (i.e. on whether edges are adjacent)



For example f_H has a $1/\varepsilon^2$ singularity if $Q_1=Q_2=0$, $Q_3 \neq 0$,
a $1/\varepsilon$ singularity if $Q_1=0$, $Q_2,Q_3 \neq 0$,
and is finite if $Q_1,Q_2,Q_3 \neq 0$

Comment 2:

- The cusp diagrams are those involving only two consecutive edges:



- Are already included (they are a subset of the 5 above)

$$\mathcal{C} \sum_{i=1}^{n} \Big(f_X(p_i, p_{i+1}; Q_{i+1i}, 0) + f_Y(p_i, p_{i+1}; Q_{i+1i}, 0) + f_Y(p_i + 1, p_i; 0, Q_{i+1i}) \Big)$$

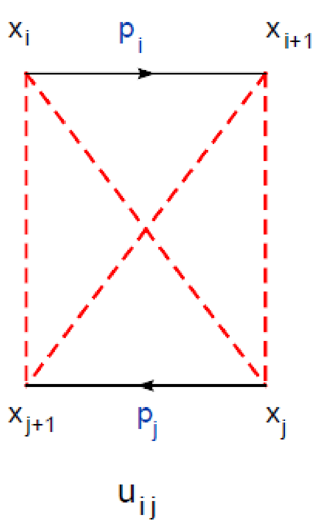
- Always compute the log of the entire Wilson loop =>
  the Remainder is obtained from this by subtracting BDS.

- DHKS conformal Ward identity =>
  Remainder function must be conformally invariant
  $\Rightarrow$ is a function of conformal cross-ratios u
  <= confirmed by our computation

- We do not impose the Gramm determinant constraint =>
  n(n-5)/2 independent on-shell cross-ratios.
  [otherwise: 3n-15 independent cross-ratios]

- Basis:

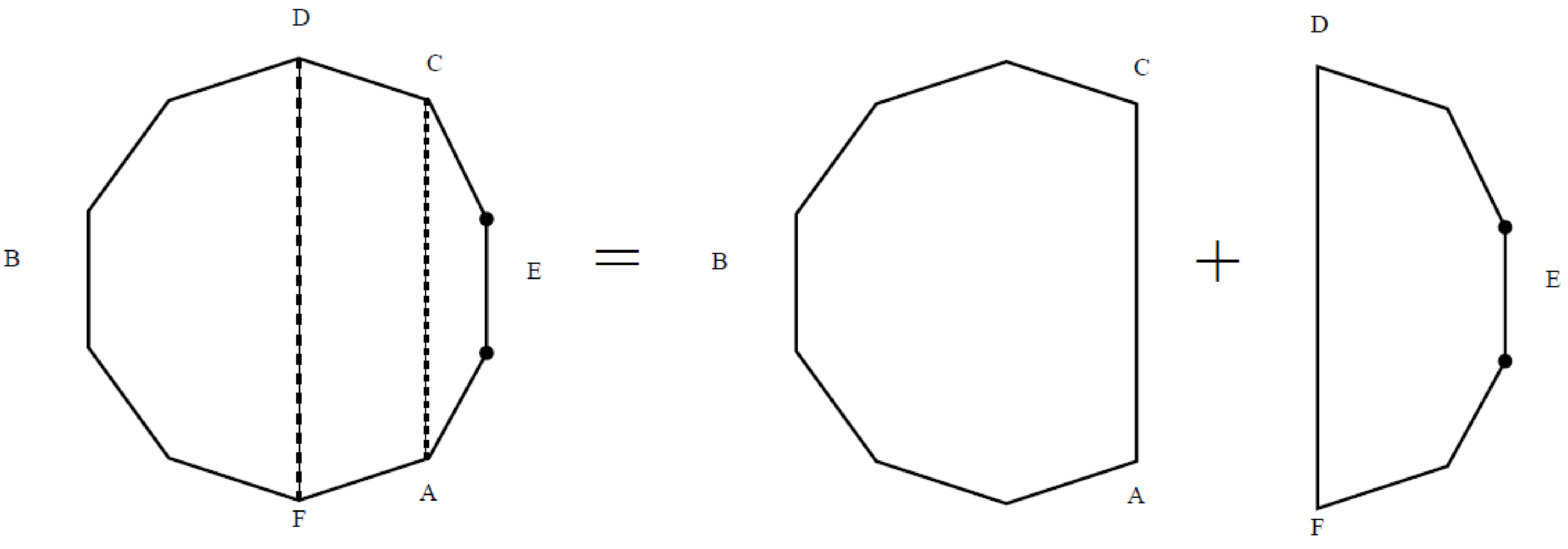
$$u_{ij} := \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$



Multi-Collinear Limits

- A nice property of R_n is the decomposition in the limits where k+1 consecutive external momenta become collinear:

$$\mathcal{R}_n \to \mathcal{R}_{n-k} + \mathcal{R}_{k+4}$$



3. n-gon computations at 2-loops

Anastasiou-Brandhuber-Heslop-VVK-Spence-Travaglini 09
Brandhuber-Heslop-VVK-Travaglini 09, Heslop-VVK'10

