Holographic Uniformization

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Outline

Motivation

- A large class of interacting $\mathcal{N}=2$ SCFTs in four dimensions from the $(2,0)$ theory on a Riemann surface Σ_g [Gaiotto-Moore-Neitzke], [Gaiotto].
- The UV theory is well defined for any metric on Σ_g. The IR CFT has information only about the complex structure deformations of Σ_g [Gaiotto].
- What happens to the conformal factor of the metric on Σ_g? Gaiotto: "It is washed out by the RG flow". There are other possibilities - other CFTs, massive theory, free theory. What happens with the metric degrees of freedom of topologically twisted supersymmetric field theories on curved manifolds?
- Direct RG analysis in the field theory is hard. For many twisted field theories there is a useful holographic description [Maldacena-Nuñez], [Gauntlett-Kim-Waldram], ...
- Can this lead to new/interesting geometric flows? This has happened in Physics before! Ricci flow arose from the RG flow equations of the 2D nonlinear sigma model. [Friedan], [Tseytlin], ... Later it was a very useful tool in Mathematics [Hamilton], [Perelman], ...

Topological twists

A supersymmetric field theory on a generic curved manifold will no longer be supersymmetric. To preserve supersymmetry perform a "topological twist", i.e. use the R-symmetry to cancel the space-time curvature [Witten]

$$A_\mu = -\frac{1}{4}\omega_\mu \,, \qquad \rightarrow \qquad \widetilde{\nabla}_\mu \varepsilon = \left(\partial_\mu + \frac{1}{4}\omega_\mu + A_\mu\right)\varepsilon = \partial_\mu \varepsilon = 0$$

Branes in string/M theory wrapping curved cycles preserve supersymmetry in this way [Bershadsky-Sadov-Vafa] . Here - topological twists of the $(2,0)$ theory in 6D and $\mathcal{N}=4$ SYM in 4D on a Riemann surface Σ_g.

Put the A_{N-1} $(2,0)$ theory on a Riemann surface. The supercharges decompose under $SO(1,3)\times SO(2)_{\Sigma_g}\times U(1)_1\times U(1)_2 \subset SO(1,5)\times SO(5)_R$. Define

$$SO(2)' = SO(2)_{\Sigma_g} + aU(1)_1 + bU(1)_2$$

For $a \pm b = \pm 1$ there are 4 invariant supercharges (1/4 BPS). For $b=0$ (or $a=0$) enhancement to 1/2 BPS, i.e. $\mathcal{N}=2$ in 4D.

These twists are realized in M-theory by N M5 branes wrapping a calibrated 2-cycle in a Calabi-Yau manifold - CY_2 for 1/2 BPS and CY_3 for 1/4 BPS.

Setup

In the large N limit the A_{N-1} $(2,0)$ theory is dual to 11D supergravity on $AdS_7 \times S^4$. To study the twist we need only modes that lie within the maximal seven-dimensional gauged supergravity [Maldacena-Nuñez]. In fact we need only the metric, two Abelian gauge fields, $A_\mu^{(i)}$, and two neutral real scalars λ_i. This is a consistent truncation of the maximal 7D theory [Liu-Minasian], [Cvetic, et al.].

The Ansatz for the supergravity fields is

$$ds^2 = e^{2f}(-dt^2 + dz_1^2 + dz_2^2 + dz_3^2) + e^{2h}dr^2 + e^{2g}\frac{dx^2 + dy^2}{y^2}$$

$$A^{(i)} = A_x^{(i)}dx + A_y^{(i)}dy + A_r^{(i)}dr\,, \qquad \lambda_i = \lambda_i(x,y,r)\,, \quad i = 1,2$$

For the 1/2 BPS twist set $A^{(2)} = 0$ and $3\lambda_1 + 2\lambda_2 = 0$ (I define $A \equiv A^{(1)}$ and $\lambda \equiv \lambda_2$). One can then derive the conditions for existence of supersymmetry.

The Riemann surface is a quotient of $\mathbb{H}_2$ by $\Gamma \in PSL(2,\mathbb{R})$. Can treat also S^2 and T^2 with the same approach.

In previous work there was no dependence on the coordinates of the Riemann surface [Maldacena-Nuñez].

BPS equations redux

Remarkably the BPS equations reduce to a single second order nonlinear elliptic PDE

$$(\partial_x^2+\partial_y^2)\Phi+\partial_\rho^2 e^{\Phi} = m^2 e^{\Phi}$$

where

$$\Phi(\rho,x,y)\equiv 2g(\rho,x,y)+4\lambda(\rho,x,y)-2\log y$$

For $m=0$ this is the $SU(\infty)$ Toda equation. It is well-known and integrable [Saveliev]. It arises in various places in Physics - continuum limit of the Toda system, self-dual gravitational solutions in 4D. It arises also in the analysis of 1/2 BPS solutions of 11D supergravity [Lin-Lunin-Maldacena]. The connection with our analysis is unclear.

The flow equation can be rewritten covariantly as

$$\partial_\rho^2 g_{ij}-2R_{ij}-m^2 g_{ij}=0$$

with g_{ij} the metric on an auxiliary Riemann surface $ds^2_{\Sigma'}=e^{\Phi}(dx^2+dy^2)$. Clearly this is very different from Ricci flow

$$\partial_\tau g_{ij}=-2R_{ij}$$

IR and UV analysis

There is a unique AdS_5 IR solution

$$e^{g} = \frac{2^{1/10}}{m}\,, \qquad e^{\lambda} = 2^{1/5}\,, \qquad e^{f} = e^{h} = \frac{2^{3/5}}{m}\frac{1}{r}$$

The metric on the Riemann surface is the constant curvature metric and it is a local attractor!

