Interacting non-BPS black holes

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Outline

- Time-like Kaluza–Klein reduction
- From solvable algebras to solvable systems
- Interacting non-BPS solutions
- Outlook

Mainly on [G. Bossard, C. Ruef, 1106.5806]

and more to come.

Time-like dimensional reduction

Kaluza–Klein Ansatz

The metric

$$ds^2 = -e^{2U}\left(dt + \omega_\mu dx^\mu\right)^2 + e^{-2U}\,\gamma_{\mu\nu}dx^\mu dx^\nu$$

where γ is the metric on V and $\omega_\mu dx^\mu$ the Kaluza–Klein vector.

And the abelian 1-form fields

$$A^\Lambda = \zeta^\Lambda\left(dt + \omega_\mu dx^\mu\right) + w_\mu^\Lambda dx^\mu$$

Convenient to parametrize G_4/K_4 by $v(\phi) \in G_4$.

Duality symmetry

The equations of motion permit to dualize

$$d\tilde{\zeta}_\Lambda = e^{2U}\mathcal{N}_{\Lambda\Xi}(\phi) \star_\gamma \left(dw^\Xi + \zeta^\Xi d\omega\right) + \mathcal{M}_{\Lambda\Xi}(\phi) d\zeta^\Xi$$

and

$$d\sigma = e^{4U} \star_\gamma d\omega + \frac{1}{2}\left(\zeta_\Lambda d\tilde{\zeta}^\Lambda - \tilde{\zeta}^\Lambda d\zeta_\Lambda\right)$$

Heisenberg gauge invariance

$$\delta\zeta^\Lambda = C^\Lambda \qquad \delta\tilde{\zeta}_\Lambda = \tilde{C}_\Lambda \qquad \delta\sigma = c + \frac{1}{2}\left(\tilde{C}_\Lambda \zeta^\Lambda - C^\Lambda \tilde{\zeta}_\Lambda\right)$$

Symmetry

$$G_4 \ltimes \left(\mathfrak{l}_4 \oplus \mathbb{R}\right)$$

Duality symmetry

Hidden symmetry $SU(2,1)$ of Maxwell–Einstein

$$\mathfrak{su}(2,1) \cong \mathbf{1}^{(-2)} \oplus \mathbb{C}^{(-1)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{u}(1)\right)^{(0)} \oplus \mathbb{C}^{(1)} \oplus \mathbf{1}^{(2)}$$

such that $U, \Phi = \zeta + i\tilde{\zeta}, \sigma$ parametrize $SU(2,1)/U(1,1)$ in a parabolic gauge

$$SU(2,1) \approx \left(\mathbb{C} \oplus \mathbb{R}\right) \times \mathbb{R}_+^* \times U(1,1)$$

as

$$\mathcal{V} = \begin{pmatrix} e^U & e^{-U}\left(\sigma - \frac{i}{2}|\Phi|^2\right) & \Phi \\ 0 & e^{-U} & 0 \\ 0 & -ie^{-U}\Phi^* & 1 \end{pmatrix}$$

equations of motion defined with

$$P = \begin{pmatrix} dU & \frac{1}{2}e^{-2U}\left(d\sigma + \frac{i}{2}(\Phi d\Phi^* - \Phi^* d\Phi)\right) & \frac{1}{2}e^{-U}d\Phi \\ \frac{1}{2}e^{-2U}\left(d\sigma + \frac{i}{2}(\Phi d\Phi^* - \Phi^* d\Phi)\right) & -dU & -\frac{i}{2}e^{-U}d\Phi \\ -\frac{1}{2}e^{-U}d\Phi^* & -\frac{i}{2}e^{-U}d\Phi^* & 0 \end{pmatrix}$$

Beyond STU truncation

$\mathcal{N} = 2$ supergravity in 8 dimensions

- ⋆ Scalars parametrizing $SL(2)/SO(2)$ and $SL(3)/SO(3)$
- ⋆ $\mathbf{2} \times \mathbf{3}$ vectors A_i^α, $\bar{\mathbf{3}}$ 2-forms B^i, a 3-form C^α

Non-supersymmetric truncation in 8 dimensions

- ⋆ Scalars parametrizing $SL(2)/SO(2)$
- ⋆ One 3-form C^α

In four dimensions one gets moduli in $SL(6)/SO(6)$ and 20 electromagnetic fields.