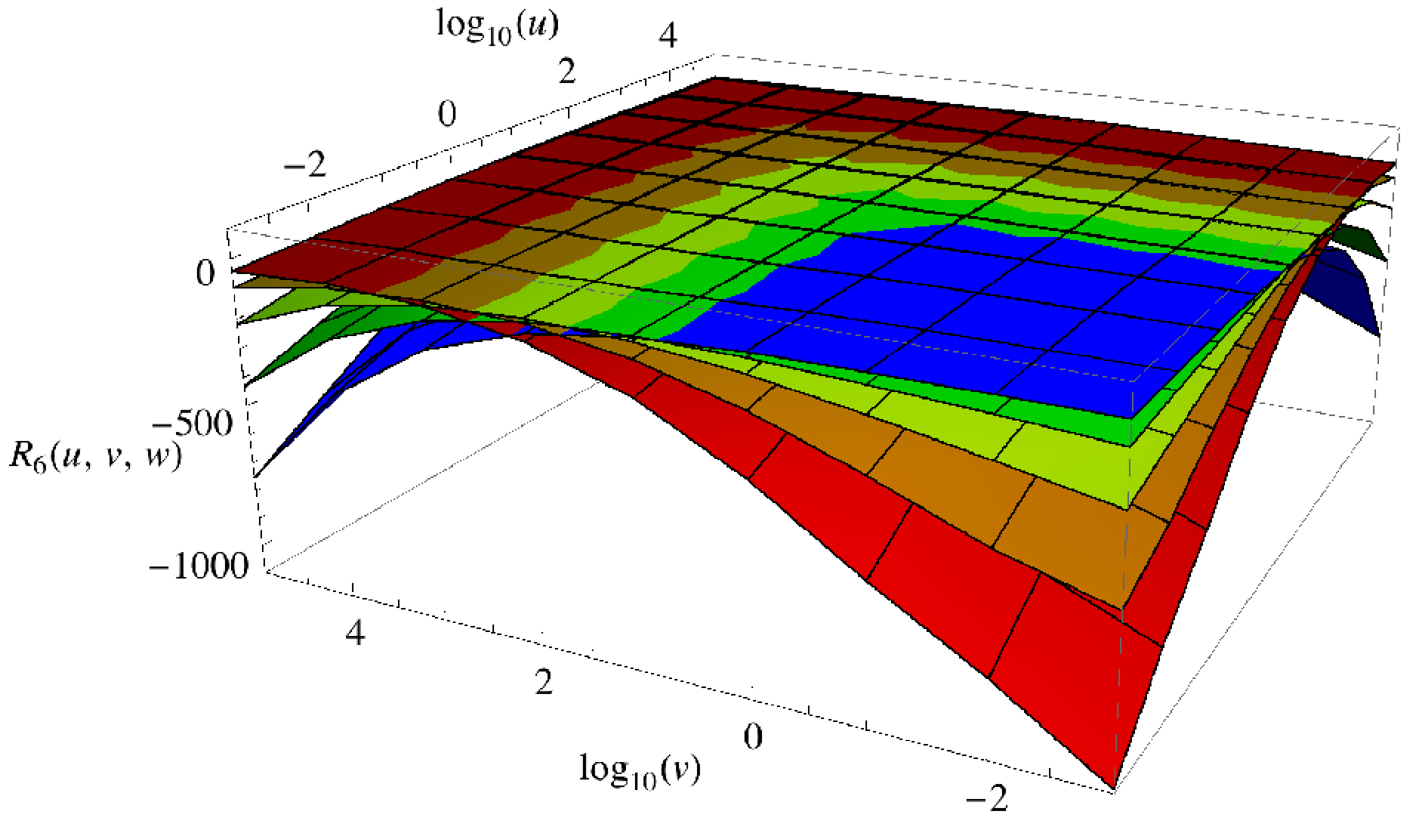
- Wilson loops computed numerically for
  n=4,5,6,7,8 and up to 30 → fully automated procedure
  at a press of a button.

[+ Very recent Del Duca-Duhr-Smirnov analytic result for n=6.]

- Number of distinct diagrams contributing to the n-gon Wilson loops does not increase with n. There is a fixed number of ``master integrals'', which we have computed.
- Verified that the remainder function depends on kinematics only via conformal cross ratios u.
- Studied and checked collinear and multi-collinear limits.

Hexagon Calculations

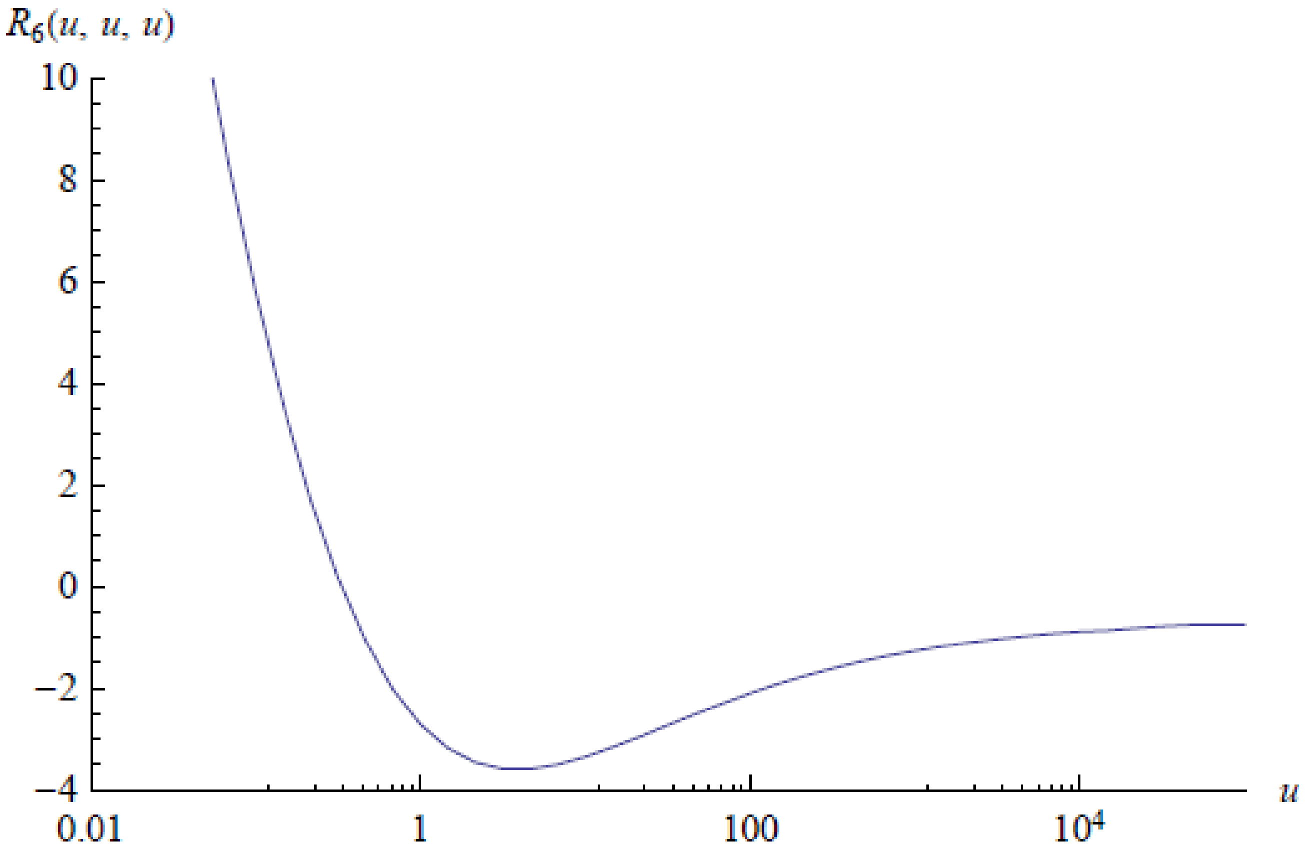
- R_6 of the hexagon at two-loops with u_1=u, u_2=v and u_3=w



- w=1 blue plot, w=10 green plot, w=100 yellow plot, w=1000 orange plot, and w=10000 red plot.

