Flux Attractors and Generating Functions

Adapting Black Hole Attractors to String Phenomenology

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Introduction

- Realistic string theory compactifications must stabilize moduli, *e.g.* by adding fluxes in the compactification.
- One aspect: ***physical variables*** (*e.g.* moduli) are determined as function of ***vacuum data*** (*e.g.* fluxes).
- Reminder: the ***black hole attractor*** mechanism determines the ***scalar field at the horizon*** in terms of ***black hole charges***.
- Goal in this talk: develop ***similarities*** and ***differences*** of black hole attractors and ***flux attractors***.
- The results will be illuminating ***conceptually*** and also (somewhat) ***practical***: they solve explicit examples.

References: FL and O'Connell: 0905.2130; FL, Robbins, O'Connell: 0912.4448; Anguelova, FL, O'Connell: 1006.4981 .

Setting: Black Hole Attractors

- Type IIB N=2 compactifications with a prepotential (no global issues).

- Just ***BPS black holes***.

- Just fixed point behavior: no flows, multicenter solutions.

- Focus on $N = 2$ ***vector fields*** interpreted as D3 branes: decouple hypers.

- ***Supergravity limit***: large black hole charges (no quantization conditions), canonical Kähler potential.

Setting: Flux Attractors

- Type IIB O3 (some O7?) flux compactifications with a prepotential (no global issues).

- Just ***no scale vacua*** (these are analogous to BPS).

- Just ***fixed point*** behavior: no cosmological evolution.

- Focus on ***orientifold projections*** of $N = 2$ vector fields but also the inevitable ***complex dilaton***.

- ***Supergravity limit***: large black hole charges (no quantization conditions, ***no tadpole condition***), canonical Kähler potential.

Black Hole Attractor: Geometry

The spacetime central charge

$$W_{\rm BH} = \int F_3 \wedge \Omega_3 \ .$$

Condition of vanishing Kähler derivative

$$D_i W_{\rm BH} = \int F_3 \wedge D_i \Omega_3 = \int F_3 \wedge \chi_i = 0 \ .$$

Geometric interpretation: the χ_i are $(2,1)$ forms so F_3 has no $(1,2)$ components.

The condition that F_3 is real then gives the ***geometric form of the attractor equation***

$$F_3 = 2{\rm Re}\left(C \Omega_3 \right) \ .$$

C is a complex constant with Kähler weight $(0,1)$.

Flux Attractor: Geometry

The ***complex*** three-form flux of type IIB on a CY orientifold

$$G_3 = F_3 - \tau H_3 \ ,$$

gives the low energy $N = 1$ GVW superpotential

$$W = \int_{\rm CY} G_3 \wedge \Omega_3 \ .$$

The $F_i = 0$ condition is equivalent to G_3 having no $(1,2)$ component:

$$F_i = D_i W = \int_{\rm CY} G_3 \wedge D_i \Omega_3 = \int_{\rm CY} G_3 \wedge \chi_i \ .$$

The $F_\tau = 0$ condition is equivalent to G_3 having no $(3,0)$ component:

$$F_\tau = D_\tau W = D_\tau \int_{\rm CY} G_3 \wedge \Omega_3 = -\frac{1}{\tau - \bar\tau} \int_{\rm CY} \bar G_3 \wedge \Omega_3 \ .$$

Flux attractor equation in geometric form (condition that G_3 is $(0,3)$+$(2,1)$:

$$G_3 = \overline{C\Omega}_3 + C^i D_i \Omega_3 \ .$$

C, C^i are complex constants with Kähler weight $(0,1)$.

Symplectic covariant formulation takes $C^i D_i \to L^I D_I$ so

$$G_3 = \overline{C\Omega}_3 + L^I D_I \Omega_3 \ .$$

The L^I are subject to the constraint

$$L^I \partial_I K = 0 \ .$$

The flux attractor equation is equivalent to the familiar ISD (Imaginary-Self-Duality) condition on the complex flux G_3:

$$*_6 G_3 = i G_3 \ .$$

Drawback: the moduli are implicit in the ISD form .

