From Twistors to Amplitudes

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based on:
• hep-th/0407214 AB-Spence-Travaglini
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• hep-th/0506068 AB-McNamara-Spence-Travaglini
Outline:

- Motivation & Aims
- Scattering Amplitudes in Gauge Theory
  - Colour Decomposition & Spinor Helicity Formalism
  - Twistor Space
  - MHV Diagrams
- Loop Amplitudes from MHV Vertices
  - Super Yang-Mills and pure Yang-Mills
- Generalized Unitarity
  - Cut-Constructibility
  - Generalised Cuts
  - Generalised Unitarity in $D = 4 - 2\varepsilon$ Dimensions
- Summary & Outlook
Why is that interesting?

- Explains unexpected simplicity of scattering amplitudes in Yang Mills & gravity
  - Simple Geometric Structure in Twistor Space
  - New Diff. Equations for Amplitudes

- New tools to calculate amplitudes
  - MHV Diagrams for trees and loops
  - Generalized Unitarity
  - New Recursion Relations
Motivation

• LHC is coming
  • Precision pert. QCD calculations
  • Long wishlist of processes to be computed
• New techniques are needed
  • Textbook methods hide simplicity of amplitudes
  • Intermediate expressions are large
  • Factorial growth of nr. of diagrams, e.g. gluon scattering

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Motivation cont’d

- Luckily we do not have to use textbook techniques
  - color decomposition
  - spinor helicity
  - unitarity
  - supersymmetry
  - string theory ...
- and since 2004
  - twistor string (inspired) techniques
Color Decomposition

\[ A^\text{tree}_n (\{p_i, \varepsilon_i, a_i\}) = ig^{n-2} \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \times \]

\[ \sum_{\sigma \in S_n/Z_n} \text{Tr} \left( T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}} \right) A^\text{tree}_n (\sigma(p_1, \varepsilon_1), \ldots, \sigma(p_n, \varepsilon_n)) \]

- at tree level Yang-Mills is planar
- only diagrams with fixed cyclic ordering contribute to the "color stripped amplitudes" \( A^\text{tree}_n \)
  - analytic structure simpler
- At loop level, also multi-traces; subleading in \( 1/N \)
  - At one-loop simple relation between planar & non-planar terms
Spinor helicity formalism

- Responsible for the existence of **compact formulas** of tree and loop amplitudes in **massless theories**

- The 4D Lorentz Group (complexified) \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \)

\[
p_\mu \iff p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} ; \quad a, \dot{a} = 1, 2
\]

**massless on-shell** \( p_\mu p^\mu = \det p_{a\dot{a}} = 0 \Rightarrow p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \quad \text{Note: In real Mink} \quad \tilde{\lambda} = \overline{\lambda} \)

- **Spinor Products**

\[
\langle i j \rangle \equiv \lambda^i_a \lambda^j_b \varepsilon^{ab} , \quad [i j] \equiv \tilde{\lambda}^i_a \tilde{\lambda}^j_b \varepsilon^{\dot{a}\dot{b}} \quad \Rightarrow \quad 2p_i \cdot p_j = \langle i j \rangle [ji]
\]

- \( \{ p^\mu_i, \varepsilon^\mu_i \} \) are redundant; the **spinor variables** \( \{ \lambda^a_i, \tilde{\lambda}^\dot{a}_i \} \) contain just the right d.o.f. to describe momentum & wavefnct./polarization of **massless particles** of arbitrary helicity h
n-Gluon Tree MHV-Amplitudes

\[ A_{\text{tree}}^{\text{MHV}} = i g^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum p_i \right) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \]

- Very Simple!
- Holomorphic, depends only on \( \lambda_i \), not on \( \tilde{\lambda}_i \)
- Correct for N=4,2,1 Super Yang-Mills, pure glue & QCD
- In N=4 SYM similar formulas for amplitudes with two gluons replaced by fermions/scalars