Hexagon Calculations

- A plot of R_6 at two-loops for $u_1=u_2=u_3$



Hexagon Calculations

- The Remainder at strong-coupling was derived more recently by Alday-Gaiotto-Maldacena'09 using integrability, and it takes a very simple form:

$$\mathcal{R}_6^{\text{strong}}(u,u,u) = \frac{\pi}{6} - \frac{1}{3\pi}\phi^2 - \frac{3}{8}(\log^2(u) + 2Li_2(u)) + \text{const}$$

$$u_{ii+3} = \frac{1}{4}\sec^2(\phi/3)$$

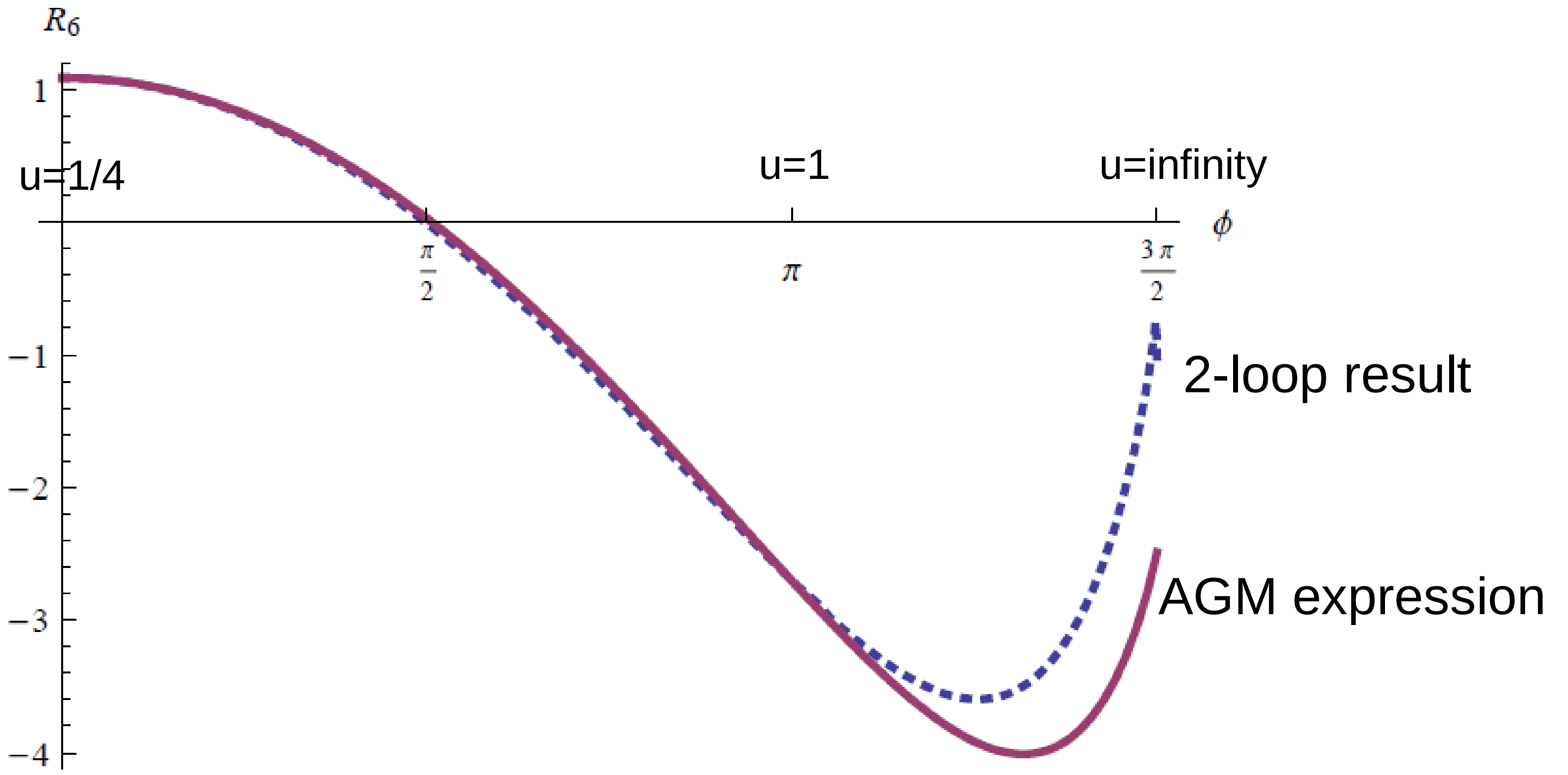
Hexagon Calculations

- It is interesting to compare these results at weak and at strong coupling. Modifying the strong coupling result by introducing 3 coefficients:

$$\mathcal{R}_6^{\mathrm{AGM}}(u,u,u) = c_1(-\frac{\pi}{6}+\frac{1}{3\pi}\phi^2) + c_2(\frac{3}{8}(\log^2(u)+2Li_2(u))) + c_3$$

- a very close (approximate) match with the weak coupling Remainder can be found (by fixing the coefficients).

Hexagon Calculations



The constant c_3 is fixed by the collinear limit $\mathcal{R}_6 \to \mathcal{R}_5$ and with a little work can be found to be $c_3 = -c_2\pi^2/12$. We plot the combined weak coupling result and the AGM expression for $c_1 = 0.263\pi^3$ and $c_2 = 0.860\pi^2$.

- The modified strong-coupling result,

$$\mathcal{R}_6^{\mathrm{AGM}}(u,u,u) = c_1\left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2\right) + c_2\left(\frac{3}{8}\left(\log^2(u) + 2Li_2(u)\right)\right) + c_3$$

and the 2-loop result for R are close,

but cannot be made identical!

$$\mathcal{R}_6^{(2)}(1,1,1) = -\pi^4/36$$

Special value at ¼ from the recent analytical 2-loop expression

Del Duca-Duhr-Smirnov'10:

$$\begin{aligned}\mathcal{R}_6^{(2)}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) &= 3\mathrm{Li}_2\left(\frac{1}{3}\right)\log^2 2 - \frac{9}{2}\mathrm{Li}_2\left(\frac{1}{3}\right)\log^2 3 - \frac{567}{4}\mathrm{Li}_3\left(\frac{1}{3}\right)\log 2 \\ &+ \frac{543}{4}\mathrm{Li}_3\left(-\frac{1}{2}\right)\log 2 + \frac{567}{8}\mathrm{Li}_3\left(\frac{1}{3}\right)\log 3 - \frac{567}{4}\mathrm{Li}_3\left(-\frac{1}{2}\right)\log 3 + \frac{1323}{16}\zeta_3\log 2 \\ &+ \frac{945}{32}\zeta_3\log 3 - \frac{39}{32}\log^4 2 - \frac{257}{64}\log^4 3 + \frac{173}{8}\log 3\log^3 2 + \frac{189}{8}\log^3 3\log 2 - \frac{543}{16}\log^2 3\log^2 2 \\ &- \frac{63}{16}\pi^2\log^2 2 - \frac{181}{64}\pi^2\log^2 3 + \frac{189}{2}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1701}{8}\mathrm{Li}_4\left(\frac{1}{3}\right) - \frac{543}{16}\mathrm{Li}_4\left(-\frac{1}{3}\right) \\ &+ \frac{555}{2}\mathrm{Li}_4\left(-\frac{1}{2}\right) - \frac{9}{2}\mathrm{Li}_2\left(\frac{1}{3}\right)^2 - \frac{567}{16}S_{2,2}\left(-\frac{1}{3}\right) - \frac{567}{4}S_{2,2}\left(-\frac{1}{2}\right) - \frac{2123\pi^4}{2880}\end{aligned}$$

