# *From Twistors to Amplitudes*

by Andreas Brandhuber

Queen Mary, University of London

based on:

- hep-th/0407214 AB-Spence-Travaglini
- hep-th/0410280 Bedford-AB-Spence-Travaglini
- hep-th/0412108 Bedford-AB-Spence-Travaglini
- hep-th/0502146 Bedford-AB-Spence-Travaglini
- hep-th/0506068 AB-McNamara-Spence-Travaglini

# Outline:

- **Motivation & Aims**
- **Scattering Amplitudes in Gauge Theory**
  - Colour Decomposition & Spinor Helicity Formalism
  - Twistor Space
  - MHV Diagrams
- **Loop Amplitudes from MHV Vertices**
  - Super Yang-Mills and pure Yang-Mills
- **Generalized Unitarity**
  - Cut-Constructibility
  - Generalised Cuts
  - Generalised Unitarity in $D = 4 - 2\varepsilon$ Dimensions
- **Summary & Outlook**

Witten 2003

weak/weak

Perturbative N=4 SYM = Topological String on Twistor Space

## Why is that interesting ?

- Explains unexpected simplicity of scattering amplitudes in Yang Mills & gravity

→ Simple Geometric Structure in Twistor Space

→ New Diff. Equations for Amplitudes

- New tools to calculate amplitudes

→ MHV Diagrams for trees and loops

Generalized Unitarity

New Recursion Relations

# Motivation

- LHC is coming
  - Precision pert. QCD calculations
  - Long wishlist of processes to be computed
- New techniques are needed
  - Textbook methods hide simplicity of amplitudes
  - Intermediate expressions are large
  - Factorial growth of nr. of diagrams, e.g. gluon scattering

| g g => n g | n=7 | n=8 | n=9 |
|---|---|---|---|
| | 559405 | 10525900 | 224449225 |

# Motivation cont'd

- Luckily we do not have to use textbook techniques
  - color decomposition
  - spinor helicity
  - unitarity
  - supersymmetry
  - string theory ...
- and since 2004
  - twistor string (inspired) techniques

# Color Decomposition

$$\mathcal{A}_n^{\text{tree}}(\{p_i, \varepsilon_i, a_i\}) = i g^{n-2} \delta^{(4)}(\sum_{i=1}^{n} p_i) \times$$
$$\sum_{\sigma \in S_n/Z_n} \text{Tr}\,(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}})\, A_n^{\text{tree}}(\sigma(p_1, \varepsilon_1), \ldots, \sigma(p_n, \varepsilon_n))$$

- at tree level Yang-Mills is planar
- only diagrams with fixed cyclic ordering contribute to the "color stripped amplitudes" $A_n^{tree}$
  - analytic structure simpler
- At loop level, also multi-traces; subleading in $1/N$
  - At one-loop simple relation between planar & non-planar terms

# Spinor helicity formalism

- Responsible for the existence of compact formulas of tree and loop amplitudes in massless theories

- The 4D Lorentz Group (complexified) $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

$$p_\mu \Longleftrightarrow p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} \; ; \;\; a, \dot{a} = 1, 2$$

massless on-shell $\quad p_\mu p^\mu = \det p_{a\dot{a}} = 0 \Rightarrow p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

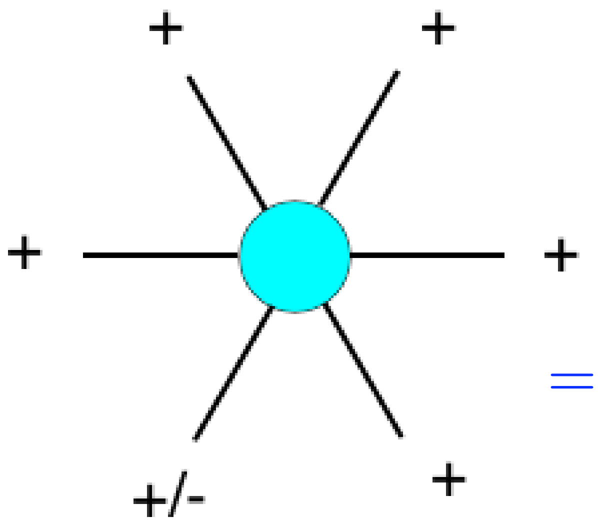
Note: In real Mink $\tilde{\lambda} = \bar{\lambda}$

- Spinor Products

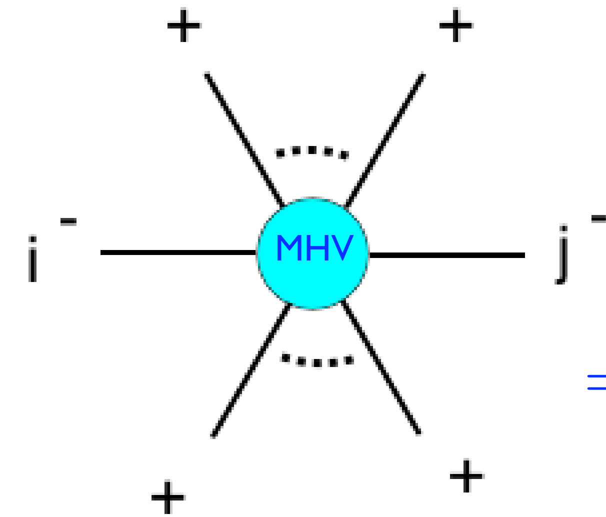
$$\langle ij \rangle \equiv \lambda^i_a \lambda^j_b \varepsilon^{ab} \; , \;\; [ij] \equiv \tilde{\lambda}^i_{\dot{a}} \tilde{\lambda}^j_{\dot{b}} \varepsilon^{\dot{a}\dot{b}} \qquad \Rightarrow \;\; 2p_i \cdot p_j = \langle ij \rangle \, [ji]$$

- $\{p^\mu_i, \varepsilon^\mu_i\}$ are redundant; the spinor variables $\left\{\lambda^a_i, \tilde{\lambda}^{\dot{a}}_i\right\}$ contain just the right d.o.f. to describe momentum & wavefnct./polarization of massless particles of arbitrary helicity h

# n-Gluon Tree MHV-Amplitudes




$= 0$




$= A_{\text{MHV}}^{\text{tree}}$

$$A_{\text{MHV}}^{\text{tree}} = i g^{n-2} (2\pi)^4 \delta^{(4)} \left(\sum p_i\right) \frac{\langle i\, j\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}$$