No Scale vs. SUSY

The superpotential does not depend on Kähler moduli so those are not stabilized. Their F-terms are

$$F_a = D_a W = W \partial_a K_t \ ,$$

with homogeneity ensuring $\sum_a |\partial_a K_t|^2 = 3$ such that

$$V = e^K \left(\sum_{A=i,\tau,a} |D_A W|^2 - 3|W|^2 \right) = e^K \left(\sum_i |D_i W|^2 + |D_\tau W|^2 \right) \ .$$

Solutions to F_i, F_τ-flatness conditions $D_i W = D_\tau W = 0$ are absolute minima of the potential and correspond to Minkowski vacua.

These “no-scale” vacua are supersymmetric exactly when $W = 0$: if $W \neq 0$, the $F_a \neq 0$ break SUSY.

Real Basis

Component form of attractor equations: expand in canonical basis of ***real*** 3-forms $\{\alpha_I, \beta^I\}$.

Moduli enter through the complex three-form

$$\Omega_3 = Z^I \alpha_I - F_I \beta^I \ .$$

The symplectic section (Z^I, F_I): a projective coordinate Z^I on moduli space and also $F_I = \frac{\partial F}{\partial Z^I}$ (F=prepotential).

Black hole charges in real basis:

$$F_3 = P^I \alpha_I - Q_I \beta^I \ .$$

Complex fluxes in real basis:

$$G_3 = m^I \alpha_I - e_I \beta^I \ .$$

BH Attractors in Components

Black hole attractor equations in component form

$$P^I = 2\mathrm{Re}[CZ^I]\,,$$
$$Q_I = 2\mathrm{Re}[CF_I]\,.$$

- Input: $2(n+1)$ ***real charges*** $\{Q_I, P^I\}$, $I = 0, \ldots, n$.

- Output: n ***complex moduli***, *eg.* $z^I = Z^I/Z^0$, $I = 1, \ldots, n$.

- ***More output***: the ***overall scale***, *eg.* CZ^0. It is equivalent to the Kähler invariant spacetime central charge CW_{BH}.

Flux Attractors in Components

Flux attractor equations in component form

$$\begin{aligned} m^I &= \overline{CZ}^I + L^I \, , \\ e_I &= \overline{CF}_I + L^J F_{JI} \, . \end{aligned}$$

- Input: $2(n+1)$ ***complex fluxes*** $\{e_I, m^I\}$, $I = 0, \ldots, n$.
- Output: $n+1$ ***complex moduli***, *eg.* $z^I = Z^I/Z^0$ ***and*** τ .
- More output: $n+1$ ***complex parameters*** L^I subject to the constraint (redundancy of L^I):

$$\overline{CF}_I L^I - \overline{CZ}^I L^J F_{IJ} = 0 \, .$$

- Even more output: the ***overall scale***, *eg.* CZ^0. It is equivalent to the Kähler invariant spacetime ***on-shell superpotential*** CW .

What do the L^I Mean?

Vanishing first derivatives of the superpotential project out $(1,2)$ and $(3,0)$ components of the superpotential.

But ***second derivatives of the superpotential are independent parameters***:

$$m_{\alpha\beta} = e^{K/2} D_\alpha D_\beta W \ .$$

The counting problem revisited: there are $2(n+1)$ input fluxes.

The output: $n+1$ VEVs of moduli ***and*** $n+1$ mass parameters.

The mass parameters can be tuned independently: they are *not* determined by the VEVs of the moduli.