Duality symmetry

Hidden symmetry $E_{6(6)}$ of the pertinent truncation

$$\mathfrak{e}_{6(6)} \cong \mathbf{1}^{(-2)} \oplus \mathbf{20}^{(-1)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_6\right)^{(0)} \oplus \mathbf{20}^{(1)} \oplus \mathbf{1}^{(2)}$$

such that $U, v(\phi), \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma$ parametrize $E_{6(6)}/(Sp(8,\mathbb{R})/\mathbb{Z}_2)$ in a parabolic gauge

$$E_{6(6)} \approx \left(\mathbf{20} \oplus \mathbb{R}\right) \times \left(\mathbb{R}_+^* \times SL(6)\right) \times Sp(8,\mathbb{R})$$

as

$$\mathcal{V}_{\mathbf{27}} = \exp\left[\zeta^\Lambda \mathbf{E}_\Lambda + \tilde{\zeta}_\Lambda \mathbf{E}^\Lambda + \sigma \mathbf{E}\right] \exp\left[U\mathbf{H}\right] v(\phi)_{\mathbf{6}\oplus\mathbf{6}\oplus\overline{\mathbf{15}}}$$

equations of motion defined with

$$P \equiv \frac{1}{2}\left(\mathcal{V}^{-1} d\mathcal{V} + (\mathcal{V}^{-1} d\mathcal{V})^\ddagger\right)$$

Extremal solutions

Under-rotating extremal solutions $V \cong \mathbb{R}^3$ (no ergosphere)

$$\gamma_{\mu\nu} = \delta_{\mu\nu} \;\; ➤ \;\; R_{\mu\nu} = \text{Tr}\; P_\mu P_\nu = 0$$

This implies either

- ⋆ P_μ nilpotent
- ⋆ P_μ admits some imaginary eigen values

At a horizon $U \to -\infty$ and imaginary eigen values produce exponentially growing oscillating modes [J. L. Hörnlund]

↪ regular extremal solutions: $\mathcal{V} \in N \subset G$

Solvable subalgebra

A solvable subalgebra $\mathfrak{n}$ inside $\mathfrak{g}$ admits a grading

$$\mathfrak{n}^{(p)} \cong \mathrm{ad}_{\mathfrak{n}}^{p-1}\,\mathfrak{n} \setminus \mathrm{ad}_{\mathfrak{n}}^{p}\,\mathfrak{n} \ .$$

which can be defined by $h \in \mathfrak{g}$ such that

$$[h, \mathfrak{n}^{(p)}] = 2p\, \mathfrak{n}^{(p)}$$

In the symmetric gauge

$$\mathcal{V} = \exp(-L) \qquad \text{for} \qquad L \in \mathfrak{n} \cap (\mathfrak{g} \ominus \mathfrak{k}^*)$$

and so we chose $h \in \mathfrak{k}^*$.

Solvable system of differential equations

The function L decomposes into $\sum_p L^{(p)}$ such that

$$L^{(p)} \in \mathfrak{n}^{(p)} \cap (\mathfrak{g} \ominus \mathfrak{k}^*)$$

and

$$\begin{aligned}
d \star dL^{(1)} &= 0 \\
d \star dL^{(2)} &= 0 \\
d \star dL^{(3)} &= -\frac{2}{3}\left[dL^{(1)}, [L^{(1)}, \star dL^{(1)}]\right] \\
d \star dL^{(4)} &= -\frac{2}{3}\left[dL^{(1)}, [L^{(1)}, \star dL^{(2)}]\right] - \frac{2}{3}\left[dL^{(1)}, [L^{(2)}, \star dL^{(1)}]\right] - \frac{2}{3}\left[dL^{(2)}, [L^{(1)}, \star dL^{(1)}]\right] \\
d \star dL^{(5)} &= \frac{2}{45}\left[dL^{(1)}, [L^{(1)}, [L^{(1)}, [L^{(1)}, \star dL^{(1)}]]]\right] + \frac{8}{45}\left[[L^{(1)}, dL^{(1)}], [L^{(1)}, [L^{(1)}, \star dL^{(1)}]]\right] \\
&\quad -\frac{2}{3}\left[dL^{(1)}, [L^{(2)}, \star dL^{(2)}]\right] - \frac{2}{3}\left[dL^{(2)}, [L^{(1)}, \star dL^{(2)}]\right] - \frac{2}{3}\left[dL^{(2)}, [L^{(2)}, \star dL^{(1)}]\right] \\
&\quad -\frac{2}{3}\left[dL^{(1)}, [L^{(1)}, \star dL^{(3)}]\right] - \frac{2}{3}\left[dL^{(1)}, [L^{(3)}, \star dL^{(1)}]\right] - \frac{2}{3}\left[dL^{(3)}, [L^{(1)}, \star dL^{(1)}]\right] \\
d \star dL^{(6)} &= \ldots
\end{aligned}$$