MHV Amplitude
Parke-Taylor;
Berends-Giels

- Very Simple!
- Holomorphic, depends only on $\lambda_i$ , not on $\tilde{\lambda}_i$
- Correct for N=4,2,1 Super Yang-Mills, pure glue & QCD
- In N=4 SYM similar formulas for amplitudes with two gluons replaced by fermions/scalars

# Twistor Space

... is a "1/2 Fourier transform" of spinor space:

$$(\lambda_a, \tilde{\lambda}_{\dot{a}}) \qquad \Rightarrow \qquad (\lambda_a, \mu_{\dot{a}})$$

- Twistor Space is complex 4 dim'l $\qquad (\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}})$
- Amplitudes are homogeneous functions on twistor space

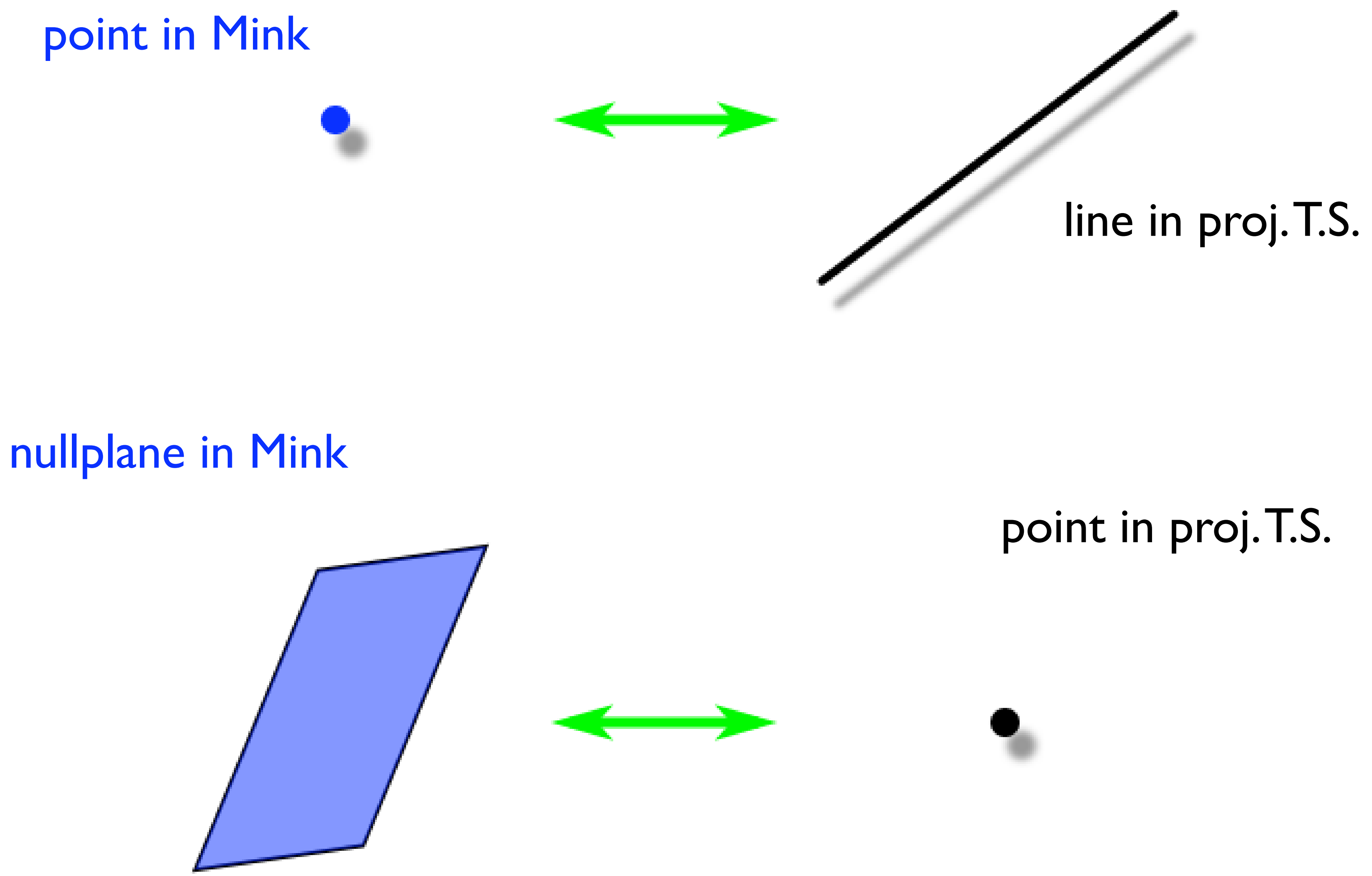
➡ Projective Twistor Space $\quad \mathbb{CP}^3$

$$(\lambda, \mu) \sim (t\lambda, t\mu)$$

# Twistor Space cont'd

- Relations between Minkowski space and projective T. S.