- Why introduce different coefficients at strong coupling?

$$\mathcal{R}_6^{\mathrm{AGM}}(u,u,u) = c_1\left(-\frac{\pi}{6} + \frac{1}{3\pi}\phi^2\right) + c_2\left(\frac{3}{8}\left(\log^2(u) + 2Li_2(u)\right)\right) + c_3$$

The terms on the RHS have different origin: the first is the free energy of an integrable system, while the second arose from subtracting the BDS expression from the cut-off world-sheet area.

Generally: Afree + (BDS-BDSlike) + Aperiods + Aextra

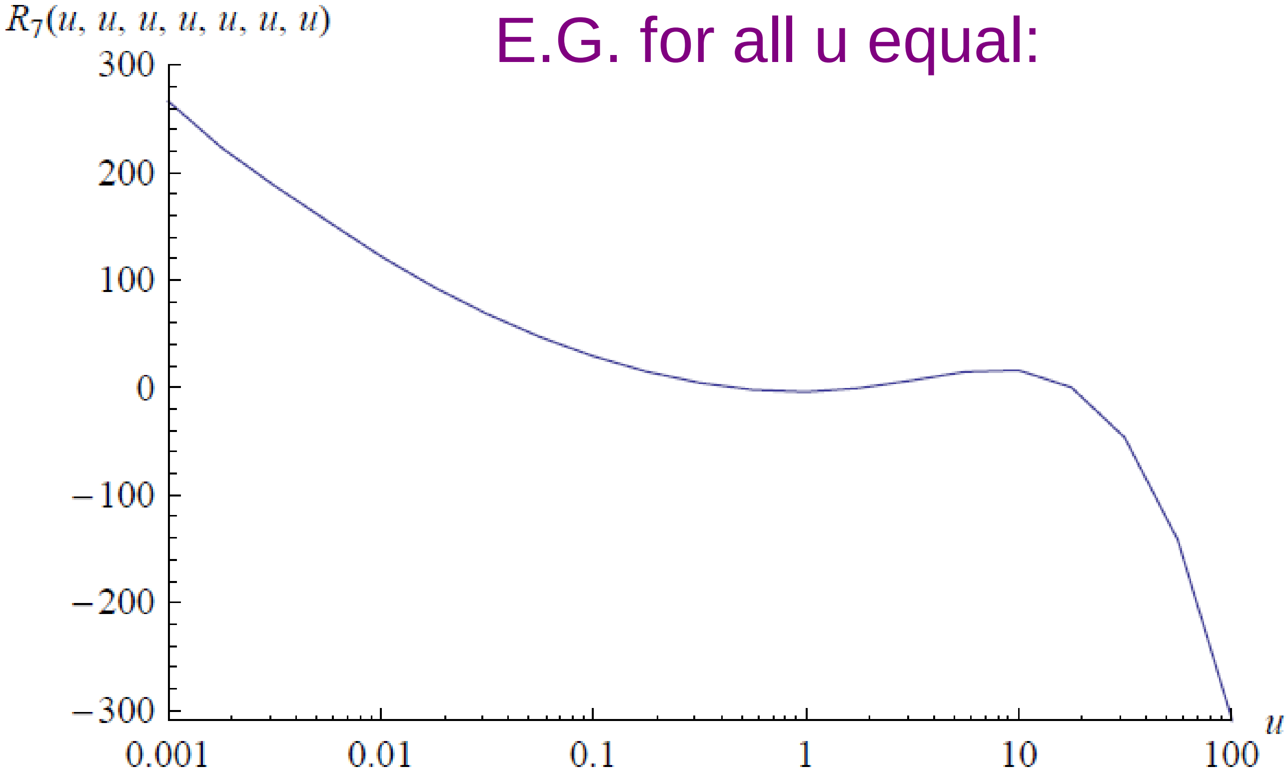
- Hope to express the strong- and the weak-coupling results as linear combinations of certain master functions and then find the coefficients.

  At present we lack the theory giving the basis of master functions (at least at weak coupling).

Seven-point Calculations

- 14 kinematic invariants in total. 7 conformal cross ratios.
- Conformal invariance checked.
- Can compute in any kinematics.

E.G. for all u equal:



Eight-point Calculations

- Twenty kinematic invariants
- Twelve conformal cross ratios:

$$u_{i\, i+3} \ , \quad i = 1, \ldots, 8 \ , \qquad u_{i\, i+4} \ , \quad i = 1, \ldots, 4$$

- Checked conformal invariance of R

- Performed computations in various kinematic settings

  e.g. for general polygons embeddable in AdS_3

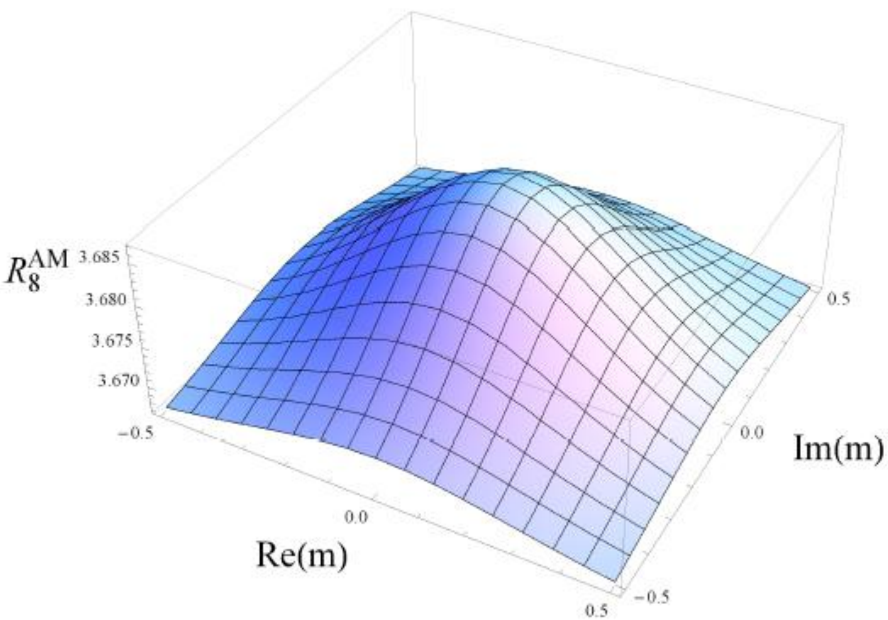
  and/or for regular polygons.