MHV Amplitude
Parke-Taylor;
Berends-Giels
Twistor Space

... is a “1/2 Fourier transform” of spinor space:

\[(\lambda_a, \tilde{\lambda}_{\dot{a}}) \implies (\lambda_a, \mu_{\dot{a}})\]

- Twistor Space is complex 4 dim’l \[(\lambda_1, \lambda_2, \mu^1, \mu^2)\]
- Amplitudes are homogeneous functions on twistor space

\[
(\lambda, \mu) \sim (t\lambda, t\mu)
\]

Projective Twistor Space \(\mathbb{CP}^3\)
Twistor Space cont’d

- Relations between Minkowski space and projective T.S.

Incidence Relation: \[ \mu \dot{a} + x^{a\dot{a}} \lambda_a = 0 \]
Amplitudes in Twistor Space

- MHV amplitudes are holomorphic (except for momentum conservation); perform $1/2$ Fourier transform

\[ \Rightarrow A_{\text{MHV}} \int dx \int \prod_i d\tilde{\lambda}_i e^{i\mu_i \tilde{\lambda}_i} e^{ix\lambda_i \tilde{\lambda}_i} \sim \prod_i \delta^{(2)}(\mu_i + x\lambda_i) \]

Hence: For MHV amplitudes all points (=ext. gluons) lie on a line in projective Twistor Space
Amplitudes in Twistor Space cont’d

- Witten’s conjecture (2003): L-loop amplitudes with \( Q \) negative helicity gluons localise on curves of degree \( = Q - 1 + L \) and genus \( \leq L \)

- Localisation properties of amplitudes in proj. T.S. translate into differential operators obeyed by the amplitudes in momentum space: \( \mu \rightarrow i\partial/\partial\tilde{\lambda} \)

- For non-MHV tree amplitudes “experiments” with diff. operators reveal:

\[ Q=3 \]

\[ Q=4 \]
MHV Diagrams

- MHV amplitude = local interaction in Mink
- CSW Rules (Cachazo-Svrcek-Witten)
  - MHV amplitudes continued off-shell as local vertices
  - Connect MHV vertices with scalar propagators: \( \frac{1}{P^2} \)
  - Sum diagrams with fixed cyclic ordering of ext. lines

Ex: \( \langle 1^- 2^- 3^- 4^+ 5^+ 6^+ \rangle \)

Off-shell continuation of spinor:

\[ \lambda_{Pa} = P_{a\dot{a}} \eta^{\dot{a}} \]
\[ \eta^{\dot{a}} \ldots \text{reference spinor} \]
MHV diagrams cont’d

some of the 5 missing diagrams of \( \langle 1^-2^-3^-4^+5^+6^+ \rangle \)

- Reproduce known and obtain new scattering amplitudes in any massless gauge theory dramatic simplifications

- Correct factorisation
  - multiparticle poles
  - collinear/soft limits
MHV diagrams - applications

- Amplitudes of gluons with fermions/scalars Georgiou-Khoze, Wu-Zhu
- Amplitudes with quarks Georgiou-Khoze, Su-Wu
- Higgs plus partons Dixon-Glover-Khoze, Badger-Glover-Khoze
- Electroweak vector boson currents Bern-Forde-Kosower-Mastrolia
From Trees to Loops (AB-Spence-Travaglini)

- Original **prognosis from twistor string theory** was negative (Berkovits-Witten), "pollution" with **Conformal SUGRA** modes

- **Try anyway:**
  - Connect **$V=Q-1+L$** MHV vertices, using the same off-shell continuation as for trees
  - Perform loop integration! **Measure**?