Solvable system of differential equations

The explicit solution can then be read from

$$\exp(-2L) = \mathcal{V}\mathcal{V}^{\ddagger} = \begin{pmatrix} \times & e^{-2U}M^{AB}\sigma + \frac{1}{2}e^{-2U}M^{CB}\zeta^{ADE}\zeta_{CDE} & \times \\ \times & e^{-2U}M^{AB} & \times \\ \times & e^{-2U}M^{AD}\zeta_{DBC} & \times \end{pmatrix}$$

and

$$d\omega = \mathrm{Tr}\ \mathbf{E} \star \mathcal{V}P\mathcal{V}^{-1} = -\mathrm{Tr}\ \mathbf{E} \sum_{k=0}^{n-1} \frac{(-2)^k}{(k+1)!} \mathrm{ad}_L{}^k \star dL$$

and similarly

$$dw^{\Lambda} = -\frac{1}{4}\mathrm{Tr}\ \mathbf{E}^{\Lambda} \star \mathcal{V}P\mathcal{V}^{-1} = \frac{1}{4}\mathrm{Tr}\ \mathbf{E}^{\Lambda} \sum_{k=0}^{n-1} \frac{(-2)^k}{(k+1)!} \mathrm{ad}_L{}^k \star dL$$

The *STU* model

The ‘supersymmetric’ orbit $\mathbf{h} = 2\mathbf{H}_0$

$$\begin{aligned} 4 \times \mathfrak{sl}_2 &\cong \mathbf{1}^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2\right)^{(0)} \oplus \mathbf{1}^{(2)} \\ \mathbf{16} &\cong \left(\overline{\mathrm{D0}} \oplus 3 \times \overline{\mathrm{D2}} \oplus 3 \times \overline{\mathrm{D4}} \oplus \overline{\mathrm{D6}}\right)^{(-1)} \oplus \left(\mathrm{D0} \oplus 3 \times \mathrm{D2} \oplus 3 \times \mathrm{D4} \oplus \mathrm{D6}\right)^{(1)} \end{aligned}$$

The subregular orbit $\mathbf{h} = 2\sum_i \mathbf{H}_i$

$$\begin{aligned} 4 \times \mathfrak{sl}_2 &\cong (3 \times \mathbf{1})^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_2\right)^{(0)} \oplus (3 \times \mathbf{1})^{(2)} \\ \mathbf{16} &\cong (\overline{\mathrm{D0}} \oplus \mathrm{D6})^{(-3)} \oplus (3 \times \overline{\mathrm{D2}} \oplus 3 \times \mathrm{D4})^{(-1)} \oplus (3 \times \overline{\mathrm{D4}} \oplus 3 \times \mathrm{D2})^{(1)} \oplus (\mathrm{D0} + \overline{\mathrm{D6}})^{(3)} \end{aligned}$$

The principal orbit $\mathbf{h} = 4\mathbf{H}_0 + 2\sum_i \mathbf{H}_i$

$$\begin{aligned} 4 \times \mathfrak{sl}_2 &\cong \mathbf{1}^{(-4)} \oplus (3 \times \mathbf{1})^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{gl}_1\right)^{(0)} \oplus (3 \times \mathbf{1})^{(2)} \oplus \mathbf{1}^{(4)} \\ \mathbf{16} &\cong \overline{\mathrm{D0}}^{(-5)} \oplus (3 \times \overline{\mathrm{D2}})^{(-3)} \oplus (\mathrm{D6} \oplus 3 \times \overline{\mathrm{D4}})^{(-1)} \oplus (\overline{\mathrm{D6}} \oplus 3 \times \mathrm{D4})^{(1)} \oplus (3 \times \mathrm{D2})^{(3)} \oplus \mathrm{D0}^{(5)} \end{aligned}$$