  Incidence Relation: $\mu^{\dot{a}} + x^{a\dot{a}}\lambda_a = 0$

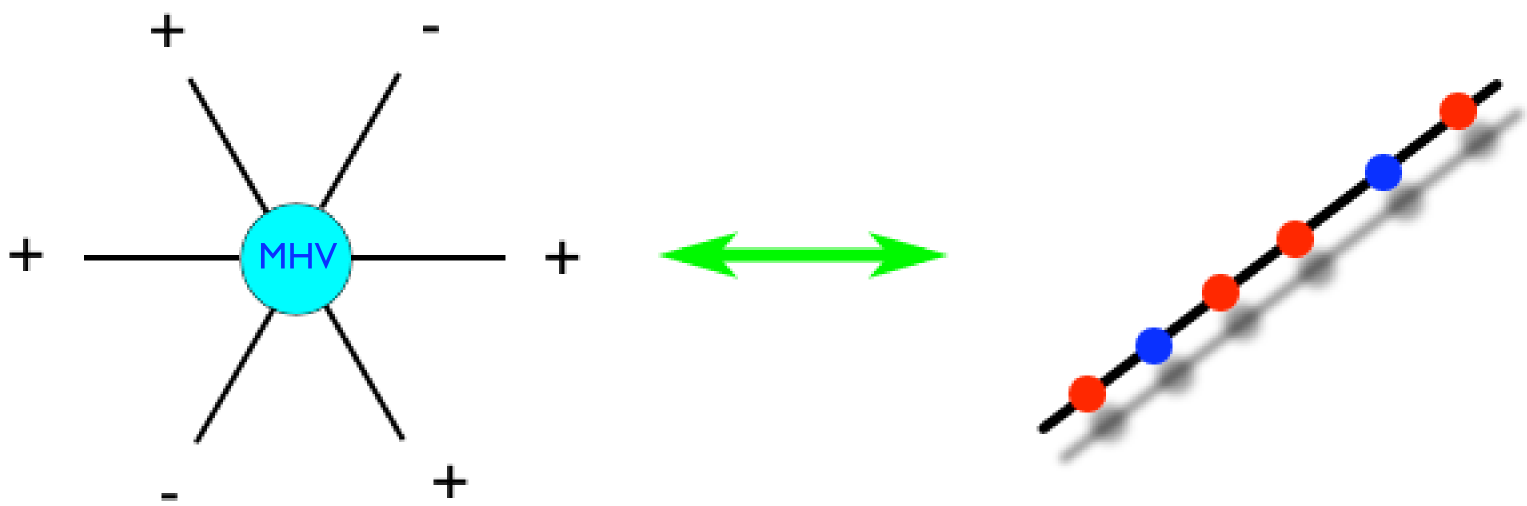


# Amplitudes in Twistor Space

- MHV amplitudes are holomorphic (except for momentum conservation); perform 1/2 Fourier transform

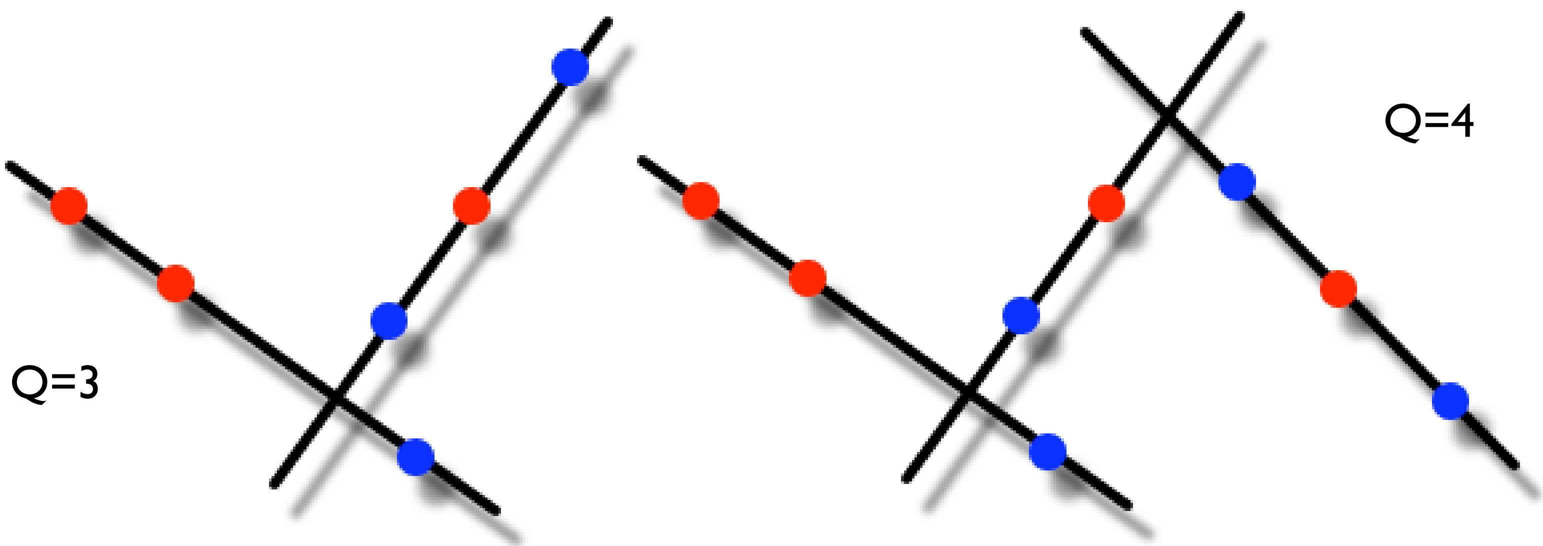
$$\Rightarrow A_{\mathrm{MHV}} \int dx \int \prod_i d\tilde{\lambda}_i e^{i\mu_i\tilde{\lambda}_i} e^{ix\lambda_i\tilde{\lambda}_i} \sim \prod_i \delta^{(2)}(\mu_i + x\lambda_i)$$

Hence: For MHV amplitudes all points (=ext. gluons) lie on a line in projective Twistor Space



# Amplitudes in Twistor Space cont'd

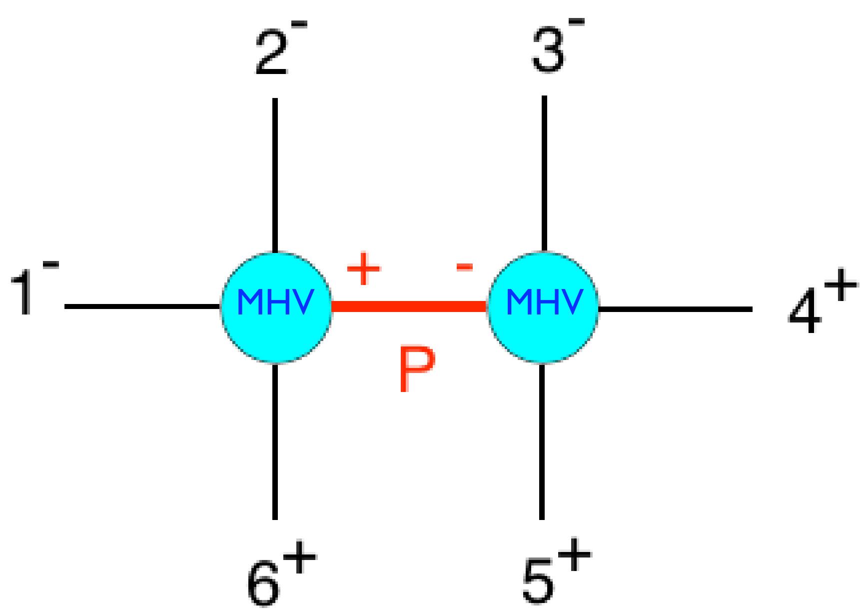
- Witten's conjecture (2003): L-loop amplitudes with Q negative helicity gluons localise on curves of degree=Q-1+L and genus<=L
- Localisation properties of amplitudes in proj. T.S. translate into differential operators obeyed by the amplitudes in momentum space: $\mu \to i\partial/\partial\tilde{\lambda}$
- For non-MHV tree amplitudes "experiments" with diff. operators reveal:



# MHV Diagrams

- MHV amplitude = local interaction in Mink
- CSW Rules (Cachazo-Svrcek-Witten)
  - MHV amplitudes continued off-shell as local vertices
  - Connect MHV vertices with scalar propagators: $\frac{1}{P^2}$
  - Sum diagrams with fixed cyclic ordering of ext. lines

Ex: $\langle 1^- 2^- 3^- 4^+ 5^+ 6^+ \rangle$



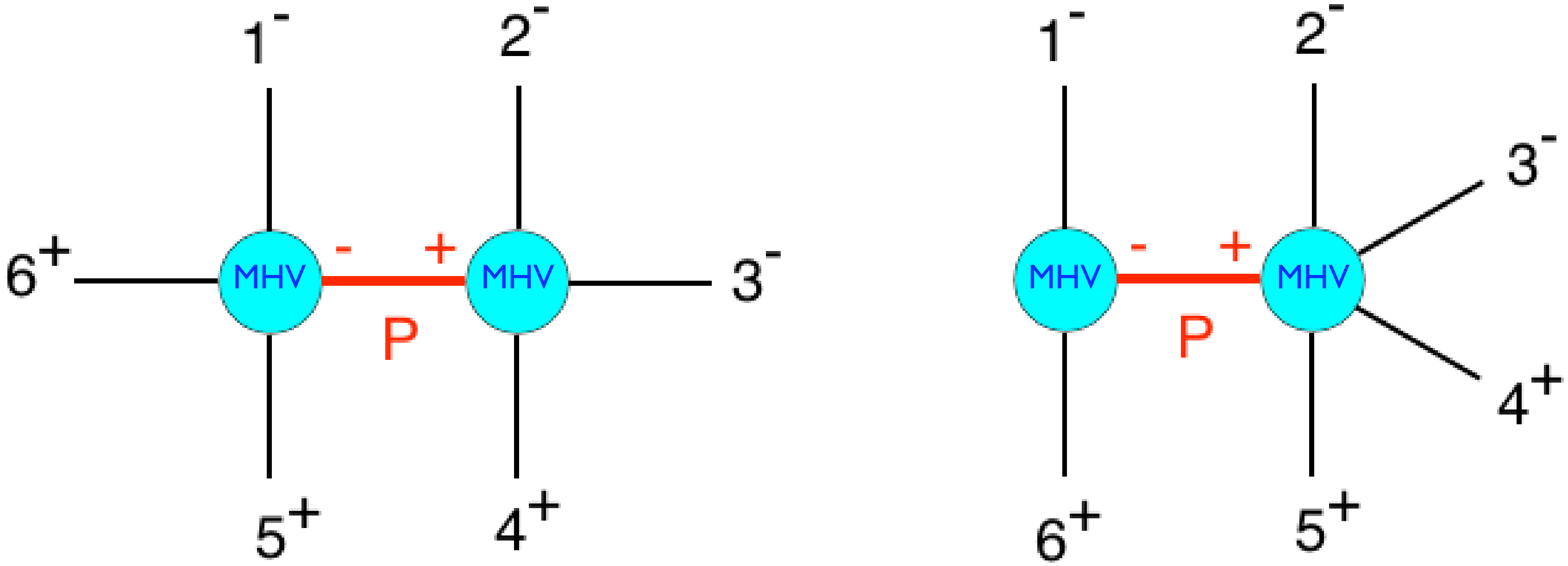

Off-shell continuation of spinor:

$$\lambda_{Pa} = P_{a\dot{a}} \eta^{\dot{a}}$$

$\eta^{\dot{a}} \ldots$ reference spinor

# MHV diagrams cont'd

some of the 5 missing diagrams of $\langle 1^- 2^- 3^- 4^+ 5^+ 6^+ \rangle$




- Reproduce known and obtain new scattering amplitudes in any massless gauge theory dramatic simplifications
- Correct factorisation
  - multiparticle poles
  - collinear/soft limits

# MHV diagrams - applications

- Amplitudes of gluons with fermions/scalars Georgiou-Khoze, Wu-Zhu
- Amplitudes with quarks Georgiou-Khoze, Su-Wu
- Higgs plus partons Dixon-Glover-Khoze, Badger-Glover-Khoze
- Electroweak vector boson currents Bern-Forde-Kosower-Mastrolia

# From Trees to Loops (AB-Spence-Travaglini)

- Original prognosis from twistor string theory was negative (Berkovits-Witten), "pollution" with Conformal SUGRA modes

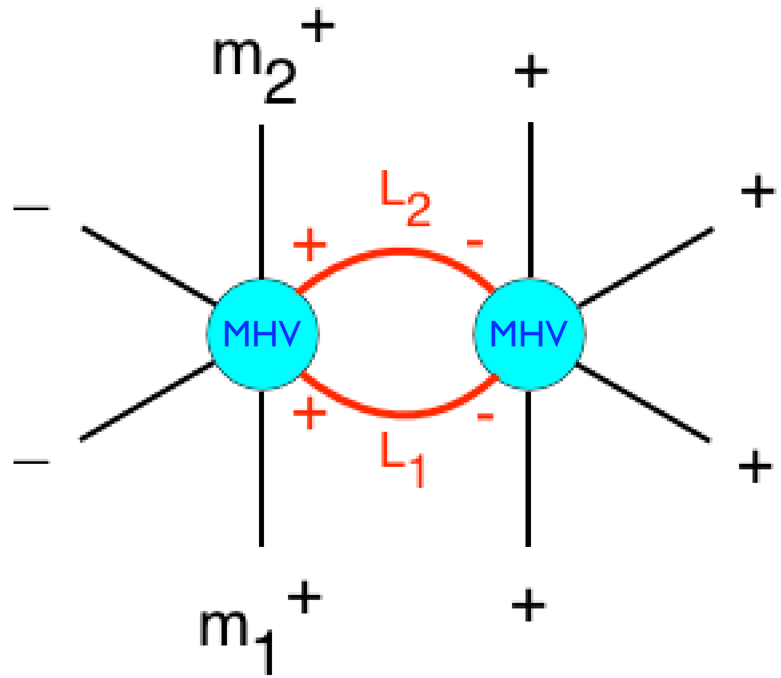
- Try anyway:

  - Connect V=Q-1+L MHV vertices, using the same off-shell continuation as for trees

  - Perform loop integration! Measure?