Octagons on the boundary of AdS_3

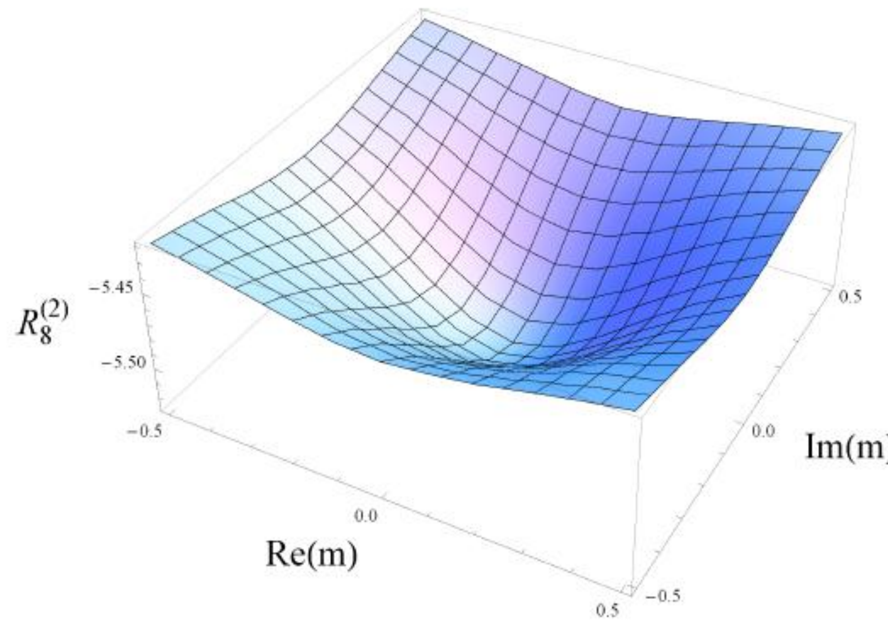
Alday-Maldacena 09

Brandhuber-Heslop-VVK-Travaglini 09

- See the talk of Paul Heslop



Strong coupling



Weak coupling 2-loop

Regular Polygons in 2+1

- 2n-gons which can be embedded into the 1+1-dim boundary of AdS3; conformally equivalent to regular polygons in 2+1 dims

$$u_{ij} = 1 \,, \qquad i - j = \text{odd}$$

$$u_{ij} = 1 - \left(\frac{\sin \frac{\pi}{n}}{\sin \frac{\pi a}{n}} \right)^2 \,, \qquad i - j = 2a \,,$$

- At 2-loops we have:

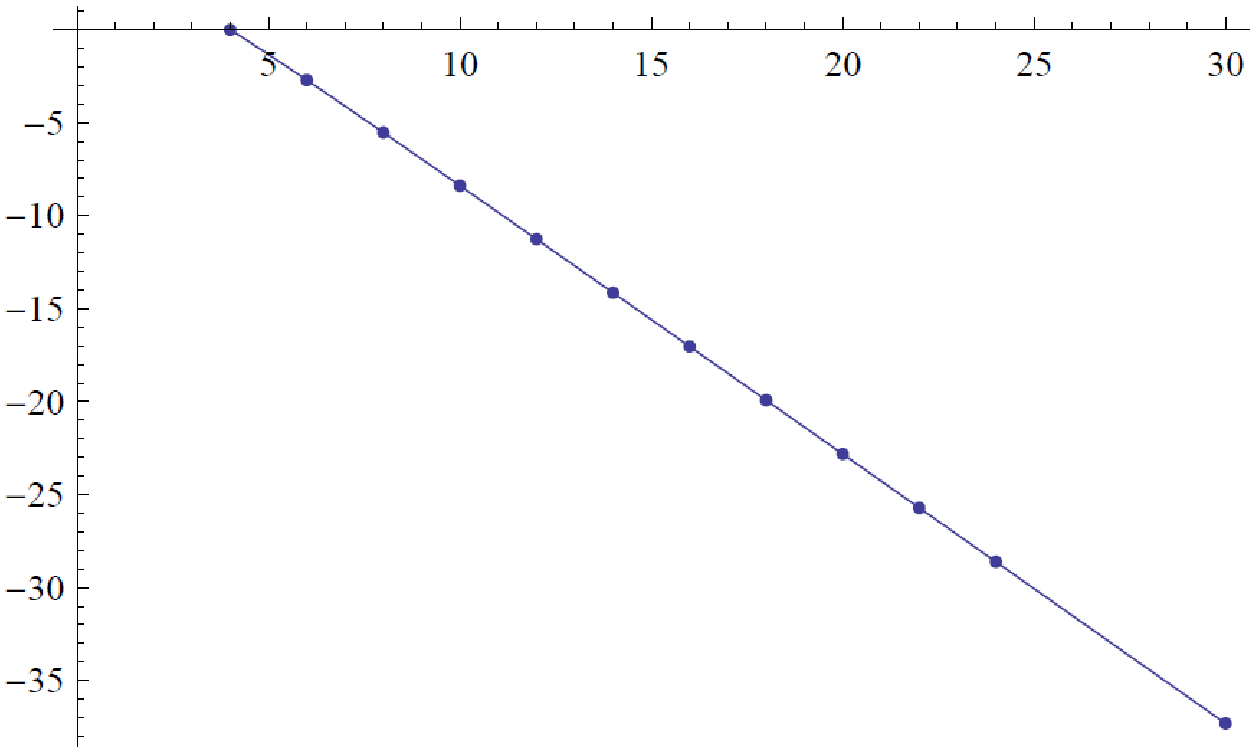
$\mathcal{R}_6^{(2)}(1,1,1) = -\pi^4/36$

Brandhuber-Heslop-VVK-Travaglini'09

$2n$	8	10	12	14	16
$\mathcal{R}_{2n}^{(2)}$	-5.528	-8.386	-11.262	-14.145	-17.035

18	20	22	24	30
-19.926	-22.821	-25.717	-28.614	-37.311

Regular Polygons in 2+1



$$\mathcal{R}^{(2)}_{\text{fit},1} \;=\; \pi\left(-\,0.9255\,n \,+\, 2.026 \,-\, \frac{0.346}{n}\right) \qquad \lesssim 0.01$$

$$\mathcal{R}^{(2)}_{\text{fit},2} \;=\; \pi\left(-\,0.9239\,n \,+\, 1.9955 \,-\, \frac{0.181}{n} \,-\, \frac{0.228}{n^2}\right) \qquad \lesssim 0.0004$$

4. The Ratio of Wilson loops

Heslop-VVK'10

- Want to work with a finite and conformally-invariant object which can be used at strong as well as weak coupling.