The *STU* model

The supersymmetric system $\mathbf{h} = 2\mathbf{H}_0$
[F. Denef]

$$\left(\mathrm{D0} \oplus 3 \times \mathrm{D2} \oplus 3 \times \mathrm{D4} \oplus \mathrm{D6}\right)^{(1)}$$

The almost-BPS system $\mathbf{h} = 4\mathbf{H}_0 + 2\sum_i \mathbf{H}_i$
[K. Goldstein and S. Katmadas]

$$(\overline{\mathrm{D6}} \oplus 3 \times \mathrm{D4})^{(1)} \oplus (3 \times \mathrm{D2})^{(3)} \oplus \mathrm{D0}^{(5)}$$

The composite non-BPS system $\mathbf{h} = 2\sum_i \mathbf{H}_i$

$$(3 \times \mathrm{D2} \oplus 3 \times \overline{\mathrm{D4}})^{(1)} \oplus (\mathrm{D0} \oplus \overline{\mathrm{D6}})^{(3)}$$

ADM mass formula in the *STU* model

⋆ BPS composite : $\dim[SL(2)^4/(SO(2)^2 \ltimes \mathbb{R})] - 8 - 1 = 0$

$$M = e^{-i\alpha} Z > 0 = |Z|$$

⋆ almost-BPS composite : $\dim[SL(2)^4] - 8 - 1 = 3$

$$M = \frac{1}{4}\Big(3e^{-i\alpha_0} Z - e^{-i\sum_i \alpha_i} \bar{Z} - \sum_i \big(e^{-i(\alpha_0+\alpha_{i+1}+\alpha_{i+2})} Z_i + e^{-i\alpha_i} \bar{Z}_i\big)\Big) > 0$$

⋆ non-BPS composite: $\dim[SL(2)^4/\mathbb{R}] - 8 - 1 = 2$

$$M = \frac{1}{2}\Big(-e^{i\sum_i \alpha_i} Z + \sum_i e^{i\alpha_i} Z_i\Big) > 0$$

Supersymmetric solutions

The metric reads [B. Bates and F. Denef]

$$e^{-4U} = I_4(\mathcal{H}) \qquad \star\, d\omega = \mathcal{H}^\Lambda d\mathcal{H}_\Lambda - \mathcal{H}_\Lambda d\mathcal{H}^\Lambda$$

and moduli

$$t^i = \frac{\frac{\partial\sqrt{I_4}}{\partial\mathcal{H}_i} - i\mathcal{H}^i}{\frac{\partial\sqrt{I_4}}{\partial\mathcal{H}_0} - i\mathcal{H}^0}$$

such that

$$\mathcal{H}^\Lambda = h^\Lambda + \sum_A \frac{p^\Lambda}{|x - x_A|} \qquad \mathcal{H}_\Lambda = h_\Lambda + \sum_A \frac{q_\Lambda}{|x - x_A|}$$

Almost BPS solutions

[I. Bena, G. Dall'Agata, S. Giusto, C. Ruef and N. P. Warner]
The metric reads

$$e^{-4U} = V\frac{1}{6}c^{ijk}L_iL_jL_k - M^2 \qquad \star\, d\omega = dM - VL_i dK^i$$

and moduli

$$t^i = K^i + \frac{3c^{ijk}L_jL_k}{Vc^{pqr}L_pL_qL_r}\left(-M + ie^{-2U}\right)$$

such that V and K^i are harmonic and
[K. Goldstein and S. Katmadas]

$$\begin{aligned} d\star dL_i &= \frac{1}{2}c_{ijk}\, d\star\left(Vd(K^jK^k) - K^jK^k dV\right) \\ d\star dM &= d\star\left(VL_i dK^i\right) \end{aligned}$$

Non-BPS composite solutions

The metric reads

$$e^{-4U} = V\frac{1}{6}c_{ijk}L^iL^jL^k - M^2 \qquad \star\, d\omega = dM - \frac{1}{2}c_{ijk}L^iL^j dK^k$$

and moduli

$$t^i = K^i + \frac{6L^i}{c_{jkl}L^jL^kL^l}\left(-M + ie^{-2U}\right)$$

such that L^i and K^i are harmonic and

$$\begin{aligned} d\star d\,M &= \frac{1}{2}c_{ijk}d\left(L^iL^j\star dK^k\right) \\ d\star d\,V &= \frac{1}{2}c_{ijk}d\left(L^i\star d(K^jK^k) - K^jK^k\star dL^i\right) \end{aligned}$$