- Simplest Ex.: MHV 1-loop amplitudes in N=4 SYM

$$\int d\mathcal{M} \sum_{m_1, m_2, h}$$



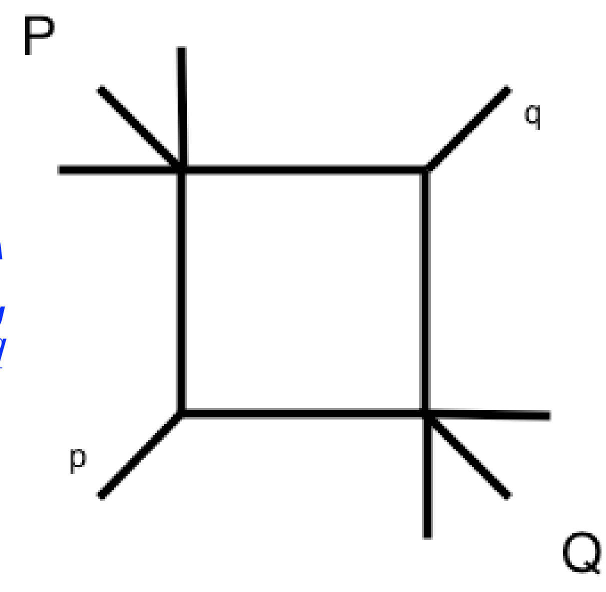
# MHV one-loop amplitudes in N=4 SYM

- Computed by Bern-Dixon-Dunbar-Kosower (1994) using four-dim'l cut-constructibility (works for SUSY, massless theories) = Unitarity

- Result is expressed in terms of "2-mass easy box functions"

$$I^{2me}(s,t,P^2,Q^2) = \int d^{4-2\varepsilon}L \frac{1}{L^2(L-p)^2(L-P-p)^2(L+Q)^2}$$

$$A^{1-loop}_{\mathrm{MHV}} = A^{tree}_{\mathrm{MHV}} \times \sum_{p,q}$$



# MHV vertices at one-loop

Loop integration (schematically):

$$A_{\mathrm{MHV}}^{1-loop} = \sum_{m_1,m_2,h} \int d\mathcal{M} A_L^{\mathrm{tree}}(-L_1, m1, \ldots, m_2, L_2) \times A_R^{tree}(-L_2, m2+1, \ldots, m_1-1, L_1)$$

Loop measure:

$$d\mathcal{M} = \frac{d^4 L_1}{L_1^2 + i\varepsilon} \frac{d^4 L_2}{L_2^2 + i\varepsilon} \delta^{(4)}(L_2 - L_1 + P_L)$$

Off-shell continutation (as before) $L_\mu = l_\mu + z\eta_\mu$

reference null-vector

Hence

$$\frac{d^4 L}{L^2 + i\varepsilon} = \frac{dz}{z} \qquad \times \qquad d^4 l \delta^{(+)}(l^2)$$

dipersive measure

phase space measure

# N=4 SYM one-loop cont'd

Putting everything together and integrating over $z' = z_1 + z_2$
we find, using $z = z_1 - z_2$

$$d\mathcal{M} = \frac{dz}{z} \times dLIPS(l_2, -l_1; P_{L;z})$$

$$P_{L;z} = P_L - z\eta$$

*dLIPS* is the 2-particle Lorentz inv. phase space measure and the corresponding integral calculates the branchcut or imaginary part of the amplitude! Note however the shift in $P_{L;z} = P_L - z\eta$

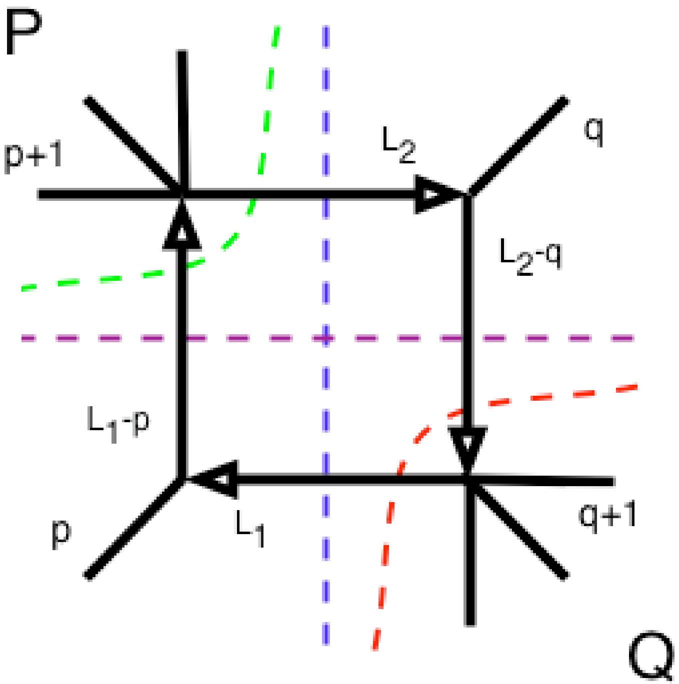
The remaining integration over $z$ is a dispersion (type) integral, which reproduces the full amplitude!