- **Simplest Ex.:** **MHV 1-loop amplitudes in N=4 SYM**

\[
\int dM \sum_{m_1,m_2,h} \]

\[
\begin{array}{c}
\text{MHV} \\
\text{MHV}
\end{array}
\]

\[
\begin{array}{c}
m_1^+ \\
+ \\
\text{L}_1 \\
+ \\
- \\
\text{L}_2 \\
+ \\
+ \\
+ \\
\end{array}
\]

\[
\begin{array}{c}
m_2^+ \\
+ \\
- \\
\end{array}
\]
MHV one-loop amplitudes in $N=4$ SYM

- Computed by Bern-Dixon-Dunbar-Kosower (1994) using four-dim’l cut-constructibility (works for SUSY, massless theories) = Unitarity

- Result is expressed in terms of “2-mass easy box functions”

\[ I^{2me}(s,t,P^2,Q^2) = \int d^{4-2\epsilon}L \frac{1}{L^2(L-p)^2(L-P-p)^2(L+Q)^2} \]

\[ A^{1-loop}_{MHV} = A^{tree}_{MHV} \times \sum_{p,q} \]

\[ \text{Diagram:} \]

\[ \text{P} \rightarrow \text{q} \]

\[ \text{P} \rightarrow \text{p} \]

\[ \text{Q} \rightarrow \text{Q} \]
MHV vertices at one-loop

Loop integration (schematically):

\[
A_{\text{MHV}}^{1\text{-loop}} = \sum_{m_1,m_2,h} \int_{-L_1}^{L_2} dM A_L^{\text{tree}} (-L_1,m_1,\ldots,m_2,L_2) \times A_R^{\text{tree}} (-L_2,m_2+1,\ldots,m_1-1,L_1)
\]

Loop measure:

\[
dM = \frac{d^4L_1}{L_1^2 + i\varepsilon L_2^2 + i\varepsilon} \delta^{(4)}(L_2 - L_1 + P_L)
\]

Off-shell continuation (as before)

\[
L_\mu = l_\mu + z\eta_\mu
\]

Hence

\[
\frac{d^4L}{L^2 + i\varepsilon} = \frac{dz}{z} \times d^4l\delta^{(+)}(l^2)
\]

dispersive measure

phase space measure

reference null-vector
N=4 SYM one-loop cont’d

Putting everything together and integrating over \( z' = z_1 + z_2 \)
we find, using \( z = z_1 - z_2 \)

\[
d\mathcal{M} = \frac{dz}{z} \times dLIPS(l_2, -l_1; P_{L;z})
\]

\( P_{L;z} = P_L - z\eta \)

\( dLIPS \) is the 2-particle Lorentz inv. phase space measure and the corresponding integral calculates the branchcut or imaginary part of the amplitude! Note however the shift in \( P_{L;z} = P_L - z\eta \)

The remaining integration over \( z \) is a dispersion (type) integral, which reproduces the full amplitude!

→ The Return of the Analytic S-Matrix
After some manipulations we find the result to be proportional to the sum over contributions from all possible cuts of all possible 2-mass easy box functions.

**Note:** only after summing over the four cuts dependence on \( \eta \) disappears!
Summary of $N=4$ SYM at one-loop

- Agrees with result of (Bern-Dixon-Dunbar-Kosower)
- Incorporates large numbers of conventional Feynman diag.
- Naturally leads to “dispersion integrals”
- Non-trivial check of MHV diagrammatic method
  - covariance (no dependence on $\eta$)
  - what about non-MHV amplitudes?
- Simpler form of “2-mass easy box function”:

  \[
  I_{2me}^{\text{me}}(s, t, P^2, Q^2) = -\frac{1}{\varepsilon^2} \left[ (-s)^{-\varepsilon} + (-t)^{-\varepsilon} - (-P^2)^{-\varepsilon} - (-Q^2)^{-\varepsilon} \right] \\
  + \text{Li}_2(1 - aP^2) + \text{Li}_2(1 - aQ^2) - \text{Li}_2(1 - as) - \text{Li}_2(1 - at),
  \]

  \[
  a = \frac{P^2 + Q^2 - s - t}{P^2Q^2 - st} = \frac{u}{P^2Q^2 - st}
  \]
Twistor Space Localisation

Using Diff. Operators $F$ and $K$ to determine **collinearity & coplanarity**, (Cachazo-Svrcek-Witten) found (a), (b) and (c)

Our computation shows that (c) should be absent!