- Don't wish to subtract (any specific form of) BDS, don't want to use the Remainder, don't want to heavily rely on any specific regularisation scheme.

- Ideally want to formulate Wilson loop/Amplitude duality in terms of something more closely related to the free-energy contribution appearing at strong coupling in

Alday-Gaiotto-Maldacena , Alday-Maldacena-Sever-Vieira'10

At strong coupling, $a \to \infty$, the quantity of interest is the area of a world sheet ending on the polygonal Wilson loop, $\sqrt{2a}A$.

A is a function of two-particle invariants s and multi-particle invariants $t^{[r>2]}$

$$s_i = (p_i + p_{i+1})^2 \ , \qquad t_i^{[r>2]} = (p_i + p_{i+1} + \ldots + p_{i+r-1})^2$$

The number of independent multi-particle invariants $t_i^{[r]}$ is $n(n-5)/2$ and is the same as the number of independent cross-ratios u_{ij}.

A is infinite (which is a reflection of the divergences of the amplitude/Wilson loop) and requires regularisation.

Alday-Gaiotto-Maldacena , Alday-Maldacena-Sever-Vieira'10

At strong coupling A can be represented as

$$A(s,t) = A_{\text{cutoff}}(s) + A_{\text{finite}}(s,t)\,, \quad A_{\text{cutoff}}(s) = 4\int_{\Sigma_0\,,\,z_{\text{AdS}}>\epsilon_c} d^2w$$

A_{cutoff} depends on the kinematics only through the two-particle invariants s_i

$$A_{\text{cutoff}} = \frac{1}{8}\sum_{i=1}^{n}\left(\log \epsilon_c^2 s_i\right)^2$$

$$-\frac{1}{4}\sum_{i=1}^{n}\left((\log s_i)^2 + \sum_{k=0}^{2K}(-1)^{k+1}\log s_i \log s_{i+1+2k}\right)$$

where $n = 4K+1$ or $n = 4K+3$, and a similar formula holds for even n.

Thus we consider the difference between two areas

$$A(s,t) - A(s,\tilde{t})$$

with the same values of s_i but different values of multi-particle invariants.

Divergent contributions A_{cutoff} cancel in the difference.

More generally,

$$\log\left(\frac{W_n}{W_n^{\text{ref}}}\right) \;:=\; w_n(s,t) - w_n^{\text{ref}}(s,\tilde{t}) \;=\; \text{finite}$$

This is finite at weak coupling as well, since the divergences again depend only of s_i

$$-\frac{1}{2\epsilon^2}\sum_{L=1}^{\infty} a^L \frac{f^{(L)}(\epsilon)}{L^2}\sum_{i=1}^{n}\left(\frac{-s_i}{\mu^2}\right)^{-L\epsilon}$$

Now use the matching between the number of independent $t_i^{[r]}$ and u_{ij} to trade all multi-particle invariants for the cross-ratios $\{s,t\} \to \{s,u\}$.

[Requres that we do not impose the Gramm det constraints and that n is not divisible by 4.]

Next note that in the decomposition

$$A(s,u) = A_{\text{cutoff}}(s) + A_{\text{finite}}(s,u)$$

the conformal anomaly equation is satisfied by A_{cutoff}.

Thus A_{finite} satisfies the homogenious equation and is a funtion only of u

$$A_{\text{finite}} = A_{\text{finite}}(u)$$

Exactly the same reasoning applies, in general, not only to the strong-coupling regime, but also at weak coupling to all orders in perturbation theory and both for scattering amplitudes and Wilson loops.

For Wilson loops the relevant quantity is

$$\log\left(\frac{W_n}{W_n^{\text{ref}}}\right) \;:=\; w_n(s,u) - w_n^{\text{ref}}(s,\tilde{u}) \;=\; \text{finite}(u,\tilde{u})$$

RHS is a function of conformal cross-ratios only.

This is ensured by the fact that $w_n(s,u)$ satisfies anomalous conformal Ward identities a particular solution of which is 'BDS-like' A_{cutoff} which crucially depends only on two-particle invariants.

The Wilson loop/MHV amplitude duality is then

$$\log\left(\frac{W_n}{W_n^{\text{ref}}}\right)(u,\tilde{u}) \;=\; \log\left(\frac{\mathcal{M}_n}{\mathcal{M}_n^{\text{ref}}}\right)(u,\tilde{u})$$

where we treat the reference variables $\tilde{u}$ as fixed.

Finite, conformally-invariant ratio.

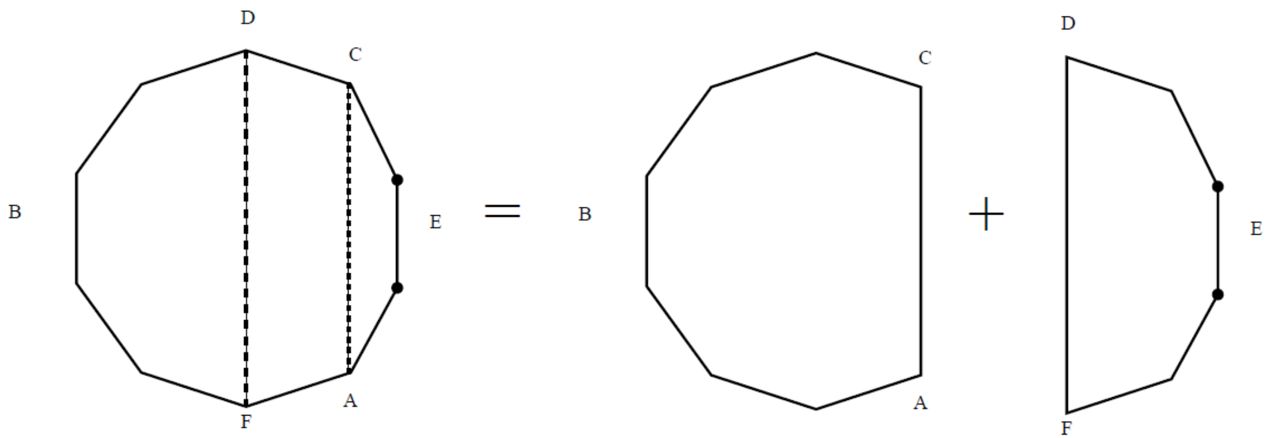
Finite => can remove the log with no loss of info

$$\frac{W_n}{W_n^{\text{ref}}} \;=\; \frac{\mathcal{M}_n}{\mathcal{M}_n^{\text{ref}}}$$