Solving BH Attractor Eq's

BH attractor equations:

$$P^I = 2\mathrm{Re}[CZ^I] \,, \qquad Q_I = 2\mathrm{Re}[CF_I] \,.$$

Solution in terms of unknown real functions $\mathcal{S}^I$, $\mathcal{S}_I$:

$$2CZ^I = P^I + i\mathcal{S}^I \,, \qquad 2CF_I = Q_I - i\mathcal{S}_I \,,$$

that ***satisfy the Maxwell relations***

$$\frac{\partial \mathcal{S}^I}{\partial P^J} = \frac{\partial \mathcal{S}_J}{\partial Q_I} \,.$$

So a ***generating function*** exists such that

$$\mathcal{S}^I = \frac{1}{\pi}\frac{\partial \mathcal{S}}{\partial Q_I} \,, \qquad \mathcal{S}_I = \frac{1}{\pi}\frac{\partial \mathcal{S}}{\partial P^I} \,.$$

The generating function $\mathcal{S}$ encodes all dependence of moduli on black hole charges.

Derivation of Maxwell Relations

F_I are generated by a prepotential so

$$F_I = \partial_I F = \frac{\partial F}{\partial Z^I} \; .$$

The attractor equations give the exact differential

$$4C^2 dF = (Q_I - i\mathcal{S}_I)(dP^I + id\mathcal{S}^I) \; .$$

So

$$d\left(\mathrm{Im} 4C^2 F\right) = -\mathcal{S}_I dP^I + Q_I d\mathcal{S}^I \; .$$

Legendre transform gives the Maxwell relation

$$d\left(Q_I \mathcal{S}^I - \mathrm{Im} 4C^2 F\right) = \mathcal{S}_I dP^I + \mathcal{S}^I dQ_I \; .$$

Bonus: the generating function is related to the prepotential

$$\mathcal{S} = Q_I \mathcal{S}^I - \mathrm{Im} 4C^2 F \; .$$

Solving Flux Attractor Equations

Solution of attractor equations in terms of potentials ϕ^I, θ_I

$$\overline{CZ}^I = \frac{1}{2}(m^I + \phi^I)\,, \qquad L^I = \frac{1}{2}(m^I - \phi^I)\,,$$
$$\overline{CF}_I = \frac{1}{2}(e_I + \theta_I)\,, \qquad L^J F_{JI} = \frac{1}{2}(e_I - \theta_I)\,.$$

The potentials satisfy ***Maxwell relations***

$$\frac{\partial \phi^I}{\partial \overline{m}^J} = -\frac{\partial \theta_J}{\partial \overline{e}_I}\,.$$

So a real ***generating function*** exists such that

$$\phi^I = 2i\tau_2 \frac{\partial \mathcal{G}}{\partial \overline{e}_I}\,, \qquad \theta_I = -2i\tau_2 \frac{\partial \mathcal{G}}{\partial \overline{m}^I}\,.$$

$\mathcal{G} = \mathcal{G}(e_I, m^I, \overline{e_I}, \overline{m}^I)$ encodes all the dependence of moduli on fluxes.

The Complex Dilaton τ

The identification of the generating function $\mathcal{G}$ above ignores:

1. Manipulations of differentials do not allow variations of τ.

2. The constraint (redundancy of L^I):

$$0 = e_I \phi^I - \theta_I m^I = e_I \frac{\partial \mathcal{G}}{\partial \overline{e}_I} + m^I \frac{\partial \mathcal{G}}{\partial \overline{m}^I} \; .$$

Pleasant bonus: these issues “cancel”.

Precise statement: extremizing $\mathcal{G}$ over τ yields the constraint!

There is a practical flexibility: can insert the physical value of τ from the outset ***or*** solve the constraint in the end.

BH Example: STU-model

$N = 2$ SUGRA model with prepotential:

$$F = \frac{Z^1 Z^2 Z^3}{Z^0} \; .$$

Geometry: type IIB on orbifold $T^6/Z^2 \times Z^2 \simeq T^2 \times T^2 \times T^2$.

The black hole attractor has 8 ***real charges***: D3 branes with one direction on each T^2 give $2^3 = 8$ distinct 3-cycles.