One-loop amplitudes not annihilated by Diff. Ops.

- **Holomorphic Anomaly** = rational function

**New tool** to calculate **new** one-loop amplitudes

New 7-point amplitude in $N=4$ SYM (Britto-Cachazo-Feng)
New 6-point amplitudes in $N=1$ SYM (Bidder, Bjerrum-Bohr, Dixon, Dunbar)
Generalisations

• In principle our approach can readily be applied to non-MHV amplitudes and theories with less supersymmetry

• MHV, one-loop amplitudes in $\mathbb{N}=1$ SYM (Bedford-AB-Spence-Travaglini)

• Contribution of a chiral multiplet (susy decomposition)

$$ A^{\mathbb{N}=1, \text{vector}} = A^{\mathbb{N}=4} - 3A^{\mathbb{N}=1, \text{chiral}} $$

• Result (BDDK) expressed in terms of (finite part of) scalar box, and triangle functions:

• MHV diagram method agrees with BDDK

• Works despite the absence of Twistor String Dual of $\mathbb{N}=1$ SYM
MHV, one-loop in N=1 SYM

\[ A_{\text{chiral}}^{1-\text{loop},\text{MHV}} = A_{\text{tree},\text{MHV}} \times I \]

\[ I = \sum_{m,s} b_{m,s}^{i,j} + \sum_{m,a} c_{m,a}^{i,j} \]
MHV, one-loop amplitudes in Yang-Mills

- **non-supersymmetric** theories are not “4D cut-constructible”
- Amplitudes contain **rational terms** that are not linked to terms containing cuts (but see later in the talk)
- From **MHV vertices** we obtain **cut-containing terms**
- **SUSY decomposition**

\[
A^g = (A^g + 4A^f + 3A^s) - 4(A^f + A^s) + A^s
\]

To be computed
Pure Yang-Mills cont’d

• Result is expressed in terms of
  • finite box functions: \( I_{\text{finite}}^{2\text{me}} = B(s, t, P^2, Q^2) \)
  • triangle functions: \( T^{(r)}(p, P, Q) = \frac{\log(Q^2/P^2)}{(Q^2 - P^2)^r} \)
• Coefficient of \( B \) is: \( (b_{m_1m_2}^{ij})^2 \)
• Agrees with 5-point result and the case of adjacent negative helicity gluons of (BDDK)
• New Result for negative helicity gluons in arbitrary position
• First step towards QCD from MHV diagrams!
Generalized Unitarity

- Very old idea, “The Analytic S(-)Matrix” (Eden-Landshoff-Olive-Polkinghorne 1966; Chew 1966); more recently (Bern-Dixon-Kosower 1997)

- 2004/05 “The Return of the Analytic S-Matrix”

- One-loop amplitudes in SUSY gauge theories are 4d cut-constructible

- One-loop amplitudes in N=4 SYM have a very simple form:

\[
\begin{array}{c}
\text{sun} \\
\end{array} = \sum c \\
\text{square}
\]

Q: can we find the rational coefficients \(c\) without integrations?
Quadruple Cuts

- **Answer:** Yes! (Britto-Cachazo-Feng)

- **Quadruple Cuts** = replace four propagators by on-shell delta functions: \( \frac{1}{L_i^2} \rightarrow \delta(+) (L_i^2) \), \( i = 1, 2, 3, 4 \)

- Loop integration localises completely! Requires complex momenta!

- The coefficients \( c \) are products of four on-shell tree amplitudes \( c = A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} \)

- \( N=4 \) SYM at one-loop is reduced to algebra!