➡ The Return of the Analytic S-Matrix

# N=4 SYM one-loop cont'd

After some manipulations we find the result to be proportional to the sum over contributions from all possible cuts of all possible 2-mass easy box functions

$$A_{\rm MHV}^{\rm tree}\times$$



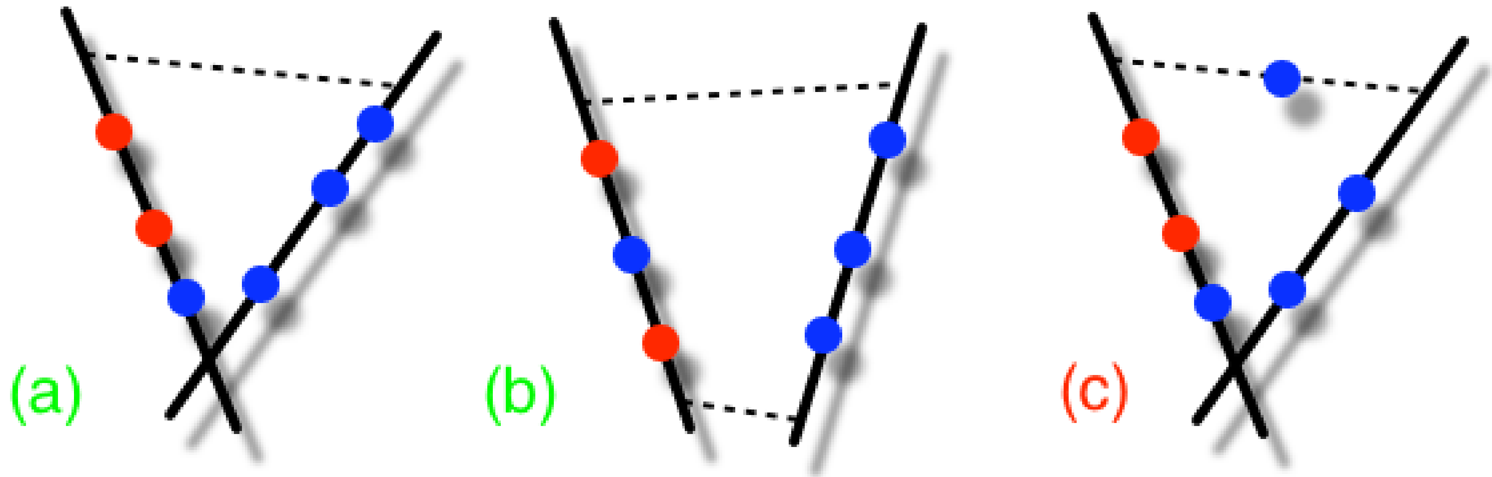

Note: only after summing over the four cuts dependence on $\eta$ disappears!

# Summary of N=4 SYM at one-loop

- Agrees with result of (Bern-Dixon-Dunbar-Kosower)
- Incorporates large numbers of conventional Feynman diag.
- Naturally leads to "dispersion integrals"
- Non-trivial check of MHV diagrammatic method
  - covariance (no dependence on $\eta$ )
  - what about non-MHV amplitudes?
- Simpler form of "2-mass easy box function":

$$I^{2me}(s,t,P^2,Q^2) = -\frac{1}{\varepsilon^2}\Big[(-s)^{-\varepsilon} + (-t)^{-\varepsilon} - (-P^2)^{-\varepsilon} - (-Q^2)^{-\varepsilon}\Big]$$
$$+\mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)\,,$$
$$a = \frac{P^2+Q^2-s-t}{P^2Q^2-st} = \frac{u}{P^2Q^2-st}$$

# Twistor Space Localisation




- Using Diff. Operators *F* and *K* to determine collinearity & coplanarity, (Cachazo-Svrcek-Witten) found (a), (b) and (c)
- Our computation shows that (c) should be absent!
- One-loop amplitudes not annihilated by Diff. Ops.
  - Holomorphic Anomaly = rational function
- New tool to calculate new one-loop amplitudes

  New 7-point amplitude in N=4 SYM (Britto-Cachazo-Feng)

  New 6-point amplitudes in N=1 SYM (Bidder, Bjerrum-Bohr, Dixon, Dunbar)

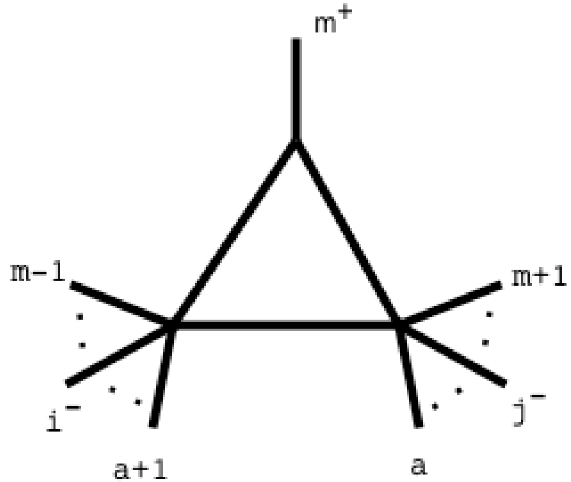
# Generalisations

- In principle our approach can readily be applied to non-MHV amplitudes and theories with less supersymmety
- MHV, one-loop amplitudes in N=1 SYM (Bedford-AB-Spence-Travaglini)
  - Contribution of a chiral multiplet (susy decomposition)
    $A^{\mathcal{N}=1,\text{vector}} = A^{\mathcal{N}=4} - 3A^{\mathcal{N}=1,\text{chiral}}$
  - Result (BDDK) expressed in terms of (finite part of) scalar box, and triangle functions:
  - MHV diagram method agrees with BDDK
  - Works despite the absence of Twistor String Dual of N=1 SYM

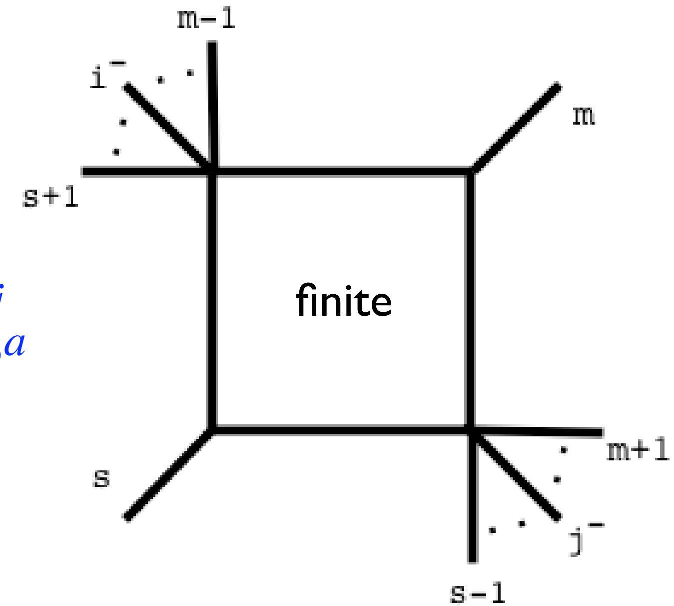
# MHV, one-loop in N=1 SYM

$$A_{chiral}^{1-loop,MHV} = A^{tree,MHV} \times I$$

$$I = \sum_{m,s} b_{m,s}^{i,j}$$




$$+ \sum_{m,a} c_{m,a}^{i,j}$$



# MHV, one-loop amplitudes in Yang-Mills

- non-supersymmetric theories are not "4D cut-constructible"
  - Amplitudes contain rational terms that are not linked to terms containing cuts (but see later in the talk)
- From MHV vertices we obtain cut-containing terms
- SUSY decomposition