Non-BPS composite solutions

The metric reads

$$e^{-4U} = V\frac{1}{6}c_{ijk}L^iL^jL^k - M^2 \qquad \star\, d\omega = dM - \frac{1}{2}c_{ijk}L^iL^j dK^k$$

with

$$L^i = l^i + \sqrt{2}\sum_A \frac{p_A^i}{|x-x_A|} \qquad K^i = k^i + \sqrt{2}\sum_A \frac{\gamma_A p_A^i}{|x-x_A|}$$

and

$$\begin{aligned} M &= \sum_A \alpha_A \frac{\cos\theta_A}{|x-x_A|^2} + \dots \\ V &= \frac{6}{c_{ijk}l^il^jl^k} - \frac{1}{2}c_{ijk}l^ik^jk^k - \sqrt{2}\sum_A \frac{p_A^0 + \dots}{|x-x_A|} + 2\sum_A \gamma_A\alpha_A \frac{\cos\theta_A}{|x-x_A|^2} + \frac{1}{2}c_{ijk}L^iK^jK^k + \dots \end{aligned}$$

Non-BPS composite solutions

The metric reads

$$e^{-4U} = V\frac{1}{6}c_{ijk}L^iL^jL^k - M^2 \qquad \star\, d\omega = dM - \frac{1}{2}c_{ijk}L^iL^jdK^k$$

with

$$L^i = l^i + \sqrt{2}\sum_A \frac{p_A^i}{|x - x_A|} \qquad K^i = k^i + \sqrt{2}\sum_A \frac{\gamma_A p_A^i}{|x - x_A|}$$

and

$$\begin{aligned} q_{A\,i} &= c_{ijk}p_A^j\left(\gamma_A l^k - k^k + \sum_{B\neq A}(\gamma_B - \gamma_A)\frac{p_B^k}{|x_A - x_B|}\right) \\ p_A^0 &= 0 \end{aligned}$$

Non-BPS composite solutions

Let us define d^i such that

$$q_i = c_{ijk} p^j d^k$$

then for $p^0 = 0$

$$I_4(q,p) = \frac{2}{3} c_{ijk} p^i p^j p^k \left(-q_0 + \frac{1}{2} c_{lpq} p^l d^p d^q \right)$$

using a Jordan algebra identity.

Non-BPS composite solutions

The ADM mass

$$M_{\mathrm{ADM}} = -\frac{1}{2}\Big(Z + (t^i - \bar{t}^i)D_i Z\Big)_0$$

identical to [E. Gimon, F. Larsen and J. Simón] for single centre

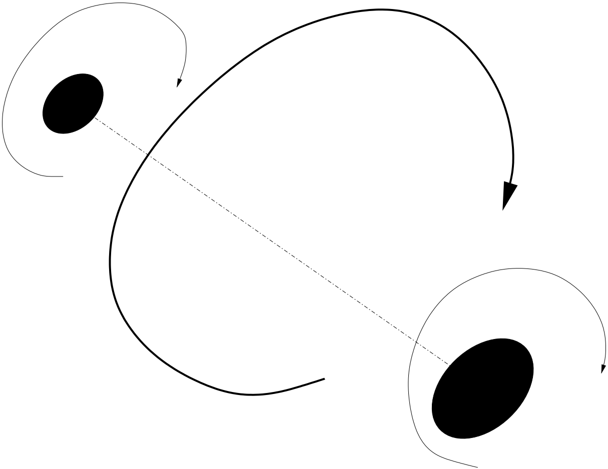
Angular momentum

$$J = -\sum_A \alpha_A + \sum_{A>B}\left(p_A^\Lambda q_{B\,\Lambda} - p_B^\Lambda q_{A\,\Lambda}\right)$$