The black hole attractor determines 3 ***complex moduli*** and the ***complex spacetime central charge***, a total of 8 real parameters.

The generating function is the black hole entropy $S = \pi\sqrt{I_4}$ where

$$\begin{aligned} I_4 \; = \; & 4(Q_0 P^1 P^2 P^3 - P^0 Q_1 Q_2 Q_3) \\ & - \sum_I (P^I Q_I)^2 - 2 P^0 Q_0 \sum_i P^i Q_i + 2 \sum_{i<j} P^i Q_j P^j Q_j \; . \end{aligned}$$

Flux Example: STU-model

Geometry: type IIB ***orientifold*** on $T^6/Z^2 \times Z^2 \simeq T^2 \times T^2 \times T^2$.

Input parameters: 16 ***real fluxes***. Each of the 8 three-cycles supports H_3 ***and*** F_3.

Primary outputs: 3 ***complex moduli*** and the ***complex dilaton***, a total of 8 real parameters.

Additional outputs: ***mass parameters*** of the moduli and also the ***gravitino mass***.

Generating Function

Only determined explicitly for ***reduced fluxes***, a subset of "just" 8 real fluxes satisfying simple reality conditions:

$$\bar{m}^0 = m^0 \; , \;\; \bar{e}_i = e_i \; , \;\; \bar{m}^i = -m^i \; , \;\; \bar{e}_0 = -e_0 \; .$$

The ***generating function for reduced fluxes***:

$$\mathcal{G} \; = \; \frac{1}{2}\left[-e_0^h m_f^0 + e_i^f m_h^i\right] + \sum_i \mathrm{sgn}(m_f^0 m_h^i)\sqrt{-e_0^h m_f^0}\sqrt{m_h^i e_i^f}$$
$$- \; \sum_{i<j} \mathrm{sgn}(m_h^i m_h^j)\sqrt{m_h^i e_i^f}\sqrt{m_h^j e_j^f} \; .$$

No obvious relation to the black hole entropy. (No "analytic continuation", it seems).

It is interesting to find the generating function for more general fluxes (not just reduced fluxes), other compactifications. . . .

The Explicit Moduli

Coupling constant:

$$\mathrm{Im}\tau = \left(-\frac{m_f^0 e_1^f e_2^f e_3^f}{e_0^h m_h^1 m_h^2 m_h^3}\right)^{1/4} .$$

Moduli:

$$z^1 = -i\left[\left(-\frac{e_0^h}{m_f^0}\right)\left(\frac{m_h^1}{e_1^f}\right)\left(\frac{e_2^f}{m_h^2}\right)\left(\frac{e_3^f}{m_h^3}\right)\right]^{1/4} ,$$

$$z^2 = -i\left[\left(-\frac{e_0^h}{m_f^0}\right)\left(\frac{e_1^f}{m_h^1}\right)\left(\frac{m_h^2}{e_2^f}\right)\left(\frac{e_3^f}{m_h^3}\right)\right]^{1/4} ,$$

$$z^3 = -i\left[\left(-\frac{e_0^h}{m_f^0}\right)\left(\frac{e_1^f}{m_h^1}\right)\left(\frac{e_2^f}{m_h^2}\right)\left(\frac{m_h^3}{e_3^f}\right)\right]^{1/4} .$$

Sign Restrictions

No-scale solutions exist only when the fluxes satisfy certain conditions.

Condition that ***complex structures are in the upper half-plane*** (coupling constants are positive) imposes

$$-\mathrm{sgn}(m_f^0 e_0^h) = \mathrm{sgn}(m_h^1 e_1^f) = \mathrm{sgn}(m_h^2 e_2^f) = \mathrm{sgn}(m_h^3 e_3^f) = +1 \, .$$

If these are violated: the superpotential is minimized in the unphysical region (T^2 volumes are not positive).

If the ***product*** of these conditions is not satisfied there are not even unphysical solutions: this is the analogue of the non-BPS branch of black holes.