$$A^g = (A^g + 4A^f + 3A^s) - 4(A^f + A^s) + A^s$$

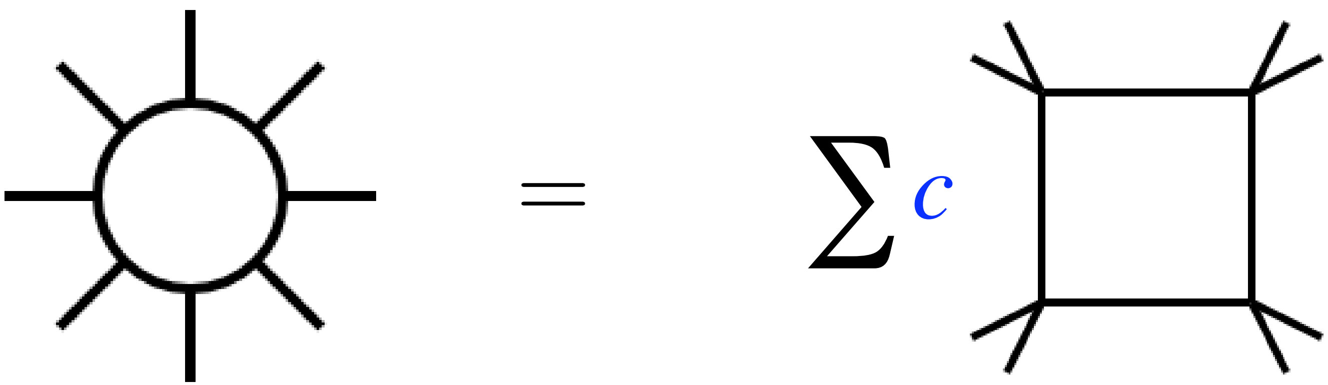
To be computed

# Pure Yang-Mills cont'd

- Result is expressed in terms of
  - finite box functions: $I^{\text{2me}}_{finite} = B(s,t,P^2,Q^2)$
  - triangle functions: $T^{(r)}(p,P,Q) = \dfrac{\log(Q^2/P^2)}{(Q^2-P^2)^r}$
- Coefficient of *B* is: $\left(b^{ij}_{m_1 m_2}\right)^2$
- Agrees with 5-point result and the case of adjacent negative helicity gluons of (BDDK)
- New Result for negative helicity gluons in arbitrary position
  - First step towards QCD from MHV diagrams !

# Generalized Unitarity

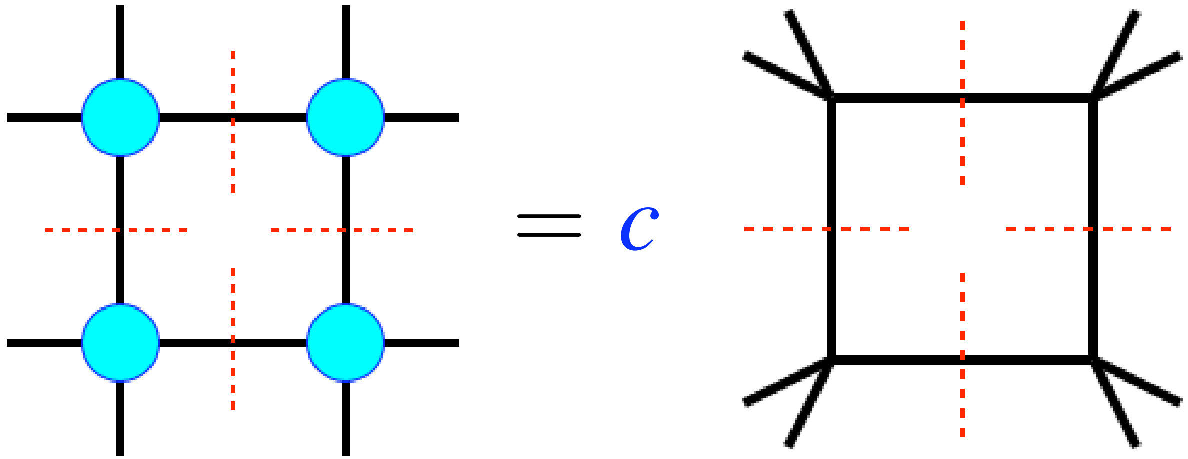
- Very old idea, "The Analytic S(-)Matrix" (Eden-Landshoff-Olive-Polkinghorne 1966; Chew 1966); more recently (Bern-Dixon-Kosower 1997)
  - 2004/05 "The Return of the Analytic S-Matrix"
- One-loop amplitudes in SUSY gauge theories are 4d cut-constructible
- One-loop amplitudes in N=4 SYM have a very simple form:



Q: can we find the rational coefficients $c$ without integrations?