A quadruple cut selects a unique box function!
Multiple cuts for amplitudes in N=1 SYM

• Similar for one-loop amplitudes in N=1 SYM, but more work

\[ \sum a + \sum b + \sum c \]

Fix coefficients \( a \) with quadruple cuts

Fix coefficients \( b \) with “triple cuts” (one remaining integration)

Fix \( c \)'s with conventional unitarity cuts

New results: all N=1, one-loop, 6-point amplitudes and an infinite series \( \langle 1^- 2^- 3^- 4^+ \ldots n^+ \rangle \)

(Bidder, Bjerrum-Bohr, Dunbar, Perkins; Britto-Buchbinder-Cachazo-Feng)
Generalised Generalized Unitarity
(AB-McNamara-Spence-Travaglini)

• **Problem:** QCD one-loop amplitudes are not 4d cut-constructible. Amplitudes contain rational terms.

• Need to work in $D = 4 - 2\varepsilon$ dimensions.

  - $R(-s)^{-\varepsilon} \Rightarrow R - R\varepsilon\log(-s) + O(\varepsilon^2)$

• This requires the knowledge of tree amplitudes with some of the legs continued to $D = 4 - 2\varepsilon$ dimensions (DR).

• We can think of this as giving a uniform mass to internal particles. This mass has to be integrated over!

  - $L^2_{4-2\varepsilon} = L^2_4 + L^2_{-2\varepsilon} = L^2_4 - \mu^2$

• Feynman integrals with powers of $[\mu^2]$ inserted lead to integrals in $D = 6 - 2\varepsilon, 8 - 2\varepsilon, \ldots$
Generalised Unitarity for YM

- The necessary amplitudes with massive particles are provided by old (Berends-Giele, BDDK) and new recursive techniques (Badger-Glover-Khoze-Svrcek). Because of the SUSY decomposition of the amplitudes we only need to consider scalars running in the loop!

- Ex: $<++++>$ one-loop amplitude in YM from quadruple cut

\[
A_{1\text{-loop}}^4 = \frac{[12] [34]}{\langle 12 \rangle \langle 34 \rangle} \mu^4
\]

\[
I_{4-2\varepsilon}^4[\mu^4] = (-\varepsilon)(1-\varepsilon)I_{4-2\varepsilon}^8[\mu^4] = -\frac{1}{6} + O(\varepsilon)
\]
Generalised Unitarity for YM cont’d

- This also works for all other 4-point amplitudes: $\langle-++++\rangle$, $\langle------\rangle$ and $\langle-----\rangle$

- This requires triple cuts but no 2-particle cuts.

- The result is expressed in terms of box and triangle functions in 4, 6 and 8 dimensions

- The 5-point amplitude $\langle++++++\rangle$ requires only quadruple cuts! Expressed in terms of 8 dim’l box and 10 dim’l pentagon integrals.

- Other 5-point and 6-point amplitudes work in progress ...
Summary & Outlook

• Exciting progress in calculating amplitudes in gauge theory and gravity

• new spectacular insights in the structure of amplitudes from twistor space

• New diagrammatic tools (twistor inspired)
  • MHV diagrams: for tree level (CSW) and loop level amplitudes (BST)

• Generalised Unitarity:
  • new efficient techniques to calculate amplitudes in supersymmetric theories, many new results (BDK, BCF)
  • cut-containing parts and rational terms (require unitarity in $D = 4 - 2\varepsilon$) of amplitudes in QCD
Summary & Outlook cont’d

• Recently: New on-shell recursion relations (Britto-Cachazo-Feng-(Witten)) Use only on-shell data, analyticity and factorisation of amplitudes

• Gauge theory amplitudes with massless and massive particles (tree level)

• Gravity (tree level)

• Rational terms in one-loop QCD amplitudes

• Coefficients of integral functions in one-loop amplitudes

Many new results expected

New complete QCD amplitudes within reach!