Finite => must be regularisation-independent.
e.g. the same $\mathcal{M}_n/\mathcal{M}_n^{\text{ref}}$ on the Coulomb phase of $\mathcal{N}=4$ SYM where infrared divergences are regulated by the masses (VEVs). As in: Alday-Henn-Plefka-Schuster'09
Henn-Naculich-Schnitzer-Spradlin'10

Multi-Collinear Limits

- k+1 consecutive external momenta become collinear:



$$\mathcal{R}_n \to \mathcal{R}_{n-k} + \mathcal{R}_{k+4}$$

one needs to keep both variables, u and $\tilde{u}$ active

$$w_n(s,u) - w_n^{\text{ref}}(s,\tilde{u}) \;\to\; w_{n-k}(s,u) - w_{n-k}^{\text{ref}}(s,\tilde{u})$$

For general non-MHV amplitudes, if we factor out the corresponding tree level amplitude, the ratio

$$\frac{\left(\mathcal{A}_n^{\text{NonMHV}}/\mathcal{A}_n^{\text{NonMHV tree}}\right)(s,u,h)}{\left(\mathcal{A}_n^{\text{ref}}/\mathcal{A}_n^{\text{ref tree}}\right)(s,\tilde{u},\tilde{h})}$$

is not only finite but also expected to be (dual) conformally invariant.

Of copurse the Wilson loop dual of this non-MHV ratio is not known outside of the strong coupling regime.

5. Continuous family of n-gons

Alday-Maldacena-Sever-Vieira'10 Heslop-VVK'10

For polygons in 4-dimensions, there is a one-parameter family of $\mathcal{Z}_n$-regular polygons for any even n.

The family depends continuously on the parameter ϕ. Varying ϕ one covers a particular slice of the u_{ij}-space.

For general n-gons, ϕ-family starts at a special regular polygon in $(1+2)$ dimensions at $\phi = 0$

as one increases ϕ, it passes through another special polygon in $(1+1)$ dimensions at $\phi = \pi(n-4)/2$ before reaching the "extreme" point where (some of) the u-variables become infinite.

At strong coupling, the ratio of Wilson loops for all ϕ-families is simply equal to the free-energy term:

$$A_{\text{free}} = -\frac{2}{n\pi}\phi^2 + \text{const}$$

We computed weak-coupling expressions for the Wilson loop ratio. For the reference Wilson loop we took the special polygon in 1+1 dimensions

$$\log\left(\frac{W_n(\phi)}{W_n^{\text{ref}}}\right)\ , \qquad \phi_{\text{ref}} = (n-4)\pi/2$$

at 1-loop (nonvanishing!) and 2-loops

for hexagons, octagons and decagons.

Hexagon phi-family

For the regular hexagon the cross-ratios are

$$u_{ii+3} = \frac{1}{4}\sec^2(\phi/3) \qquad i = 1, 2, 3$$

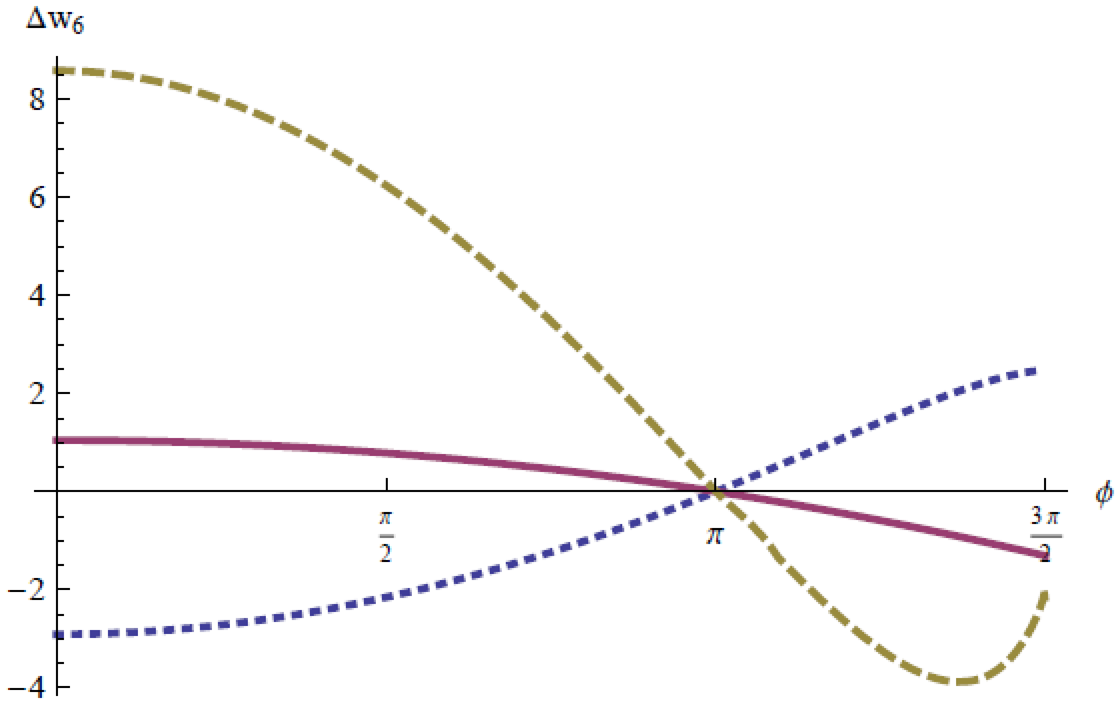
where ϕ varies between 0 and $3\pi/2$.

The Wilson loop ratio has a non-trivial contribution already at one loop

$$w_6^{(1)} - w_6^{(1)\mathrm{ref}} = -\frac{\gamma_K^{(1)}}{2}\left\{\frac{3}{8}\left[\log^2(u) + 2Li_2(1-u)\right]\right\}$$

Two-loop computation is carried out numerically.

Hexagon phi-family



Δw_6 at one loop (dotted), two loops (dashed) and at strong coupling (solid line). The two special regular polygons are at $\phi = 0$ and at $\phi = \pi$, the latter being chosen as the reference point.