# Quadruple Cuts

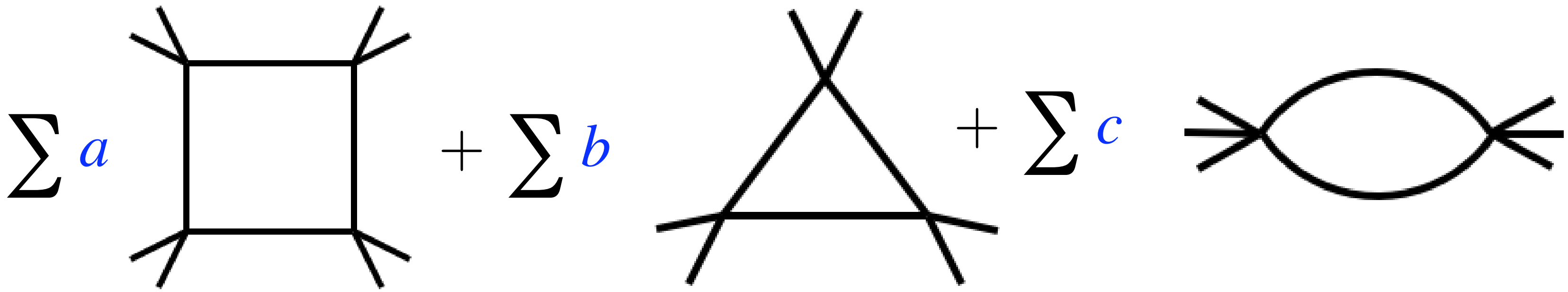
- Answer: Yes! (Britto-Cachazo-Feng)
- Quadruple Cuts = replace four propagators by on-shell delta functions: $1/L_i^2 \to \delta^{(+)}(L_i^2)\ ,\ i = 1,2,3,4$
- Loop integration localises completely! Requires complex momenta!
- The coefficients *c* are products of four on-shell tree amplitudes $c = A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$
- N=4 SYM at one-loop is reduced to algebra!



A quadruple cut selects a unique box function!

# Multiple cuts for amplitudes in N=1 SYM

- Similar for one-loop amplitudes in N=1 SYM, but more work




Fix coefficients *a* with quadruple cuts

Fix coefficients *b* with "triple cuts" (one remaining integration)

Fix *c*'s with conventional unitarity cuts

New results: all N=1, one-loop, 6-point amplitudes and an infinite series $\langle 1^- 2^- 3^- 4^+ \ldots n^+ \rangle$

(Bidder, Bjerrum-Bohr, Dunbar, Perkins; Britto-Buchbinder-Cachazo-Feng)

# Generalised Generalized Unitarity

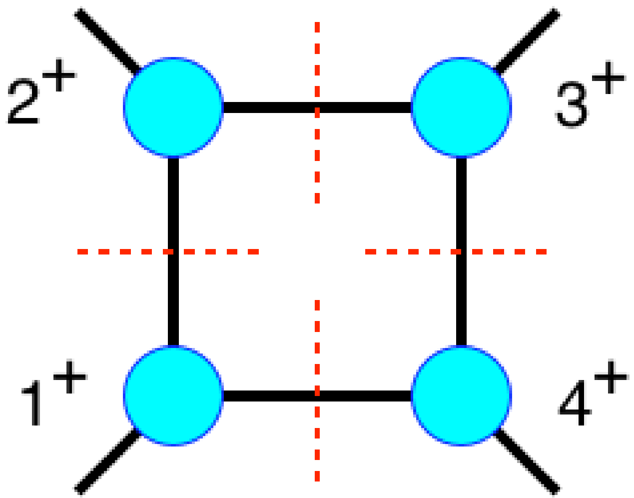
(AB-McNamara-Spence-Travaglini)

- Problem: QCD one-loop amplitudes are not 4d cut-constructible   Amplitudes contain rational terms
- Need to work in $D = 4 - 2\varepsilon$ dimensions
  - $R(-s)^{-\varepsilon} \Rightarrow R - R\,\varepsilon \log(-s) + O(\varepsilon^2)$
- This requires the knowledge of tree amplitudes with some of the legs continued to $D = 4 - 2\varepsilon$ dimensions (DR)
- We can think of this as giving a uniform mass to internal particles. This mass has to be integrated over!
  - $L^2_{4-2\varepsilon} = L^2_4 + L^2_{-2\varepsilon} = L^2_4 - \mu^2$
- Feynman integrals with powers of $[\mu^2]$ inserted lead to integrals in $D = 6 - 2\varepsilon\,,\ 8 - 2\varepsilon\,, \ldots$

# Generalised Unitarity for YM

- The necessary amplitudes with massive particles are provided by old (Berends-Giele, BDDK) and new recursive techniques (Badger-Glover-Khoze-Svrcek) . Because of the SUSY decomposition of the amplitudes we only need to consider scalars running in the loop!

- Ex: <++++> one-loop amplitude in YM from quadruple cut




$$= \frac{[12]}{\langle 12\rangle}\frac{[34]}{\langle 34\rangle}\mu^4$$

$$\Rightarrow A_4^{1-\mathrm{loop}} = \frac{[12]}{\langle 12\rangle}\frac{[34]}{\langle 34\rangle} I_4^{4-2\varepsilon}[\mu^4]$$

$$I_4^{4-2\varepsilon}[\mu^4] = (-\varepsilon)(1-\varepsilon) I_4^{8-2\varepsilon}[\mu^4] = -\frac{1}{6} + O(\varepsilon)$$

# Generalised Unitarity for YM cont'd

- This also works for all other 4-point amplitudes: <−+++>, <−−++> and <−+−+>
  - This requires triple cuts but no 2-particle cuts.
  - The result is expressed in terms of box and triangle functions in 4, 6 and 8 dimensions
- The 5-point amplitude <+++++> requires only quadruple cuts! Expressed in terms of 8 dim'l box and 10 dim'l pentagon integrals.
- Other 5-point and 6-point amplitudes work in progress ...

# Summary & Outlook

- Exciting progress in calculating amplitudes in gauge theory and gravity
- new spectacular insights in the structure of amplitudes from twistor space
- New diagrammatic tools (twistor inspired)
  - MHV diagrams: for tree level (CSW) and loop level amplitudes (BST)
- Generalised Unitarity:
  - new efficient techniques to calculate amplitudes in supersymmetric theories, many new results (BDK,BCF)
  - cut-containing parts and rational terms (require unitarity in $D = 4 - 2\varepsilon$) of amplitudes in QCD

# Summary & Outlook cont'd

- Recently: New on-shell recursion relations (Britto-Cachazo-Feng-(Witten)) Use only on-shell data, analyticity and factorisation of amplitudes
  - Gauge theory amplitudes with massless and massive particles (tree level)
  - Gravity (tree level)
  - Rational terms in one-loop QCD amplitudes
  - Coefficients of integral functions in one-loop amplitudes

Many new results expected

New complete QCD amplitudes within reach !