Such input fluxes do not correspond to solutions with Minkowski geometry — they constitute the ***de Sitter branch***.

Restoring General Fluxes

The explicit STU-solution involves just 8 fluxes and (therefore) purely imaginary moduli.

The model has large symmetry: $SL(2)^4$

This symmetry generalizes the $SL(2)^3$ of STU black holes to include S-duality acting on the τ modulus.

The formulae for moduli can be ***uniquely*** generalized to the entire $SL(2)^4$ orbit, but the details have not been carried out.

Explicit expressions appear complicated because there are 4 invariants (for BHs there is just one, the quartic invariant I_4 that determines the entropy completely).

Invariants

The simplest invariant is quadratic and very well known:

$$I_2 = \int F_3 \wedge G_3 = -e_0^h m_f^0 + e_i^f m_h^i$$

But there are 4 ***independent invariants*** of $SL(2)^4$: the quadratic I_2, two quartic $I_4^{1,2}$, and one sextic I_6.

The general solution to the STU-example of the flux attractor equations are determined uniquely by these invariants (at least locally).

Generating Function: Properties

$\mathcal{G}$ is homogeneous of degree $(1,1)$: scaling of e_I, m^I scales $\mathcal{G}$ so

$$\mathcal{G} = e_I \frac{\partial \mathcal{G}}{\partial e_I} + m^I \frac{\partial \mathcal{G}}{\partial m^I} = \frac{i}{2\tau_2}(e_I \overline{\phi}^I - m^I \overline{\theta}_I) \ .$$

The RHS is real (but not manifestly so).

The ***on-shell superpotential*** is related to the generating function

$$-\frac{i}{2\tau_2} CW = \int F_3 \wedge H_3 - \mathcal{G} \ .$$

The ***on-shell*** Kähler potential is similarly related to fluxes, and so is the ***gravitino mass***

$$m^2_{3/2} = e^K |W|^2 = -i\overline{CW} = 2\tau_2 \left(\int F_3 \wedge H_3 - \mathcal{G} \right) \ .$$

Positivity

The generating function $\mathcal{G}$ defined in analogy with the entropy function $\mathcal{S}$ is not positive definite.

The positive function in the flux ensemble is the ***gravitino mass*** (squared), considered ***as function of all fluxes***.

In the STU-example:

$$m_{3/2}^2 = \frac{1}{2}\left(\mathrm{sgn}(m_f^0)\sqrt{-e_0^h m_f^0} + \sum_i \mathrm{sgn}(m_h^i)\sqrt{m_h^i e_i^f}\right)^2 .$$

Generalizations

- This talk: standard type IIB (GKP) scenario (with Kähler moduli unstabilized).

- Generalization: stabilize Kähler moduli perturbatively using ***Geometric Fluxes***. (with O'Connell and Robbins).

- Consider ***Heterotic Compactifications***. (with Anguelova and O'Connell)

- Consider G_2 ***Compactifications of M-theory***. (unpublished, with O'Connell and Robbins)

Generating Function: Interpretation

The generating function is the flux attractor analogue of the black hole entropy.

Is it (or one of its relatives) related to a counting problem for flux vacua?

A candidate: "counting" of flux vacua usually involves enumeration of distinct flux quantum numbers.

It is natural to expect that ***a given set of semi-classical flux quantum numbers can be realized microscopically in many ways***.

The generating function $\mathcal{G}$ may count such realizations semi-classically.

The upshot: a thermodynamic interpretation of the generating function may give a short-cut to the ***measure on the landscape***.

Summary

- ***Studied IIB orientifold compactifications*** with ISD fluxes

- ***Analyzed attractor equations***: the input fluxes determine scalar moduli ***and*** mass parameters.

- Solved attractor equations in terms of a ***Generating Function***, a scalar function of the fluxes.

- Explicit computations for a canonical example with ***16 fluxes***.

- Speculated that the generating function constitutes a ***measure on moduli space***.