Octagon phi-family

For the regular octagon the cross-ratios are

$$u_{ii+3} = \frac{1}{1+\sqrt{2}\cos(\phi/4)} \qquad i = 1, \ldots, 8$$

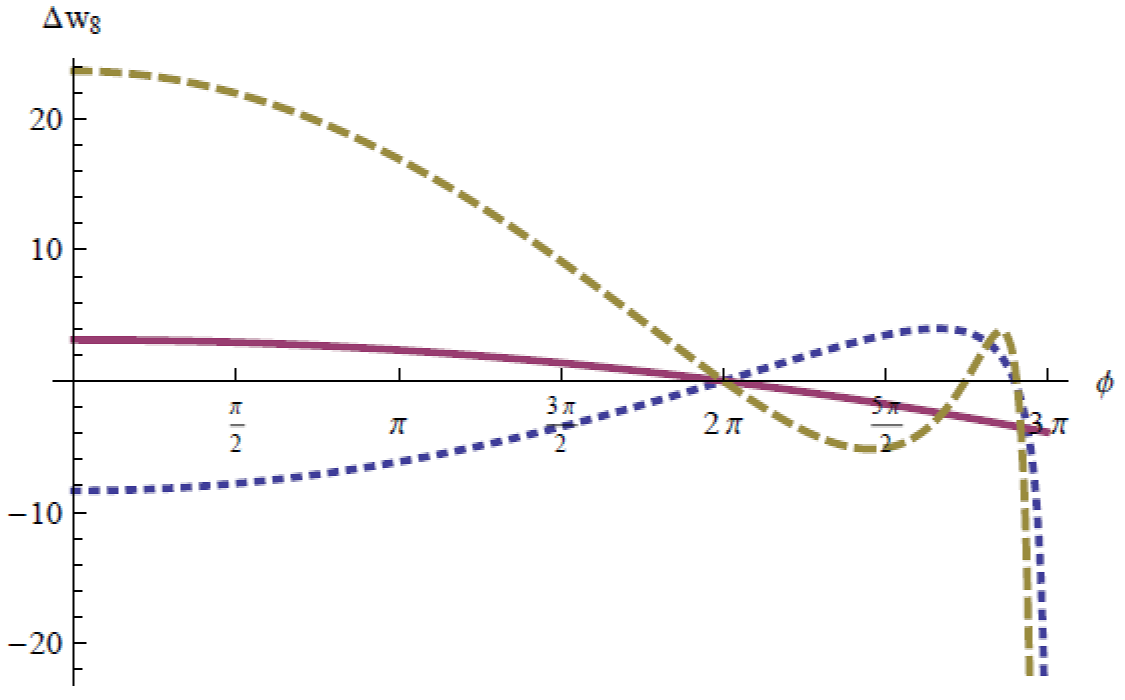
$$u_{ii+4} = \frac{1}{2} \qquad i = 1, \ldots, 4$$

where ϕ varies between 0 and 3π.

The Wilson loop ratio at one loop is

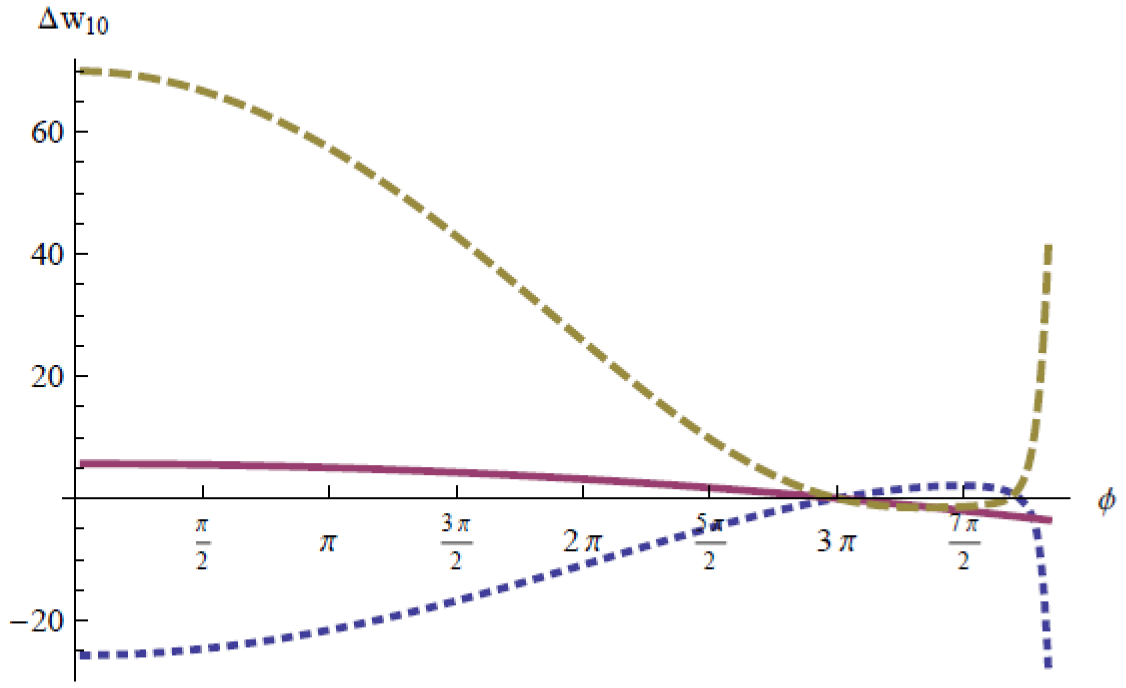
$$\Delta w_8^{(1)} = -\gamma_K^{(1)} \left(\mathrm{Li}_2(1-u_{14}) + \log\left(\frac{u_{14}}{2}\right) \log(u_{14}) \right)$$

Octagon phi-family



Δw_8 at one loop (dotted), two loops (dashed) and at strong coupling (solid line). The two special regular polygons are at $\phi = 0$ and at $\phi = 2\pi$, the latter being chosen as the reference point.

Decagon phi-family



$\Delta w_1 0$ at one loop (dotted), two loops (dashed) and at strong coupling (solid line). The two special regular polygons are at $\phi = 0$ and at $\phi = 3\pi$, the latter being chosen as the reference point.

Conclusions

- Wilson loops at 2-loop level are under control.
- If the duality with amplitudes continues to hold
  → numerical control over planar n-point 2-loop MHV amplitudes.

- Wilson loop/Amplitudes duality can be formulated via $\mathcal{R}_n = \mathcal{R}_n^{WL}$
  or in terms of the ratios: $\frac{W_n}{W_n^{\mathrm{ref}}} = \frac{\mathcal{M}_n}{\mathcal{M}_n^{\mathrm{ref}}}$ Then:

1. All loops are involved including 1-loop
2. No dependence survives beyond gamma-cusp in subleading terms

$$f^{(L)}(\epsilon) := f_0^{(L)} + f_1^{(L)}\epsilon + f_2^{(L)}\epsilon^2$$

3. Any regularisation applies and the logarithm looses no information.

- Similarities between weak-coupling and strong-coupling results.
  [Apply integrable methods at weak coupling??]