Horizon area

$$A_A = 4\pi\sqrt{-I_4(q_{A\,\Lambda}, p_A^\Lambda) - \alpha_A^2}$$

Two-centre solution



Two-centre solution stability

[A. Ceresole, G. Dall'Agata, S. Ferrara and A. Yeranyan]

$$
\begin{aligned}
M_{\rm ADM} &= W[Z_\Lambda(q_\Lambda, p^\Lambda), \beta^s] \\
&= W[Z_\Lambda(q_{A\,\Lambda}, p_A^\Lambda), \beta^s] + W[Z_\Lambda(q_{B\,\Lambda}, p_B^\Lambda), \beta^s] \\
&< W[Z_\Lambda(q_{A\,\Lambda}, p_A^\Lambda), \beta^*(Z_\Lambda)] + W[Z_\Lambda(q_{B\,\Lambda}, p_B^\Lambda), \beta^*(Z_\Lambda)]
\end{aligned}
$$

with

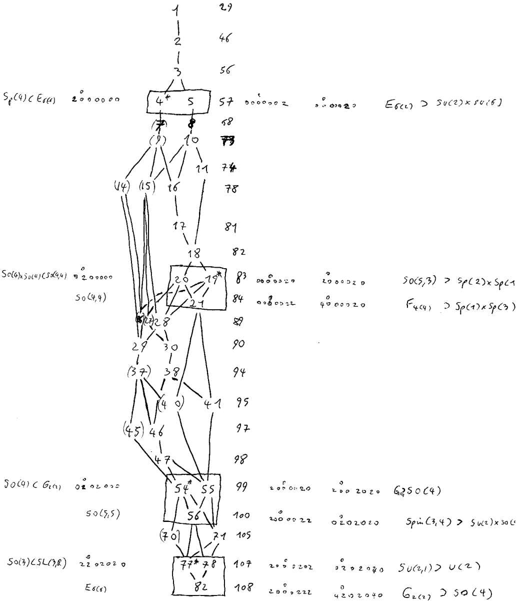
$$
\left.\frac{\partial W[\beta]}{\partial \beta}\right|_{\beta=\beta^s} \neq 0 \qquad \left.\frac{\partial W[\beta]}{\partial \beta}\right|_{\beta=\beta^*} = 0
$$

therefore

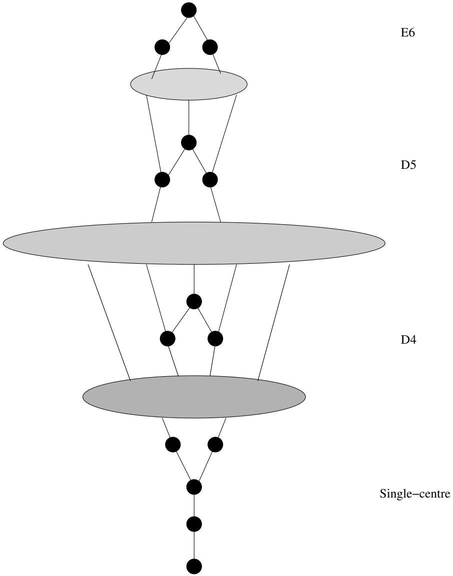
$$
M_{\rm ADM} < M_{{\rm ADM}\,A} + M_{{\rm ADM}\,B}
$$

$$
\beta^s = \beta^* \Longleftrightarrow q_i = -c_{ijk}\, p^j\, \mathrm{Re}[(1 - is)t_0^k]
$$

$E_{8(8)}$ closure diagram $\cap\, \mathfrak{e}_{8(8)} \ominus \mathfrak{so}^*(16)$



$E_{8(8)}$ closure diagram $\cap\, \mathfrak{e}_{8(8)} \ominus \mathfrak{so}^*(16)$



Outlooks

⋆ It is very appealing that the most general systems have 56 harmonic functions for $\mathcal{N} = 8$ and generalise the three, BPS, almost BPS and non-BPS systems.

However, the same Ansatz of harmonic functions does not permit to find regular solutions in higher degree orbits

↪ Generalisations including non-physical poles or multipole harmonics?

⋆ Would help to have the explicit solution

$$\Delta f_{ABC} = \frac{1}{|x - x_A|} \nabla \frac{1}{|x - x_B|} \nabla \frac{1}{|x - x_C|}$$

⋆ Need to study the general solution to find the duality invariant equations for the distances in terms of the charges, and domain of stability.

⋆ Generalisation to arbitrary (non-symmetric) cubic pre-potential.