The UV expansion is ($\zeta \equiv e^{-\frac{m}{2}\rho}$)

$$g(\zeta,x,y) = -\log(\zeta) + g_0(x,y) + g_2(x,y)\zeta^2 + g_{4\ell}(x,y)\zeta^4\log\zeta + g_4(x,y)\zeta^4 + \mathcal{O}(\zeta^5)$$

$$\lambda(\zeta,x,y) = \lambda_2(x,y)\zeta^2 + \lambda_{4\ell}(x,y)\zeta^4\log\zeta + \lambda_4(x,y)\zeta^4 + \mathcal{O}(\zeta^5)$$

The scalar λ is dual to a dimension 4 operator in the $(2,0)$ CFT, $\lambda_2(x,y)$ is related to the source and $\lambda_4(x,y)$ controls the vev. All other functions are fixed in terms of $g_0(x,y)$ and $\lambda_4(x,y)$.

The metric on the Riemann surface is arbitrary in the UV!

Holographic RG flows are somewhat different from Wilsonian RG flows. We need to choose the “correct” $\lambda_4(x,y)$ to flow to the IR fixed point. This is a well-known feature of holographic RG flows [Gubser-Freedman-Pilch-Warner].

Exact solution and linearized analysis - I

The MN exact solution is

$$e^{\varphi_{MN}} = \frac{e^{2m\rho} + 2e^{m\rho} + C}{m^2 e^{m\rho}}$$

Only for $C = 0$ the solution flows to the fixed point in the IR. For other values of C the RG flow is to the Coulomb/Higgs branch, i.e. C is proportional to the vev of the dimension 4 operator.

One can expand around the MN solution, linearize and solve for the small perturbations

$$\lambda = \lambda_{MN}(\rho) + \varepsilon\tilde{\lambda}(\rho, x, y) , \qquad g = g_{MN}(\rho) + \varepsilon\tilde{g}(\rho, x, y)$$

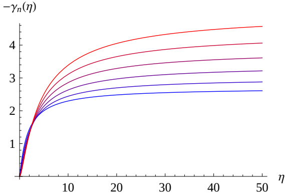
To make progress expand in eigenfunctions of the Laplacian on Σ_g, $\Delta Y^{(n)}(x, y) = -\mu_n Y^{(n)}(x, y)$, $\mu_n \geq 0$

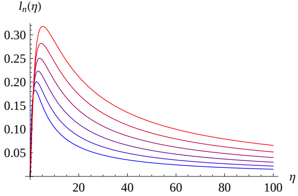
$$\tilde{\lambda} = \sum_n \ell_n(\rho) Y^{(n)}(x, y) , \qquad \tilde{g} = \sum_n \gamma_n(\rho) Y^{(n)}(x, y)$$

An analytic solution for $\ell_n(\rho)$ and $\gamma_n(\rho)$ in terms of hypergeometric functions.

Exact solution and linearized analysis - II

It is convenient to work with $\eta = e^{m\rho}$. The UV is at $\eta \to \infty$ and the IR is at $\eta \to 0$.





This is the expected uniformizing behavior! Any (small) local perturbation of the metric on Σ_g is damped and the constant curvature metric is an IR attractor.

There is a rigorous proof that the uniformizing flows exist globally, i.e. for any metric on the Riemann surface in the UV. The proof uses some technology from nonlinear functional analysis.

Other twists

- A similar analysis can be performed for a (special) 1/4 BPS twist of the $(2,0)$ theory as well as twists of $\mathcal{N}=4$ SYM.
- In all cases the gravitational description is given by a metric, an Abelian gauge field and a single neutral scalar. The BPS equations always imply the equations of motion and reduce to a single, second order, nonlinear elliptic PDE.
- When there is an *AdS* IR vacuum it is an attractor and the holographic RG flow uniformizes the metric on the Riemann surface.
- For all cases we studied there is a rigorous existence proof for the existence of the uniformizing solutions.
- There is a one-parameter family of $\mathcal{N}=1$ twists of the $(2,0)$ theory on a Riemann surface (including T^2 and S^2). All of them flow to $\mathcal{N}=1$ SCFTs in the IR. An infinite discrete family of 4D $\mathcal{N}=1$ SCFTs with explicit gravity duals [Bah-Beem-NB-Wecht]! The central charge of these SCFTs scales as N^3. Interesting to explore these further. Are there generalized $\mathcal{N}=1$ quivers à la Gaiotto? Is there an analog of AGT?

Outlook

- Study Riemann surfaces with punctures along the lines of Gaiotto-Maldacena. This should be straightforward but technically involved.
- Formulate the holographic RG flow as an initial value problem? Holographic analogue of Wilsonian RG flow? Recent attempts to address these questions [Heemskerk-Polchinski], [Faulkner-Liu-Rangamani].
- Study M5 and D3 branes on 3-manifolds and obtain similar geometric flows [Anderson-Beem-NB-Rastelli]. The constant curvature case has already been studied and there are 1/4 BPS and 1/8 BPS $AdS_4 \times M_3$ solutions [Acharya-Gauntlett-Kim], [Gauntlett-Kim-Waldram]. Recent work on the dual 3D SCFTs with $\mathcal{N} = 2$ supersymmetry [Dimofte-Gaiotto-Gukov], [Cecotti-Cordova-Vafa].
- Can these flows teach us anything new about geometry/topology? Can physics help with the familiar singularities of Ricci flow [Perelman], ...? String theory is very good in helping us understand/resolve some singularities - geometric transition, enhançon,...

THANK YOU!